Reconstruction of Predictively Encoded Signals Over Noisy Channels Using A Sequence MMSE Decoder

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Abstract

In this work, we consider the problem of decoding a predictively encoded signal over a noisy channel when there is a residual redundancy (captured by a γ-order Markov model) in the sequence of transmitted data. Our objective is to minimize the mean squared error in the reconstruction of the original signal (input to the predictive source coder). The problem is formulated and solved through Minimum Mean Squared Error (MMSE) decoding of a sequence of samples over a memoryless noisy channel. The related previous works include several MAP and MMSE-based decoders. The MAP-based approaches are suboptimal when the performance criterion is the mean squared error. On the other hand, the previously known MMSE-based approaches are suboptimal since they are designed to efficiently reconstruct the data samples received (the prediction residues) rather than the original signal. The proposed scheme is setup by modeling the source coder produced symbols and their redundancy with a trellis structure. Methods are presented to optimize the solutions in terms of complexity. Numerical results and comparisons are provided which demonstrate the effectiveness of the proposed techniques.

Keywords

Joint source channel coding, residual redundancies, source decoding, MMSE estimation, MAP detection, forward backward recursion, Markov sources, predictive quantization, DPCM

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I. Introduction

Motivated by the fundamental work of Shannon [1], researchers have performed enormous endeavors on separate treatment of source and channel coders. However, in practise, due to strict design constraints such as limited transmission bandwidth, restricted delay and limitations on the complexity of the systems involved, the joint design of source and channel coders has found increasing interest. Several paths have been taken toward the joint design of source and channel coders in the literature. These methods include optimized rate allocation, unequal error protection, optimized index assignment, channel optimized quantization, and more recently, exploiting the source residual redundancies. For a comprehensive review of these techniques the interested reader is referred to [2]-[7].

The work presented in this manuscript falls into the category of joint source channel coders which use the *residual redundancy* [8] in the output of the source coder for improved reconstruction over noisy channels. This redundancy is due to the suboptimal source coding which is caused by, e.g., a constraint on complexity or delay. In general, this redundancy can be used for enhanced channel decoding, e.g., [9]-[13] or for effective source decoding, e.g., [14]-[19]. This is formulated in the form of a *Maximum A Posteriori* (MAP) detection or a *Minimum Mean Squared Error* (MMSE) estimation problem. The residual redundancy is utilized both at the source and channel decoders in [20] which demonstrates an improved performance. In the same direction, iterative source and channel decoding schemes are presented in [21][22].

This manuscript considers the problem of reconstruction of a predictively quantized signal over a noisy channel when there is a residual redundancy in the source coder output stream. In fact, it is shown in [8], that there is always a residual redundancy in the output of a predictive quantizer due to a mismatch between the encoder prediction model and that of the source.

In predictive coding schemes, the signal that is quantized, $\mathbf{Y}_n$, is the *prediction residue* or the difference between the original signal, $\mathbf{X}_n$ and its estimate produced using a prediction function. In moving average (MA) systems (see Figure 1), the prediction function operates based on μ previous quantized prediction residues, $(\tilde{\mathbf{Y}}_{n-\mu}, \ldots, \tilde{\mathbf{Y}}_{n-1})$, whereas in auto regressive (AR) systems (see Figure 2), μ' previous quantized signals, $(\tilde{\mathbf{X}}_{n-\mu'}, \ldots, \tilde{\mathbf{X}}_{n-1})$ are used for prediction. Alternatively, ARMA systems use both sets of data[1]. For a comprehensive review of predictive

[1] In this work, the terms AR, MA and ARMA predictive systems indicate using either a linear or non-linear predictive function, unless specifically specified.

quantization refer to [23][24].

The output of a predictive coder is the index I_n corresponding to the quantized prediction residue $\tilde{\mathbf{Y}}_n$. In a basic predictive decoder, the block labeled "reverse mapping" in Figure 1, is simply the inverse of the index generation function at the encoder, which ignores any residual redundancy in the source coder output. Recently, researchers have replaced this block with more sophisticated systems which exploit the residual redundancy for improved reconstruction. Sayood and Borkenhagen in [8], proposed a MAP-based decoder. In [16], for a DPCM encoded speech, an MMSE-based scheme is employed that aims at minimizing the error in reconstructing the prediction residue, $\mathbf{Y}_n$ at the receiver. Both schemes of [8] and [16] utilize a first-order Markov model to capture the residual redundancy. For reconstruction of a DPCM encoded image, several schemes have been suggested which exploit the residual redundancy both in the horizontal and the vertical directions [25]-[27]. A scheme, called Maximal SNR decoding, is suggested in [25] which searches for the residue codeword that minimizes a simplified expression of the reconstructed signal SNR. This simplification reduces the objective function to one that represents the error in the reconstruction of the prediction residues. Another scheme for the reconstruction of DPCM encoded images is the MMSE-based decoder proposed in [26] which uses a Markov mesh for the reconstruction of the prediction residues. In general, the MAP-based approaches are suboptimal when the performance criterion is the mean squared error and the previously known MMSE-based approaches are suboptimal, since they aim at minimizing the error in reconstruction of the prediction residues, rather than the original signal (input to the source coder).

In this work, our objective is to design a source decoder (not a reverse mapping unit) which minimizes the mean squared error in the reconstruction of the original signal, when the residual redundancy is captured by a γ-order Markov model ($\gamma \geq 1$) and a delay of $\delta, \delta \geq 0$ is allowed in the decoding process. The problem is formulated and solved through minimum mean squared error decoding of a sequence of samples over a memoryless noisy channel, which was previously recognized to be an open problem by Phamdo and Farvardin [14]. The solution is setup by modeling the stream of encoder produced symbols and their redundancy with a trellis structure. The proposed solution is optimized to minimize the computational complexity.

The organization of the article is as follows. The notations, system and channel model used are presented in section II. In section III, the Sequence MMSE decoder is presented. In section

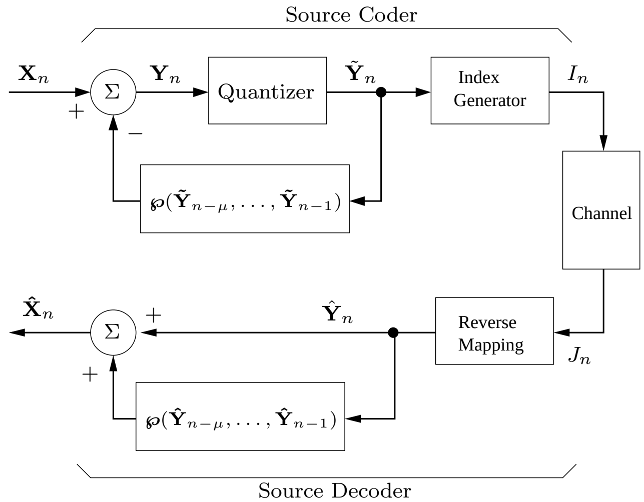

Fig. 1. Conventional DPCM encoder and decoder with a (linear or non-linear) moving average prediction

IV, the application of the proposed Sequence MMSE decoding scheme for the reconstruction of an auto regressive DPCM encoded source is discussed. The systems considered for comparison, numerical results and analysis are presented in section V.

II. Preliminaries

A. Notations

The notations used in this article are as follows. The capital letters, e.g., I, represent random variables, while the small letters, e.g., i, represent a realization. We replace the probability $P(I = i)$ by $P(I)$ in most instances when it does not lead to a confusion. The vectors are shown bold faced, e.g., $\mathbf{X}$. The lower index indicates the time instant, e.g., $\mathbf{X}_n$ is the vector $\mathbf{X}$ at time instant n. The upper index in parenthesis indicates components of a vector or bit positions of an integer, e.g., $\mathbf{X}_n = [X_n^{(1)}, \ldots, X_n^{(N)}]$ where N is the dimension of the vector $\mathbf{X}_n$. A sequence of variables over time, e.g., $(I_{n_1}, \ldots, I_{n_2})$, $n_1 \leq n_2$ is denoted by $\underline{I}_{n_2}^{n_1}$. For simplicity, we represent $\underline{I}_n^1$ by $\underline{I}_n$. The N dimensional Cartesian product of a set $\mathcal{J}$ is represented by $\mathcal{J}^N$ that consists of N dimensional vectors whose components are taken from $\mathcal{J}$.

B. System Overview

The block diagram of the system is shown in Figure 1. The source coder is a mapping from an N-dimensional Euclidean space, $\mathcal{R}^N$, into a finite index set $\mathcal{J}$ of M elements. It is composed of two components: a predictive quantizer and an index generator. The predictive quantizer maps the input sample $\mathbf{X} \in \mathcal{R}^N$ to one of the reconstruction points or *codewords* in the codebook $\mathcal{C} \subset \mathcal{R}^N$. The example of predictive quantizer shown in Figure 1 uses a moving average prediction system. The index generator then maps the codeword selected by the quantizer to an *index (symbol)* I in the index set $\mathcal{J}$. The bitrate of the quantizer $r = \lceil \log_2 M \rceil$ bits/symbol (or $\lceil \log_2 M \rceil / N$ bits/dim).

We assume that the quantized sample $\tilde{\mathbf{X}}_n$ corresponding to the predictive quantizer input $\mathbf{X}_n$ can be described as a function $\mathbf{f}$ of the last $\mu + 1$ encoded symbols, i.e.,

$$\tilde{\mathbf{X}}_n = \mathbf{f}(I_{n-\mu}, \ldots, I_{n-1}, I_n) \tag{1}$$
$$I_{n-k} \in \mathcal{J}, 0 \le k \le \mu$$

where μ denotes the memory length of the predictor. A concrete example is the MA predictive quantizer of Figure 1, for which we have,

$$\tilde{\mathbf{X}}_n = \tilde{\mathbf{Y}}_n + \wp(\tilde{\mathbf{Y}}_{n-\mu}, \ldots, \tilde{\mathbf{Y}}_{n-1}) \tag{2}$$

and noting that the index generator is a simple one to one mapping function, equation (1) holds. In section IV, we demonstrate that a DPCM scheme with an auto regressive predictor can also be cast into the model of equation (1).

At the receiver, for each transmitted r-bit index $I = i$, a vector J with r components is received which provides information about I. The reconstructor (source decoder) maps J to an output sample $\hat{\mathbf{X}}$. In this reconstruction, the source decoder may use the previously received samples or also some of the future samples.

C. Channel Model

The channels considered in this work are described by a pdf $P(J_n|I_n)$. We assume that the channel is memoryless without intersymbol interference in the sense that, for a sequence of transmitted symbols $\underline{I}_n = (I_1, I_2, \ldots, I_n)$ and the corresponding received signals $\underline{J}_n$, the following equality is valid.

$$P(J_n = j_n | \underline{I}_n = \underline{i}_n, \underline{J}_{n-1} = \underline{j}_{n-1}) = P(J_n = j_n | I_n = i_n). \tag{3}$$

This results in the followings,

$$P(J_n = j_n | \underline{I}_n = \underline{i}_n) = P(J_n = j_n | I_n = i_n), \tag{4}$$

$$P(\underline{J}_n = \underline{j}_n | \underline{I}_n = \underline{i}_n) = \prod_{k=1}^{n} P(J_k = j_k | I_k = i_k). \tag{5}$$

An example is a BPSK modulation over a channel with AWGN which produces soft outputs as,

$$j_n^{(m)} = s\left(i_n^{(m)}\right) + \eta_n^{(m)}, \qquad m = 1, \ldots, r. \tag{6}$$

where $i_n^{(m)}, m = 1, \ldots, r$ are the bit components of i_n or the source coder output and $j_n^{(m)}$ are the corresponding channel soft outputs, $\boldsymbol{\eta}_n = [\eta_n^{(1)}, \ldots, \eta_n^{(r)}]$ is a vector of i.i.d. Gaussian noise samples and $s(.) \in \{\sqrt{E_b}, -\sqrt{E_b}\}$ is a mapping of bits to channel signals. The relationship between the transmitted and the received symbols is then given by the following conditional pdf,

$$P(J_n = j_n | I_n = i_n) = \prod_{m=1}^{r} P(j_n^{(m)} | i_n^{(m)}). \tag{7}$$

In this work, we refer to such a channel as the Soft Output Channel model. The Binary Symmetric Channel model is also based on the equation (6), when a hard decision is made on the received soft outputs. If the resulting bit error probability is denoted by ϵ, then the relationship between the transmitted and the received symbols is given by,

$$P(J_n = j_n | I_n = i_n) = (\epsilon)^{h(i_n, j_n)} (1 - \epsilon)^{r - h(i_n, j_n)}, \tag{8}$$

where j_n is the received binary codeword in $\mathcal{J}$ and $h(i_n, j_n)$ is the Hamming distance between indices i_n and j_n. In the following, for the development of the proposed source decoders, we assume that the probability distribution of $P(J_n | I_n)$ is given and the memoryless channel assumption of equation (3) is valid.

III. A Sequence MMSE Decoder

Consider the case where due to the sub-optimality of the predictive source coder there is a residual redundancy in its output stream. This redundancy is in the form of a memory in the sequence of the transmitted symbols or also in the form of a non-uniform symbol probability distribution. Our objective is to design a source decoder that exploits this residual redundancy to effectively reconstruct the original source samples at the receiver. The source decoder is designed to produce the minimum mean squared error estimate of the source sample $\mathbf{X}_n$ given the received sequence $\underline{J}_{n+\delta} = [J_1, J_2, \ldots, J_{n+\delta}]$, where $\delta \geq 0$ is the delay allowed in the decoding process. Based on the fundamental theorem of estimation, this is given by,

$$\hat{\mathbf{x}}_n = E[\mathbf{X}_n | \underline{J}_{n+\delta}] \tag{9}$$

which minimizes the expected squared error of estimation,

$$E[(\mathbf{X}_n - \hat{\mathbf{X}}_n)'(\mathbf{X}_n - \hat{\mathbf{X}}_n)] \tag{10}$$

The equation (9) can be expanded as follows,

$$\hat{\mathbf{x}}_n = \sum_{\underline{I}_{n+\delta} \in \mathcal{J}^{n+\delta}} E[\mathbf{X}_n | \underline{I}_{n+\delta}] P(\underline{I}_{n+\delta} | \underline{J}_{n+\delta}), \tag{11}$$

in which $E[\mathbf{X}_n | \underline{I}_{n+\delta}]$ forms the *decoder codebook*. Therefore, equation (11) presents an optimal decoder that at time n requires a sum over $M^{n+\delta}$ elements of the decoder codebook. In this case, both computational complexity and the memory requirement grow exponentially with time, leading to an impractical scheme. Assuming that the source $\mathbf{X}$ has a memory that asymptotically decays with time, for sufficiently large values of τ, $\tau \in \mathcal{Z}$, the decoder codebook can be approximated by,

$$E[\mathbf{X}_n | \underline{I}_{n+\delta}] \approx E[\mathbf{X}_n | \underline{I}_{n+\delta}^{n-\tau}]. \tag{12}$$

and therefore, the MMSE decoder given by,

$$\hat{\mathbf{x}}_n = \sum_{\underline{I}_{n+\delta}^{n-\tau}} E[\mathbf{X}_n | \underline{I}_{n+\delta}^{n-\tau}] P(\underline{I}_{n+\delta}^{n-\tau} | \underline{J}_{n+\delta}). \tag{13}$$

is asymptotically optimal and yet feasible. This decoder is in fact the same as the Asymptotically Optimum MMSE decoder derived in [6] for a memoryless source coder and it shows that the same formulation is applicable here.

The decoder codewords $E[\mathbf{X}_n | \underline{I}_{n+\delta}^{n-\tau}]$ (for sufficiently large values of τ) provide a finer reconstruction of the source samples as compared to the quantized signal at the encoder (given by equation (1) as a function of only $\underline{I}_n^{n-\mu}$). Now, we turn our attention to derive a simplified MMSE decoder for source coders with memory. Specifically, we are interested in a source decoder which uses a decoder codebook similar to its corresponding encoder (quantization) codebook. This is of particular interest since it leads to a less complex decoder with significantly smaller memory requirement, specially in symmetric communication systems where the encoder codebook is already available at the receiver. Consequently, we consider the following sequence MMSE decoder,

$$\hat{\mathbf{x}}_n = \sum_{\underline{I}_n^{n-\mu} \in \mathcal{J}^{\mu+1}} \mathbf{f}(\underline{I}_n^{n-\mu}) P(\underline{I}_n^{n-\mu} | \underline{J}_{n+\delta}). \tag{14}$$

which provides the MMSE estimate as a weighted average of the reconstruction values $\mathbf{f}(\underline{I}_n^{n-\mu})$. Each weight or the probability $P(\underline{I}_n^{n-\mu} = \underline{i}_n^{n-\mu} | \underline{J}_{n+\delta})$ is the a posteriori probability of a sequence of symbols calculated in every time instant. In the following section, methodologies are presented to calculate the required a posteriori probabilities. Subsequently, in section IV a decoder for reconstruction of an AR DPCM coded signal over a noisy channel based on the proposed Sequence MMSE decoder is presented.

A. Calculating the Weights

To calculate the a posteriori probabilities required in the proposed Sequence MMSE decoder of equation (14), we assume that the encoder output symbols form a γ-order Markov model due to the residual redundancies. These symbols are then modeled by a trellis structure. In this structure, the states at time n are defined by the ordered set,

$$S_n = (I_{n-\gamma+1}, I_{n-\gamma+2}, \ldots, I_{n-1}, I_n), \quad (15)$$
$$I_{n-k} \in \mathcal{J}, 0 \le k < \gamma.$$

Hence, there are M^γ states in each time step (stage), $S_n \in \mathcal{J}^\gamma$. Each branch leaving the state at time step n corresponds to one particular symbol $I_{n+1} = i_{n+1}$. Therefore, there are M branches leaving each state. Each branch is identified by the pair $(S_n = s_n, S_{n+1} = s_{n+1})$ of the two states that the branch connects together. Having defined the trellis structure as such, there will be one a priori probability $P(I_{n+1} = i_{n+1} | S_n = s_n)$ corresponding to each branch which characterizes the γ-order Markov property of the source coder symbols. The states now form a first-order Markov sequence. Using this property and the memoryless assumption of the channel (see equations (3)-(5)), in line with the BCJR algorithm [28], the probability of a particular state S_n given the observed sequence $\underline{J}_{n+\delta}$ is calculated recursively by the following forward backward equation,

$$P(S_n | \underline{J}_{n+\delta}) = C \,.P(S_n | \underline{J}_n) \,.\, P(\underline{J}_{n+\delta}^{n+1} | S_n) \quad (16)$$

where C is a factor which normalizes the sum of the probabilities to one. The term $P(S_n | \underline{J}_n)$ is the forward term and is given by,

$$P(S_n | \underline{J}_n) = C.P(J_n | I_n) \,. \sum_{S_{n-1} \to S_n} P(I_n | S_{n-1}) P(S_{n-1} | \underline{J}_{n-1}) \quad (17)$$

where the summation is over a subset of M states in time step $n-1$ which are connected to the state S_n. The term $P(\underline{J}_{n+\delta}^{n+m+1} | S_{n+m})$ in equation (16) is the backward term and is calculated recursively by,

$$P(\underline{J}_{n+\delta}^{n+1}|S_n) = \sum_{I_{n+1}\in\mathcal{J}} P(J_{n+1}|I_{n+1})\,.\,P(I_{n+1}|S_n)\,.\,P(\underline{J}_{n+\delta}^{n+2}|S_{n+1}) \tag{18}$$

where the recursion starts from,

$$P(J_{n+\delta}|S_{n+\delta-1}) = \sum_{I_{n+\delta}\in\mathcal{J}} P(J_{n+\delta}|I_{n+\delta})\,.\,P(I_{n+\delta}|S_{n+\delta-1}) \tag{19}$$

and continues backward in each time step. The details of the derivation of these equations are provided in [6][7]. We note that in each time step, the forward recursion of equation (17) proceeds one step forward through the trellis while the backward term is recomputed over the entire backward window as indicated in equations (18) and (19). The presented trellis structure and either of the forward and backward equations are used in the following sections for calculation of the required probabilities (weights) in equation (14). Depending on the relative value of encoder memory μ to the residual redundancy order γ this is performed in two ways as described below.

A.1 Calculating the weights for $\mu < \gamma$

For the scenario with $\mu < \gamma$, we can calculate the probabilities required in equation (14), by performing $\gamma - \mu - 1$ summations over *any of the state probabilities* $P(S_{n+m}|\underline{J}_{n+\delta})$ *as long as* S_{n+m} *includes* $\underline{I}_n^{n-\mu}$ *or equivalently,* $0 \leq m \leq \gamma - \mu - 1$. However, it is shown that the number of computations required for the forward and backward recursions per time step (denoted by NC_{fwd} and NC_{bwd} respectively) is given by,

$$NC_{fwd} = (2M+3)\,M^{\gamma} \tag{20}$$

$$NC_{bwd} = 3(\delta - m)\,M^{\gamma+1} \tag{21}$$

where $\delta - m$ is the number of backward recursions required per time step. Therefore, we can select the value of m such that it minimizes the overall computational burden which consists of the computations required for the forward and the backward terms. Noting that only NC_{bwd} depends on m, we solve the following for the optimum value of m,

$$\begin{aligned} &\text{Minimize} \quad NC_{bwd} = 3(\delta - m)\,.\,M^{\gamma+1} \\ &\text{subject to} \quad 0 \leq m \leq \gamma - \mu - 1; \quad 0 \leq m \leq \delta \end{aligned} \tag{22}$$

case 1. $\delta < \gamma - \mu$ In the cases where the delay is smaller than the difference of the assumed residual redundancy order and the encoder memory, we are able to choose $m = \delta$ and eliminate the backward term. The probabilities in equation (14) are calculated using (17) and the following,

$$P(\underline{I}_n^{n-\mu}|\underline{J}_{n+\delta}) = \ldots \sum_{I_{n+k}} \ldots P(S_{n+\delta}|\underline{J}_{n+\delta}), \qquad (23)$$
$$k = \delta - \gamma + 1,\ \ldots,\ \delta, \quad k \neq -\mu, \ldots, 1, 0.$$

where equation (23) indicates $\gamma - \mu - 1$ summations over the probabilities of states at time step $n + \delta$, $S_{n+\delta} = (I_{n-\gamma+\delta+1}, \ldots, I_{n+\delta})$.

case 2. $\delta \geq \gamma - \mu$ Alternatively, when the delay is larger than $\gamma - \mu$, the NC_{bwd} is minimized when $m = \gamma - \mu - 1$, i.e., $\delta + \mu - \gamma + 1$ backward recursions are required. The probabilities in equation (14) are now given by,

$$P(\underline{I}_n^{n-\mu}|\underline{J}_{n+\delta}) = \sum_{I_{n+1}} \sum_{I_{n+2}} \cdots \sum_{I_{n+\gamma-\mu-1}} P(S_{n+\gamma-\mu-1}|\underline{J}_{n+\delta}) \qquad (24)$$

and equations (17) to (19).

A.2 Calculating the weights for $\mu \geq \gamma$

For the scenario with the residual redundancy order smaller than the encoder memory $\mu \geq \gamma$, the sequence $\underline{I}_n^{n-\mu} = (I_{n-\mu}, \ldots, I_{n-1}, I_n)$ whose a posteriori probability is required, in fact corresponds to a sequence of states within the trellis structure of the source coder produced symbols as described before. Consequently, the desired probabilities can be calculated using the probability of the corresponding sequence of states. We have,

$$P(\underline{I}_n^{n-\mu}|\underline{J}_{n+\delta}) = P(\underline{S}_n^{n-\mu+\gamma-1}|\underline{J}_{n+\delta}) \qquad (25)$$

This can be written in the following forward backward form where we have used the assumption of redundancy order of γ, to replace $P(\underline{J}_{n+\delta}^{n+1}|\underline{S}_n^{n-\mu+\gamma-1})$ with $P(\underline{J}_{n+\delta}^{n+1}|S_n)$.

$$P(\underline{S}_n^{n-\mu+\gamma-1}|\underline{J}_{n+\delta}) = C.P(\underline{S}_n^{n-\mu+\gamma-1}|\underline{J}_n).P(\underline{J}_{n+\delta}^{n+1}|S_n) \qquad (26)$$

The value C is a factor which normalizes the sum of probabilities to one. The second term or the backward term is given by the equations (18) and (19). The forward term is given by,

$$P(\underline{S}_n^{n-\mu+\gamma-1}|\underline{J}_n) = \left[\prod_{k=-\mu+\gamma}^{0} P(J_{n+k}|I_{n+k}).P(I_{n+k}|S_{n+k-1})\right] P(S_{n-\mu+\gamma-1}|\underline{J}_{n-\mu+\gamma-1}) \qquad (27)$$

Alternative ways to calculate the required a posteriori probabilities $P(\underline{I}_n^{n-\mu}|\underline{J}_{n+\delta})$ for the Sequence MMSE decoder of equation (14) is possible by using the extended trellis structure described in [6][7].

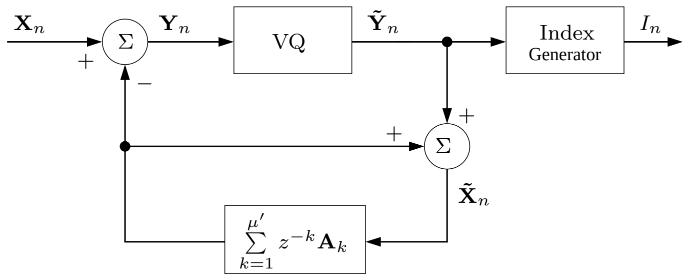

Fig. 2. DPCM encoder with linear auto regressive prediction

IV. Reconstruction of Predictively Encoded Signals

In this section, we consider the MMSE reconstruction of a linear auto regressive DPCM coded signal over a noisy channel. This focus is due to the popularity of these systems and the fact that the ideas employed in this case can be easily applied to the other cases including moving average (linear or nonlinear) predictive encoding systems.

Figure 2, demonstrates the block diagram of a DPCM encoder with a linear auto regressive prediction. In this system, the quantized sample $\tilde{\mathbf{X}}_n$ is given by

$$\tilde{\mathbf{X}}_n = \tilde{\mathbf{Y}}_n + \sum_{k=1}^{\mu'} \mathbf{A}_k \tilde{\mathbf{X}}_{n-k}. \tag{28}$$

By recursive replacement of $\tilde{\mathbf{X}}_{n-k}$ in equation (28), it is straight forward to see that $\tilde{\mathbf{X}}_n$ can be described as a function of the sequence of prediction residues $(\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2, \ldots, \tilde{\mathbf{Y}}_n)$. Consequently, the equation (1) holds and a solution based on the proposed Sequence MMSE Decoder exists. However, this implies that the length of the sequence to be decoded, $\underline{\tilde{\mathbf{Y}}}_n$ grows with time. A manageable solution is created by defining an *effective memory* length, i.e., assuming that the sample $\tilde{\mathbf{X}}_n$ depends *effectively* only on $\tilde{\mathbf{Y}}_n$ and μ previous prediction residue values, $(\tilde{\mathbf{Y}}_{n-\mu}, \ldots, \tilde{\mathbf{Y}}_{n-1}, \tilde{\mathbf{Y}}_n)$. Therefore, we can finalize the reconstructed value of the residues beyond $n - \mu$ or equivalently their corresponding output $\hat{\mathbf{X}}_{n-\mu-1}$. This idea is supported by the fact that in DPCM systems error in one sample is effectively propagated to only a limited number of future samples. Using this concept, we now consider the case of a first-order AR DPCM system in more detail.

For a first-order AR predictive coder, we have,

$$\begin{aligned}\mathbf{X}_n &= \mathbf{Y}_n + \mathbf{A}\tilde{\mathbf{X}}_{n-1} && (29)\\ &= \mathbf{Y}_n + \sum_{k=1}^{\mu}\mathbf{A}^k\tilde{\mathbf{Y}}_{n-k} + \mathbf{A}^{\mu+1}\tilde{\mathbf{X}}_{n-\mu-1} && (30)\\ &= \mathbf{Z}_n + \mathbf{A}^{\mu+1}\tilde{\mathbf{X}}_{n-\mu-1}, && (31)\end{aligned}$$

where

$$\mathbf{Z}_n \triangleq \mathbf{Y}_n + \sum_{k=1}^{\mu}\mathbf{A}^k\tilde{\mathbf{Y}}_{n-k} \tag{32}$$

Using equation (31), the MMSE estimate (equation (9)) is now given by

$$\begin{aligned}\hat{\mathbf{x}}_n &= E[\mathbf{X}_n|\underline{J}_{n+\delta}]\\ &= E[\mathbf{Z}_n|\underline{J}_{n+\delta}] + \mathbf{A}^{\mu+1}E[\tilde{\mathbf{X}}_{n-\mu-1}|\underline{J}_{n+\delta}] && (33)\end{aligned}$$

Subsequently, assuming an effective memory length of μ, we approximate the second term by $\mathbf{A}^{\mu+1}\hat{\mathbf{x}}_{n-\mu-1}$. Next, we reach a recursive formula for MMSE decoding of a first-order AR DPCM system.

$$\hat{\mathbf{x}}_n = \hat{\mathbf{z}}_n + \mathbf{A}^{\mu+1}\hat{\mathbf{x}}_{n-\mu-1}, \tag{34}$$

where

$$\hat{\mathbf{z}}_n = E[\mathbf{Z}_n|\underline{J}_{n+\delta}] \tag{35}$$

can be calculated using the Asymptotically Optimum MMSE decoder of equation (13) or the (simplified) Sequence MMSE decoder of equation (14). The latter is motivated by the fact that

$$\begin{aligned}\tilde{\mathbf{Z}}_n &= \mathbf{f}(I_{n-\mu},\ldots,I_{n-1},I_n)\\ &= \sum_{k=0}^{\mu}\mathbf{A}^k E[\mathbf{Y}_{n-k}|I_{n-k}] && (36)\end{aligned}$$

as required by the assumption of equation (1). Subsequently, the solution based on the Sequence MMSE decoder for reconstruction of a first-order DPCM encoded signal over a noisy channel is given by

$$\hat{\mathbf{x}}_n = \sum_{\underline{I}_n^{n-\mu}}\left[\sum_{k=0}^{\mu}\mathbf{A}^k E[\mathbf{Y}_{n-k}|I_{n-k}]\right]P(\underline{I}_n^{n-\mu}|J_{n+\delta}) + \mathbf{A}^{\mu+1}\hat{\mathbf{x}}_{n-\mu-1} \tag{37}$$

in which the decoder codebook is determined by the encoder (quantization) codebook. Note that the $E[\mathbf{Y}|I]$ in equations (36) and (37) is the encoder (quantization) codeword assuming an LBG vector quantizer [30]. It is noteworthy that for $\mu = 0$ the solution collapses to that of the MMSE reconstruction of prediction residues,

$$\hat{\mathbf{x}}_n = \sum_{I_n \in \mathcal{J}} E[\tilde{\mathbf{Y}}_n | I_n] P(I_n | \underline{J}_{n+\delta}) + \mathbf{A}\hat{\mathbf{x}}_{n-1}. \tag{38}$$

In equation (37), the assumption of an effective memory length of μ reflects the fact that $\hat{\mathbf{x}}_n$ is composed of the decoded sample $\hat{\mathbf{x}}_{n-\mu-1}$ and the *soft* (un-decoded) information on symbols $\underline{I}_n^{n-\mu}$ which are positioned within the effective memory of the predictive decoder. This soft information is encapsulated in a posteriori probabilities $P(\underline{I}_n^{n-\mu} | J_{n+\delta})$. In section V, we investigate the performance of this decoder where it is referred to as the SMMSE decoder.

V. Performance Analysis

To analyze the performance of the proposed decoders, we use a synthesized source similar to [8]. The source here is a tenth-order Gauss-Markov source with the coefficients given in Table I. The coefficients are matched to the LPC coefficients of a 20ms segment of speech. The source

coefficient (1-5)	1.1160	0.5365	-0.1830	-0.5205	-0.0535
coefficient (6-10)	-0.3159	0.3263	-0.0194	0.2841	-0.2006

TABLE I

Coefficients of the synthesized source

$\mathbf{X}_n = [X_{(n-1)N+1}, \ldots, X_{nN}]$ is the input to a first-order linear auto regressive DPCM encoder. The quantizer is an M point N dimensional LBG VQ [30]. In our particular example we consider $M = 8, N = 1$ and a predictor designed for noisy channels [29]. The Index Assignment is natural binary and we use a Soft Output Channel model as described in section II-C.

Table III presents, the value $R(M, \gamma)$ (in bits) defined as

$$R(M, \gamma) \triangleq \log_2 M - H(I_n | S_{n-1}) \tag{39}$$

where $S_n = (I_{n-\gamma+1}, \ldots, I_n)$, as an indication of the available redundancy at the output of the source coder and hence the gain to be achieved using different redundancy model orders γ. A similar expression up to a scaling for the case of a first-order Markov model is presented in [8] and referred to as the error correction capability index. As given in this Table, the redundancy due to the non-uniform distribution ($\gamma = 0$) is 0.34 bits. The redundancy exploited by means of a first, second and third order Markov model is 1.15, 1.40 and 1.44 bits respectively.

Redundancy Order γ	0	1	2	3
$R(M, \gamma)$ (bits)	0.34	1.15	1.40	1.44

TABLE III

Redundancy of the source coder output, $R(M, \gamma)$ (bits), at different redundancy model orders γ, $(M = 8, N = 1)$.

A. Systems for Comparison: ML, MMSE, SMAP, MSNR

Several schemes are considered for comparison to the proposed Sequence MMSE decoder. As mentioned before, all these schemes reconstruct the prediction residues $\hat{\mathbf{y}}_n$ (or select the corresponding index $\hat{i}_n$), which is then fed to an ordinary DPCM decoder. In our experiment set up this can be written as,

$$\hat{\mathbf{x}}_n = \mathbf{A}\hat{\mathbf{x}}_{n-1} + \hat{\mathbf{y}}_n \tag{40}$$

As base-lines for comparisons we consider the Maximum Likelihood decoder given by,

$$\hat{i}_n = \arg \min_{I_n \in \mathcal{J}} P(J_n|I_n) \tag{41}$$

and the basic MMSE decoder [6] given by,

$$\hat{\mathbf{y}}_n = \sum_{I_n \in \mathcal{J}} E[\mathbf{Y}_n|I_n]P(J_n|I_n) \tag{42}$$

Both the ML decoder and the MMSE decoder do not utilize any of the available residual redundancy for reconstruction.

The Sequence MAP (SMAP) decoder detailed in [6] is also considered for reconstruction of predictively encoded signals. The SMAP decoder exploits the residual redundancy in the source coder output with a Markov model of order γ. It decodes the prediction residues corresponding to the most probable transmitted sequence of symbols using a Viterbi-style decoder in which the trellis is constructed as described in section III and the metric corresponding to branch (S_{k-1}, S_k) is given by $\log[P(J_k|I_k)P(I_k|S_{k-1})]$. Related and similar sequence MAP decoders are available in [8][14][25] as well.

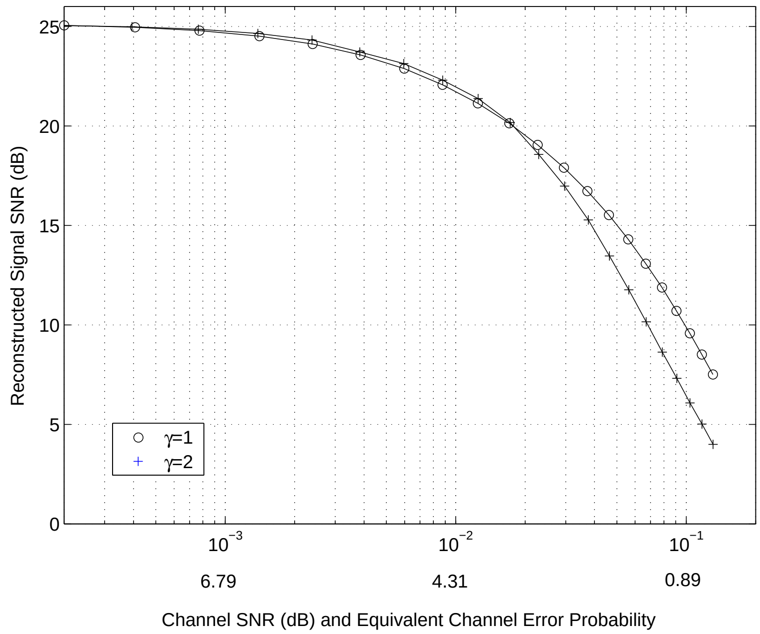

Fig. 3. Performance of the Maximal SNR decoder over a Soft Output Channel when different levels of residual redundancy are exploited at the decoder ($M = 8$).

The Maximal SNR (MSNR) receiver presented in [25] is also considered for comparison. The MSNR decoding rule is given by,

$$\hat{i}_n = \arg\min_{i_n \in \mathcal{J}} \sum_{i_k \in \mathcal{J}} d(i_n, i_k)\, P(J_n | I_n = i_k) P(I_n = i_k | \underline{I}_{n-1}^{n-\gamma} = \hat{\underline{i}}_{n-1}^{n-\gamma}) \tag{43}$$

where,

$$d(i_n, i_k) = (\tilde{\mathbf{Y}}(i_n) - \tilde{\mathbf{Y}}(i_k))(\tilde{\mathbf{Y}}(i_n) - \tilde{\mathbf{Y}}(i_k))' \tag{44}$$

and $\tilde{\mathbf{Y}}(i_n)$ denotes the residue codeword corresponding to the index i_n; here we have $\tilde{\mathbf{Y}}(i_n) = E[\mathbf{Y}_n | I_n = i_n]$. The MSNR decoder suffers from error propagation, since it is designed based on the assumption of the correctness of the previously decoded signals. This is observed from Figure 3 which shows that the performance of the MSNR decoder degrades with the increase of redundancy order γ at high channel error rates. In fact, it appears that by increasing γ, the gain due to the extra use of the residual redundancy is removed by the loss due to the error propagation. Note that in our experiments, for the cases where two or more symbols produce the same value for the distortion function of equation (43), we adopted a rule to select the symbol with the highest a posteriori probability $P(J_n | I_n)$. We found that a trivial selection among these codewords, specially at very low error rates, results in error propagation and degrades the

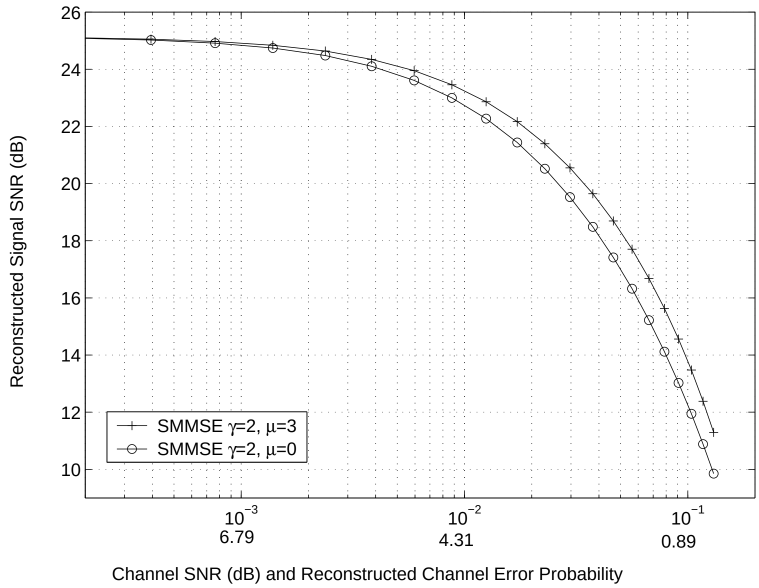

Fig. 4. Performance of the Sequence MMSE decoder, effect of μ or the effective memory length of the decoder ($\gamma = 2, \delta = 0$)

performance.

B. SMMSE Decoder Numerical Results

Figure 4 demonstrates the effect of the effective decoder memory length μ, on the performance of the proposed Sequence MMSE decoder. It is seen that increasing μ noticeably enhances the performance. For the case with $\mu = 3$ a gain of more than 1.5dB in reconstructed signal SNR is achieved over the case with $\mu = 0$.

Figure 5 provides a performance comparison between the proposed Sequence MMSE decoder and the Sequence MAP decoder. Also, the performance of the Maximal SNR decoder of Equation (43) for $\gamma = 1$ and the basic MMSE decoder of Equation (42) as well as the ML decoder of Equation (41) are depicted in the same figure.

It is seen that the proposed Sequence MMSE decoder provides an effective solution for reconstruction of predictive coded signals transmitted over a noisy channel. It outperforms the sequence MAP decoder by nearly 2dB. The Sequence MMSE decoder also gains as high as 8dB compared to the Maximum Likelihood decoder and as high as 4dB compared to the MSNR decoder.

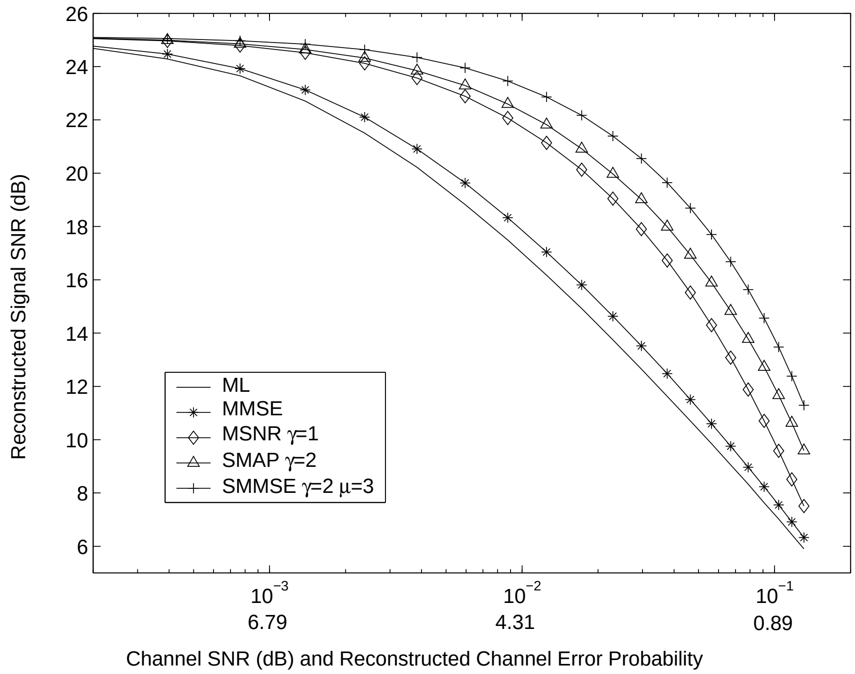

Fig. 5. Performance comparison of the Sequence MMSE decoder with Sequence MAP ($\delta = 0$), Maximal SNR, basic MMSE and ML decoders

VI. Conclusions

In this manuscript, the problem of reconstruction of predictively encoded signals over noisy channels is considered. Due to sub-optimality of the source coder, there is a residual redundancy in its output stream which is modeled by a γ-order Markov model. We present a Sequence MMSE decoder which is formulated to minimize the mean squared error in the reconstruction of original signal (input to the source coder) at the receiver. This is different from the previous approaches which aim at decoding of the data samples received over the channel (prediction residues, output of the source coder). Numerical results are presented which demonstrate the effectiveness of the proposed algorithm.

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