need to show that $v_f(\frac{n-1}{2} - k - 1) = 0$ and $v_f(\frac{n+1}{2} + k + 1) = 1$ provided that (P_k) is true.

If $v_f(\frac{n+1}{2} + k + 1) = 0$, then

$$\begin{aligned} & f(x_1, \ldots, x_n) = 1 \\ \Longrightarrow & \mathrm{wt}(x_1, \ldots, x_n) \in \Big\{0, 1, 2, \ldots, \frac{n-1}{2} - k - 1, \frac{n+1}{2}, \\ & \frac{n+1}{2} + 1, \ldots, \frac{n+1}{2} + k, \frac{n+1}{2} + k + 2, \ldots, n\Big\} \end{aligned} \quad (2)$$

By Lemma 2, we have $g(x_1, \ldots, x_{2k+3}) \in B_{2k+3}$ such that $\deg(g) \leq k + 1$ and

$$\begin{aligned} & g(x_1, \ldots, x_{2k+3}) = 1 \\ \Longrightarrow & \mathrm{wt}(x_1, \ldots, x_{2k+3}) \in \{1, 2, \ldots, k+1, 2k+3\}. \end{aligned}$$

For $h(x_{2k+4}, \ldots, x_n) = (x_{2k+4} + x_{2k+5}) \ldots (x_{n-1} + x_n)$, we have $\deg(h) = \frac{n-2k-3}{2}$ and

$$\begin{aligned} & h(x_{2k+4}, \ldots, x_n) = 1 \\ \Longrightarrow & \mathrm{wt}(x_{2k+4}, \ldots, x_n) = \frac{n-2k-3}{2} = \frac{n-1}{2} - k - 1. \end{aligned}$$

Therefore, $\deg(gh) \leq k + 1 + \frac{n-2k-3}{2} = \frac{n-1}{2}$, and

$$\begin{aligned} & gh(x_1, \ldots, x_n) = 1 \\ \Longrightarrow & \mathrm{wt}(x_1, \ldots, x_n) \in \Big\{\frac{n-1}{2} - k \\ & \frac{n-1}{2} - k + 1, \ldots, \frac{n-1}{2}, \frac{n-1}{2} + k + 2\Big\}. \end{aligned} \quad (3)$$

Since two sets on the right-hand side of (2) and (3) are disjoint, we get $fgh = 0$ which contradicts $\deg(gh) \leq \frac{n-1}{2}$ and $\mathrm{AI}(f) = \frac{n+1}{2}$. Therefore, we proved $v_f(\frac{n+1}{2} + k + 1) = 1$.

Consider the symmetric Boolean function $f'(x_1, \ldots, x_n) = f(x_1 + 1, \ldots, x_n + 1) + 1 \in B_n$. It is easy to see that $v_{f'}(i) = v_f(n-i) + 1 (0 \leq i \leq n)$ so that (P_k) is true for f'. Since $\mathrm{AI}(f') = \mathrm{AI}(f) = \frac{n+1}{2}$ we know that $v_{f'}(\frac{n+1}{2} + k + 1) = 1$ by the proof above. Therefore $v_f(\frac{n-1}{2} - k - 1) = 0$ and (P_{k+1}) is true for f. This completes the proof. □

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On the Capacity of Time-Varying Channels With Periodic Feedback

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***Abstract*—The capacity of time-varying channels with periodic feedback at the transmitter is evaluated. It is assumed that the channel-state information (CSI) is perfectly known at the receiver and is fed back to the transmitter at the regular time intervals. The system capacity is investigated in two cases: 1) finite-state Markov channel, and 2) additive white Gaussian noise channel with time-correlated fading. In the first case, it is shown that the capacity is achievable by multiplexing multiple codebooks across the channel. In the second case, the channel capacity and the optimal adaptive coding is obtained. It is shown that the optimal adaptation can be achieved by a single Gaussian codebook, while adaptively allocating the total power based on the side information at the transmitter.**

***Index Terms*—Adaptive modulation, channel capacity, Gaussian channel, periodic feedback, time-correlated fading.**

I. INTRODUCTION

Communications theory over time-varying channels has been widely studied from different perspectives regarding the availability of the channel-state information (CSI) at the transmitter and/or the receiver. Communication with perfect CSI at the transmitter is studied by Shannon in [1], where the capacity is expressed as that of an equivalent memoryless channel without side information at either the transmitter or the receiver. Communication with perfect CSI at

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Color version of Fig. 1 is available online at http://ieeexplore.ieee.org.

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the receiver is investigated, for example, in [2]. With the assumption of perfect CSI at both the transmitter and the receiver, the capacity of finite-state Markov channels (FSMCs) and compound channels is studied in [3] and [4], respectively. In practice, the assumption of perfect CSI is not practical due to estimation inaccuracy, limited feedback channel capacity, or feedback delay. Communication with imperfect side information is well investigated in the literature [5]–[9]. In [5], the capacity of FSMCs is evaluated based on the assumed statistical relationship of the channel state and side information at the transmitter. The channel capacity, when feedback delay is taken into account, is studied in [10], [11]. The optimal transmission and feedback strategies with finite feedback alphabet cardinality is investigated in [12].

In this correspondence, we consider a point-to-point time-varying wireless channel with perfectly known CSI at the receiver. It is assumed that the channel is constant during a channel use and varies from one channel use to the next, based on a Markov random process. The CSI is provided at the transmitter through a noiseless feedback link at regularly spaced time intervals. Every T channel use, the CSI of the current channel is fed back to the transmitter. We obtain the channel capacity of the system and show that it is achievable by multiplexing T codebooks across the channel. It is worth mentioning that for FSMCs, the results of [5] apply directly to compute the channel capacity, if the side information at the transmitter and receiver are jointly stationary. However, in our model, the side information at the transmitter is not stationary.

Adaptive transmission is an efficient technique to increase the spectral efficiency of the time-varying wireless channels by adaptively modifying the transmission rate, power, etc., according to the state of the channel seen by the receiver. Adaptive transmission, which requires accurate channel estimates at the receiver and a reliable feedback path between the receiver and transmitter, was first proposed in the late 1960s [13]. Practical implementation of adaptive transmission schemes has been the subject of numerous research works (see [14]–[17] and references therein). In specific, a variable-rate and variable-power MQAM modulation scheme for high-speed data transmission over fading channels is studied in [14], [15], where the transmission rate and power are optimized to maximize the spectral efficiency. We utilize the introduced feedback model to obtain the capacity of additive white Gaussian noise (AWGN) channel with time-correlated fading. It is shown that the capacity is achievable using a single codebook with adaptively allocating power based on the side information at the transmitter. Also, the optimum power allocation is derived.

The rest of the correspondence is organized as follows. In Section II, the system model is described and the channel capacity is obtained. The capacity of time-correlated fading channel with periodic feedback is derived in Section III. The impact of channel correlation and feedback error on the capacity is evaluated in Section IV. Finally, the correspondence is concluded in Section V.

Throughout this correspondence, upper case letters represent random variables; lower case letters denote a particular value of the random variable; a_m^n represents the sequence $(a_m, \ldots, a_n)$ and a^* is the complex conjugate of a.

II. Markov Channel With Feedback State

We consider a channel with discrete input $X_n \in \mathcal{X}$ and discrete output $Y_n \in \mathcal{Y}$ at time instant n. The channel state is characterized as a finite-state first-order Markov process

$$\Pr\left(u_n | u_1^{n-k}\right) = \Pr(u_n | u_{n-k}). \tag{1}$$

The channel output at time n is assumed to depend only on the channel input and state at time n, i.e., $\Pr(y_n | x_1^N, u_1^N) = \Pr(y_n | x_n, u_n)$. Hence, the block transition probability of the channel is

$$\Pr\left(y_1^N | x_1^N, u_1^N\right) = \prod_{n=1}^{N} \Pr\left(y_n | x_n, u_n\right) \tag{2}$$

which implies that the channel is memoryless given the state process $U_n \in \mathcal{U}$. It is assumed that CSI is perfectly known at the receiver. The CSI is provided at the transmitter through a noiseless feedback link periodically at every T symbols, i.e., $U_1, U_{T+1}, U_{2T+1}, \ldots$ are sent over the feedback link and instantly received at the transmitter. Assume that the codeword length, N, is an integer factor of T and $M \triangleq \frac{N}{T}$. Let us define $V_i \triangleq U_{T(i-1)+1}$ for $1 \le i \le M$, and $\tilde{n} \triangleq \lfloor \frac{n}{T} \rfloor + 1$.

Encoding and Decoding: Assume that $W \in \mathcal{W}$ is the message to be sent by the transmitter and $A_w = 2^{NR}$ is the cardinality of $\mathcal{W}$. A codeword of length N is a sequence of the encoding function φ_n which maps the set of messages to the channel input alphabets. The input codeword at time n depends on the message w and the CSI at the transmitter up to time n, i.e., $v_1^{\tilde{n}}$

$$x_n = \varphi_n\left(w, v_1^{\tilde{n}}\right). \tag{3}$$

The decoding function ϕ maps a received sequence of N channel outputs using CSI at the receiver to the message set such that the decoded message is $\hat{w} = \phi(y_1^N, u_1^N)$.

Theorem 1: The capacity of a finite state Markov channel with periodic feedback is given by

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{v}\Pr(v)\max_{q_t(x|v)}\sum_{u}P_t(u|v)I(X;Y|u,v) \tag{4}$$

where T is the feedback period, $P_t(u|v) = \Pr_{u_i|u_{i-t+1}}(u|v)$, and $q_t(x|v)$ is the random coding probability distribution function (pdf) parametrized with subscript t to reflect the dependency on time.

A. Achievability

We state a result on the capacity of FSMCs, which we then apply in the proof. It is shown that the capacity of FSMCs with perfectly known CSI, U, at the receiver and side information V at the transmitter is [5]

$$C = \sum_{v}\Pr(v)\max_{q(x|v)}\sum_{u}\Pr(u|v)I(X;Y|u,v) \tag{5}$$

where U, and V are jointly stationary and ergodic with joint pdf $\Pr(U,V)$, and V is a deterministic function of U.

We consider the channel as T parallel subchannels where the tth subchannel $(1 \le t \le T)$ occurs in time instances $(i-1)T+t$, $1 \le i \le M$. Noting that the channel state of the tth subchannel $\{U_{(i-1)T+t}\}_{i=1}^{M}$ and the side information at the transmitter $\{V_i\}_{i=1}^{M} = \{U_{(i-1)T+1}\}_{i=1}^{M}$ are jointly stationary and ergodic, we define $P_t(u|v) = \Pr_{u_i|u_{i-t+1}}(u|v)$ for $1 \le t \le T$. Using (5), the achievable rate of the tth subchannel is

$$R_t = \sum_{v}\Pr(v)\max_{q_t(x|v)}\sum_{u}P_t(u|v)I(X;Y|u,v). \tag{6}$$

T codebooks are designed corresponding to R_t for $1 \le t \le T$ and multiplexed across the T subchannels, i.e., at time instants $(i-1)T+t$ for $1 \le i \le M$, the channel inputs from the tth codebook are sent over the channel. Therefore, the achievable rate is

$$R = \frac{1}{T}\sum_{t=1}^{T}\sum_{v}\Pr(v)\max_{q_t(x|v)}\sum_{u}P_t(u|v)I(X;Y|u,v). \tag{7}$$

B. Converse

In this part, we prove the converse to the capacity theorem. The proof is motivated by the proof in [5]. From the Fano's inequality [18], we have

$$H\left(W|Y_1^N, U_1^N\right) \le P_e \log A_w + h(P_e) = N\epsilon_N \tag{8}$$

where $P_e = \Pr(W \neq \hat{W})$, and $\epsilon_N \to 0$ as $N \to \infty$

$$\begin{aligned} H\left(W|Y_1^N, U_1^N\right) &= H\left(W|U_1^N\right) - I\left(W; Y_1^N|U_1^N\right) \\ &= NR - I\left(W; Y_1^N|U_1^N\right). \end{aligned} \tag{9}$$

Using (8) and (9), we can write

$$R \le \frac{1}{N} I\left(W; Y_1^N|U_1^N\right) + \epsilon_N. \tag{10}$$

Then we have

$$\begin{aligned} &I\left(W; Y_1^N|U_1^N\right) \\ &= \sum_{n=1}^{N} I\left(W; Y_n|U_1^N, Y_1^{n-1}\right) \\ &= \sum_{n=1}^{N} H\left(Y_n|U_1^N, Y_1^{n-1}\right) - H\left(Y_n|U_1^N, Y_1^{n-1}, W\right) \\ &\le \sum_{n=1}^{N} H\left(Y_n|U_n, V_1^{\tilde{n}}\right) - H\left(Y_n|U_1^N, Y_1^{n-1}, W\right) \\ &\overset{a}{\le} \sum_{n=1}^{N} H\left(Y_n|U_n, V_1^{\tilde{n}}\right) - H\left(Y_n|U_n, X_n, V_1^{\tilde{n}}\right) \qquad (11) \\ &= \sum_{n=1}^{N} I\left(X_n; Y_n|U_n, V_1^{\tilde{n}}\right) \qquad (12) \end{aligned}$$

where (a) follows from the fact that the channel output is independent of the message and past channel outputs given the state of the channel and the channel input. On the other hand, for a given n, we have

$$\begin{aligned} &I\left(X_n; Y_n|U_n, V_1^{\tilde{n}}\right) \\ &= \sum_{u_n, v_1^{\tilde{n}}} \Pr\left(u_n|v_{\tilde{n}}, v_1^{\tilde{n}-1}\right) \Pr\left(\left|v_1^{\tilde{n}-1}\right| v_{\tilde{n}}\right) \\ &\quad \times \Pr(v_{\tilde{n}}) I\left(X_n; Y_n|u_n, v_1^{\tilde{n}-1}, v_{\tilde{n}}\right) \\ &\overset{b}{=} \sum_{u_n, v_{\tilde{n}}} \Pr(u_n|v_{\tilde{n}}) \Pr(v_{\tilde{n}}) \\ &\quad \times \sum_{v_1^{\tilde{n}-1}} \Pr\left(v_1^{\tilde{n}-1}\middle| v_{\tilde{n}}\right) I\left(X_n; Y_n|u_n, v_1^{\tilde{n}-1}, v_{\tilde{n}}\right) \\ &\overset{c}{\le} \sum_{u_n, v_{\tilde{n}}} \Pr(u_n|v_{\tilde{n}}) \Pr(v_{\tilde{n}}) \max_{q(x_n|v_{\tilde{n}})} I(X_n; Y_n|u_n, v_{\tilde{n}}) \qquad (13) \end{aligned}$$

where (b) follows from the property in (1), and (c) results from the concavity of mutual information with respect to the input distribution, and

$$q(x_n|v_{\tilde{n}}) \triangleq \sum_{v_1^{\tilde{n}-1}} \Pr(v_1^{\tilde{n}-1}|v_{\tilde{n}}) \Pr(x_n|v_1^{\tilde{n}}).$$

Replacing $n = (\tilde{n}-1)T + t$ in (13) and using (12), we have

$$\begin{aligned} &I\left(W; Y_1^N|U_1^N\right) \\ &\le \sum_{\tilde{n}=1}^{M} \sum_{t=1}^{T} \sum_{v_{\tilde{n}}} \sum_{u_{(\tilde{n}-1)T+t}} \Pr\left(u_{(\tilde{n}-1)T+t}|v_{\tilde{n}}\right) \Pr(v_{\tilde{n}}) \\ &\quad \times \max_{q(x_{(\tilde{n}-1)T+t}|v_{\tilde{n}})} I\left(X_{(\tilde{n}-1)T+t};\right. \\ &\quad \left. Y_{(\tilde{n}-1)T+t}|u_{(\tilde{n}-1)T+t}, v_{\tilde{n}}\right) \qquad (14) \\ &= M \sum_{t=1}^{T} \sum_{u,v} P_t(u|v) \Pr(v) \max_{q_t(x|v)} I(X; Y|u, v) \qquad (15) \end{aligned}$$

where (15) follows from the fact that $\{V_i\}_{i=1}^M$, and $\{U_{(i-1)T+t}\}_{i=1}^M$ are jointly stationary and ergodic and the right-hand side (RHS) of (14) does not depend on $\tilde{n}$. Using (10) and (15), we have

$$R \le \frac{1}{T} \sum_{t=1}^{T} \sum_{v} \Pr(v) \max_{q_t(x|v)} \sum_{u} P_t(u|v) I(X; Y|u, v) + \epsilon_N. \tag{16}$$

□

III. Gaussian Channel

In this section, we consider a point to point transmission over a time-correlated fading channel. It is assumed that the channel gain is constant over each channel use (symbol) and varies from symbol to symbol, following a first-order Markovian random process. The signal at the receiver is

$$r_n = h_n x_n + z_n \tag{17}$$

where $h_n \in \mathbb{C}$ is the fading gain and z_n is AWGN with zero mean and unit variance. It is assumed that the CSI is perfectly known to the receiver. Every T channel use, the instantaneous fading gain is sent to the transmitter through a noiseless feedback link, i.e., $|h_1|, |h_{T+1}|, \ldots, |h_{(M-1)T+1}|$ are fed back and instantly received at the transmitter.

Let us define $u_n \triangleq |h_n|^2$ for $1 \le n \le N, v_i \triangleq |h_{(i-1)T+1}|^2$ for $1 \le i \le M$, and $P_t(u|v) \triangleq \Pr_{u_i|u_{i-t+1}}(u|v)$. The average input power is subject to the constraint $\mathbb{E}[|x_n|^2] \le \mathcal{P}$. In the following, $\mathbb{E}_t[g(U,V)]$ denotes the expectation value over $g(u,v)$ where U and V have joint pdf $P_t(u,v)$.

Theorem 2: The capacity of time-correlated fading channel with periodic feedback is

$$\max_{\bar{\rho}_1 \cdots \bar{\rho}_T} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_t[\log(1 + U\bar{\rho}_t(V))] \tag{18}$$

subject to $\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[\bar{\rho}_t(V)] \le \mathcal{P}$, where T is the feedback period.

First, we recount some results on the capacity of single user channels, which is applied in the proof. A general formula for the capacity of single user channels which is not necessarily information stable or stationary is obtained in [19]. Consider input X and output Y as sequences of finite-dimensional distribution, where Y is induced by X via a channel which is an arbitrary sequence of finite-dimensional conditional output distribution from input alphabets to the output alphabets. The general formula for the channel capacity is as follows:

$$C = \sup_X \underline{I}(X; Y) \tag{19}$$

where $\underline{I}(X;Y)$ is defined as the liminf in probability of the normalized information density [19]

$$i_N\left(X_1^N; Y_1^N\right) = \frac{1}{N} \log \frac{\Pr\left(Y_1^N|X_1^N\right)}{\Pr\left(Y_1^N\right)}. \tag{20}$$

Assume that the channel state information, Q, is available at the receiver. Considering Q as an additional output, the channel capacity is $C = \sup_X \underline{I}(X;Y,Q)$. If Q is not available at the transmitter and is consequently independent of X, then the capacity is [20]

$$C = \sup_X \underline{I}(X;Y|Q) \tag{21}$$

where $\underline{I}(X;Y|Q)$ is the liminf in probability of the normalized conditional information density

$$i_N\left(X_1^N;Y_1^N|Q_1^N\right) = \frac{1}{N}\log\frac{\Pr\left(Y_1^N|X_1^N,Q_1^N\right)}{\Pr\left(Y_1^N|Q_1^N\right)}. \tag{22}$$

Now, we are ready to prove Theorem 2, where the proof is motivated by the proof in [5].

A. Achievability

Noting (17), the processed received signal at time n is

$$y_n = r_n \frac{h_n^*}{|h_n|} = |h_n|x_n + z_n' \tag{23}$$

where $z_n' = \frac{h_n^*}{|h_n|}z_n$, which has the same distribution as z_n. The transmitter sends

$$x_n = \sqrt{\rho_n(v_{\tilde{n}})}s_n \tag{24}$$

over the channel where s_n is an independent and identically distributed (i.i.d.) Gaussian codebook with zero mean and unit variance, and $\rho_n : \mathbb{R}_+ \to \mathbb{R}_+$ is the power allocation function. Using (23) and (24), we can write

$$y_n = \sqrt{q_n}s_n + z_n' \tag{25}$$

where $q_n = \rho_n(v_{\tilde{n}})|h_n|^2 = \rho_n(v_{\tilde{n}})u_n$. Noting (25), we have a channel with input S and output Y and channel state Q, which is known at the receiver. Since Q_1^N is independent of S_1^N, we can use (21) to obtain the achievable rate

$$\begin{aligned} i_N&\left(S_1^N;Y_1^N|Q_1^N\right)\\ &= \frac{1}{N}\log\frac{\Pr\left(Y_1^N|S_1^N,Q_1^N\right)}{\Pr\left(Y_1^N|Q_1^N\right)}\\ &\overset{d}{=} \frac{1}{N}\sum_{n=1}^N \log\frac{\Pr(Y_n|S_n,Q_n)}{\Pr(Y_n|Q_n)}\\ &= \frac{1}{N}\sum_{n=1}^N\left(\log(1+Q_n) + \frac{|Y_n|^2}{1+Q_n} - |Z_n'|^2\right) \end{aligned} \tag{26}$$

where (d) results from the fact that S_1^N, and Z'^N_1 are i.i.d. sequences and the last line follows from the fact that Y_n conditioned on Q_n is Gaussian with zero mean and variance $1+Q_n$. Note that as $N\to\infty$, $\frac{1}{N}\sum_{n=1}^N \frac{|Y_n|^2}{1+Q_n} = \frac{1}{N}\sum_{n=1}^N |Z_n'|^2 = 1$ with probability one. Therefore, with probability one, we have

$$\begin{aligned} i_N&\left(S_1^N;Y_1^N|Q_1^N\right)\\ &= \frac{1}{N}\sum_{n=1}^N \log(1+Q_n)\\ &= \frac{1}{MT}\sum_{t=1}^T\sum_{i=1}^M \log\left(1+Q_{(i-1)T+t}\right)\\ &= \frac{1}{T}\sum_{t=1}^T \frac{1}{M}\sum_{i=1}^M \log\left(1+U_{(i-1)T+t}\rho_{(i-1)T+t}(V_i)\right). \end{aligned} \tag{27}$$

Noting that $\{U_{(i-1)T+t}\}_{i=1}^M$ and $\{V_i\}_{i=1}^M$ are jointly stationary and ergodic for $1\le t\le T$, we define $P_t(u,v)$ to be their joint pdf. We set $\rho_{(i-1)T+t} = \bar{\rho}_t$ for $1\le i\le M$, and $1\le t\le T$. As $M\to\infty$ in (27), the sample mean converges in probability to the expectation. Therefore, the achievable rate is

$$R = \frac{1}{T}\sum_{t=1}^T \mathbb{E}_t[\log(1+U\bar{\rho}_t(V))]. \tag{28}$$

B. Converse

Using (11), we have

$$\begin{aligned} I&\left(W;Y_1^N|U_1^N\right)\\ &\le \sum_{n=1}^N H\left(Y_n|U_n,V_1^{\tilde{n}}\right) - H\left(Y_n|U_n,X_n,V_1^{\tilde{n}}\right)\\ &\le \sum_{n=1}^N \mathbb{E}\left[\log\left(1+U_n\mathbb{E}\left[|X_n|^2|V_1^{\tilde{n}}\right]\right)\right]. \end{aligned} \tag{29}$$

The above inequality relies on the facts that

$$\begin{aligned} H&\left(Y_n|U_n,X_n,V_1^{\tilde{n}}\right)\\ &= H(Z_n) = \log 2\pi e \end{aligned} \tag{30}$$

and

$$\begin{aligned} H&\left(Y_n|U_n,V_1^{\tilde{n}}\right)\\ &\le \mathbb{E}\left[\log\left(2\pi e\left(1+U_n\mathbb{E}\left[|X_n|^2|V_1^{\tilde{n}}\right]\right)\right)\right]. \end{aligned} \tag{31}$$

The upper-bound in (31) is achieved if X_n conditioned on $V_1^{\tilde{n}}$ has a Gaussian distribution. We set $x_n = \sqrt{f_n(v_1^{\tilde{n}})}s_n$ where $f_n : \mathbb{R}_+^{\tilde{n}} \to \mathbb{R}_+$, and S_1^N is an i.i.d. Gaussian sequence with zero mean and unit variance. On the other hand

$$\begin{aligned} \mathbb{E}&\left[\log\left(1+U_nf_n\left(V_1^{\tilde{n}}\right)\right)\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\log\left(1+U_nf_n\left(V_1^{\tilde{n}}\right)\right)|U_n,V_{\tilde{n}}\right]\right]\\ &\overset{d}{\le} \mathbb{E}\left[\log\left(1+\mathbb{E}\left[U_nf_n\left(V_1^{\tilde{n}}\right)|U_n,V_{\tilde{n}}\right]\right)\right]\\ &= \mathbb{E}\left[\log\left(1+U_n\mathbb{E}\left[f_n\left(V_1^{\tilde{n}}\right)|V_{\tilde{n}}\right]\right)\right] \end{aligned} \tag{32}$$

where (d) follows from the concavity of the logarithm. Let us define $\rho_n(V_{\tilde{n}}) \triangleq \mathbb{E}[f_n(V_1^{\tilde{n}}))|V_{\tilde{n}}]$. By using (29) and (32), we obtain

$$\begin{aligned} \frac{1}{N}I&\left(W;Y_1^N|U_1^N\right)\\ &\le \frac{1}{N}\sum_{n=1}^N \mathbb{E}[\log(1+U_n\rho_n(V_{\tilde{n}}))]\\ &= \frac{1}{T}\sum_{t=1}^T\frac{1}{M}\sum_{i=1}^M \mathbb{E}\left[\log\left(1+U_{(i-1)T+t}\rho_{(i-1)T+t}(V_i)\right)\right]\\ &\le \frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[\log\left(1+\frac{1}{M}\sum_{i=1}^M U_{(i-1)T+t}\rho_{(i-1)T+t}(V_i)\right)\right]. \end{aligned} \tag{33}$$

Using (33) and noting the fact that $\{U_{(i-1)T+t}\}_{i=1}^M$, and $\{V_i\}_{i=1}^M$ are jointly stationary and ergodic for $1\le t\le T$, we can write

$$\frac{1}{N}I\left(W;Y_1^N|U_1^N\right) \le \frac{1}{T}\sum_{t=1}^T \mathbb{E}_t[\log(1+U\bar{\rho}_t(V))] \tag{34}$$

where $\bar{\rho}_t(\cdot) \triangleq \frac{1}{M}\sum_{i=1}^M \rho_{(i-1)T+t}(\cdot)$. Combining (10) and (34), we conclude that

$$R \le \frac{1}{T}\sum_{t=1}^T \mathbb{E}_t[\log(1+U\bar{\rho}_t(V))] \tag{35}$$

subject to $\frac{1}{T}\sum_{t=1}^T \mathbb{E}[\bar{\rho}_t(V)] \le \mathcal{P}$. □

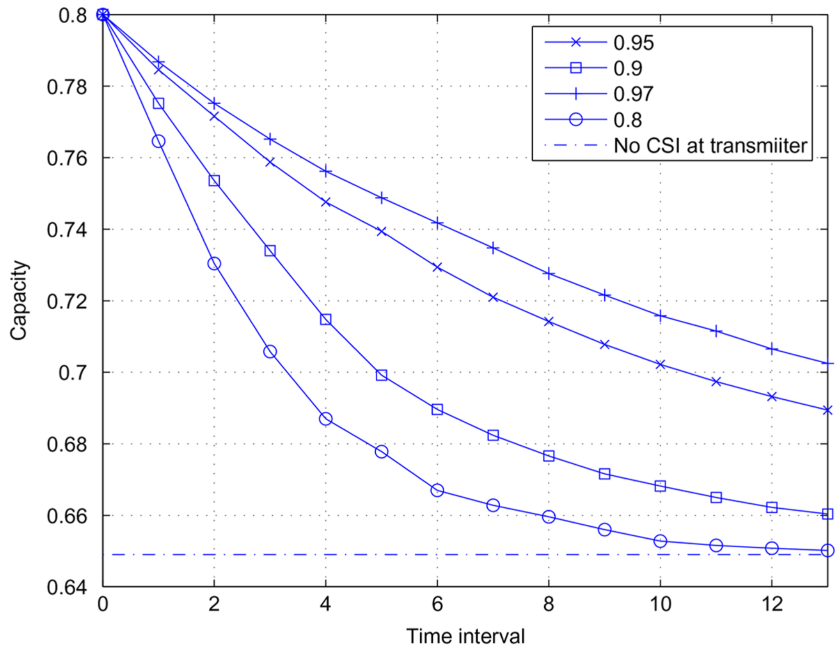

Fig. 1. Capacity of time-correlated Rayleigh fading channel versus T for SNR $= 1$ and channel correlation coefficients $\alpha = 0.97, 0.95, 0.9, 0.8$. The dash-dot line is the capacity with no side information at the transmitter.

Remark: In Section II, we proved that the capacity of Markov channels is generally achieved by using multiple code multiplexing technique. However, for AWGN channel with time-correlated fading, the proof relies on using one Gaussian codebook, where the symbols are adaptively scaled by the appropriate power allocation function based on the side information at the transmitter.

IV. Performance Evaluation

We study the impact of the channel correlation and feedback period on the capacity of the time-correlated Rayleigh fading channel. Let us assume that time-correlated Rayleigh-fading channel is a Markov random process with the following pdf [21]:

$$\Pr(u) = \begin{cases} e^{-u}, & u \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{36}$$

$$P_1(u|v) = \delta(v)$$

$$P_t(u|v) = \Phi(u, v, \alpha^{t-1}) \tag{37}$$

where

$$\Phi(u, v, \sigma) = \begin{cases} \frac{1}{1-\sigma^2} \exp\left(-\frac{u+\sigma^2 v}{1-\sigma^2}\right) \mathcal{I}_0\left(\frac{2\sigma\sqrt{uv}}{1-\sigma^2}\right), & u \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{38}$$

In (38), $0 < \sigma < 1$ describes the channel correlation coefficient and $\mathcal{I}_0(\cdot)$ denotes the modified Bessel function of order zero. Noting that the capacity in (18) is a strictly concave region of $\bar{\rho}_t, 1 \leq t \leq T$, we numerically solve the convex optimization problem. In Fig. 1, the capacity is depicted versus the feedback period for various channel correlation coefficients and compared to the capacity when no CSI is available at the transmitter.

V. Conclusion

We have obtained the capacity of finite state Markov channel with periodic feedback at the transmitter. Also, the channel capacity and optimal adaptive coding is derived for the time-correlated fading channel with periodic feedback. It is shown that the optimal adaptation can be achieved by a single Gaussian codebook, while scaling by the appropriate power.

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Random Access Broadcast: Stability and Throughput Analysis

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***Abstract*—A wireless network in which packets are broadcast to a group of receivers through use of a random access protocol is considered in this work. The relation to previous work on networks of interacting queues is discussed and subsequently, the stability and throughput regions of the system are analyzed and presented. A simple network of two source nodes and two destination nodes is considered first. The broadcast service process is analyzed assuming a channel that allows for packet capture and multipacket reception. It is proved that the stability and throughput regions coincide in this small network. The same problem for a network with N sources and M destinations is considered next. The channel model is simplified in that packet capture and multipacket reception is no longer permitted. Bounds on the stability region are developed using the concept of stability rank and the throughput region of the system is compared to the bounds. Our results show that as the number of destination nodes increases, the stability and throughput regions diminish. Additionally, a previous conjecture that the stability and throughput regions coincide for a network of arbitrarily many sources is supported for a broadcast scenario by the results presented in this work.**

***Index Terms*—ALOHA, multipacket reception, random access, queueing, stability, throughput, wireless broadcast.**

I. Introduction

The stability and throughput of finite-user random access systems for *unicast* transmission have been studied extensively. The throughput analysis is included in some of Abramson's early work on the topic [1], while the stability problem was first introduced by Tsybakov and Mikhailov [2]. The finite-user stability problem proves to be much more difficult than the throughput problem, and as such, the history of the stability problem is more rich and interesting. In [2], sufficient conditions for ergodicity were found using the transition probabilities of the Markov chain corresponding to queue lengths. Later work [3] described the use of stochastic dominance as a means of characterizing the stability region. Additionally, stability conditions based on the joint queue statistics were provided in [4]. Despite these and other attempts, a computable result for the stability region for arbitrarily many sources N remains unsolved. In [5] the concept of stability rank was introduced and provided the tightest known bounds to the stability region for N sources. The exact stability region for N sources and an arrival process which is correlated among the sources was obtained in [6].

Most works, including all of those mentioned above, study random access under the collision channel model, in which transmission by more than one source results in failed reception of all packets. Some recent works have incorporated the probabilistic nature of reception and the possibility of multipacket reception (MPR) into the channel model. The stability of infinite-user random access with MPR was first examined in [7]. More recently, the finite-user problem was examined in [8] and it was shown that the possibility of MPR results in an increase in the stable throughput of the system. The benefit to stability was so dramatic that for channels with sufficiently strong reception capabilities, random access was shown to outperform time-division multiple-access (TDMA) schemes. This result provides motivation for a renewed interest in random access.

In this correspondence we introduce *multiple destination nodes* and broadcast transmission into the network and study the resulting stability and throughput performance of random access. The introduction of multiple destinations is a necessary first step in understanding the behavior of ad hoc and multihop networks, where random access presents an advantage over TDMA due to its distributed nature. In particular, we analyze the performance of a random access *broadcast* system in which a source node sends a common packet to all destination nodes. Broadcast transmission is useful for control of the network (e.g., route discovery, timing synchronization) and for a number of applications.

II. Model and Formulation

Consider a system $\mathcal{S}$ consisting of N source nodes, $s_1, s_2, \ldots, s_N$, and M destination nodes $d^{(1)}, d^{(2)}, \ldots, d^{(M)}$. Packets arrive to s_n according to a Bernoulli process with rate $\lambda_n, n = 1, 2, \ldots, N$ packets per slot. The arrival process is independent from source to source and independent and identically distributed (i.i.d.) over slots. Packets that are not immediately transmitted are stored in an infinite buffer maintained at each source. All source nodes compete in a random fashion for access to the channel in order to transmit a packet of information to *all* destination nodes. When source n has a packet to transmit, it does so with probability p_n in the first available slot. This scenario is depicted in Fig. 1. Each packet is intended for all M destinations. We assume that instantaneous and error-free acknowledgements (ACKs) are sent from the destinations and that each source–destination pair has a dedicated channel for ACKs. If the source has not yet received an ACK from all M destinations, the packet is retransmitted. This policy of relentless retransmissions is assumed throughout the present work. We note that this policy is suboptimal in terms of stable throughput. For instance, in the case of a single destination $M = 1$, collision resolution algorithms such as the one in [9] have been shown to provide a higher stable throughput in the infinite-user case. We choose to focus our attention on random access with retransmissions as a first, non-trivial step in investigating the stability of random access broadcast.

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