$$S_i(t) + n_i(t) = [A + f(t)] \cos \omega_c t + n_c(t) \cos \omega_c t - n_s(t) \sin \omega_c t \tag{399}$$

The envelop of this signal is

$$r(t) = \sqrt{\{[A + f(t)] + n_c(t)\}^2 + \{n_s(t)\}^2} \tag{400}$$

For high input signal-to-noise this can be approximated by use of binomial expansion to give

$$r(t) \approx A + f(t) + n_c(t) \tag{401}$$

The detector output gives

$$S_o = \overline{f^2(t)} \tag{402}$$

$$N_o = \overline{n_c^2(t)} = \overline{n_i^2(t)} = N_i \tag{403}$$

also

$$S_i = \overline{\{[A + f(t)] \cos \omega_c t\}^2} = \frac{1}{2}A^2 + \frac{1}{2}\overline{f^2(t)} \tag{404}$$

Combining Eqs. 402, 403 and 404, we get

$$\frac{S_o}{N_o} = \frac{2\overline{f^2(t)}}{A^2 + \overline{f^2(t)}} \frac{S_i}{N_i} \quad \text{for AM, large signal-to-noise ratio} \tag{405}$$

Note that Eq. 405 is identical to a previous result using synchronous detection.

In the particular case of sinusoidal modulation, $f(t) = mA \cos \omega_m t$ and Eq. 405 can be written as

$$\frac{S_o}{N_o} = \frac{2m^2}{2 + m^2} \frac{S_i}{N_i} \quad \text{for AM, large signal-to-noise ratio, sinusoidal modulation} \tag{406}$$

The maximum improvement in the signal-to-noise ratio is $\frac{2}{3}$ at 100% modulation.

7 Angle Modulation

Consider a sinusoid of the form,

$$\phi(t) = a(t)\cos[\omega_c t + \gamma(t)] \tag{407}$$

In the case of the amplitude modulation, we kept $\gamma(t)$ constant and changed $a(t)$. In the case of the angle modulation, $a(t)$ is constant and $\gamma(t)$ is changed in proportion to the input signal $f(t)$.

Consider a sinusoid $A = \cos\theta(t)$.

$$\begin{aligned} \text{Instantaneous Phase:} \quad &\equiv \theta(t) \\ \text{Instantaneous Frequency:} \quad &\equiv \omega_i(t) = \frac{d\theta}{dt} \end{aligned} \tag{408}$$

Phase Modulation (PM): instantaneous phase is proportional to $f(t)$, i.e., $\theta(t) = \omega_c t + k_p f(t) + \theta_0$ resulting in: $\omega_i = d\theta/dt = \omega_c + k_p df/dt$.

Frequency Modulation (FM): instaneous frequency is proportional to $f(t)$, i.e., $\omega_i = d\theta/dt = \omega_c + k_f f(t)$ resulting in:

$$\theta(t) = \int_0^t \omega_i(\tau)d\tau = \omega_c t + \int_0^t k_f f(\tau)d\tau + \theta_0 \tag{409}$$

Note that a PM modulator can be used to generate an FM signal if we replace $f(t)$ by $\int_0^t f(\tau)d\tau$. Similarly, an FM modulator can be used to generate a PM signal if we replace $f(t)$ by $df(t)/dt$.

As amplitude modulation is a simple multiplication, it has linearity property, i.e., if $g(t) = f_1(t) + f_2(t)$, then $g(t)\cos\omega_c t = f_1(t)\cos\omega_c t + f_2(t)\cos\omega_c t$. However, this property does not hold in the case of the angle modulation (PM and FM).

7.1 Narrowband FM

Consider the modulation of a sinusoid signal,

$$f(t) = a\cos\omega_m t \tag{410}$$

For an FM signal, we have,

$$\omega_i = \omega_c + k_f f(t) = \omega_c + ak_f \cos\omega_m t = \omega_c + \Delta\omega \cos\omega_m t \tag{411}$$

where $\Delta\omega = ak_f$ is the peak frequency deviation.

The phase of this signal is equal to,

$$\theta(t) = \omega_c t + \frac{\Delta\omega}{\omega_m}\sin\omega_m t = \omega_c t + \beta\sin\omega_m t \tag{412}$$

where $\beta = \Delta\omega/\omega_m$. The resulting FM signal is equal to:

$$\begin{aligned}\phi_{FM}(t) &= A\cos(\omega_c t + \beta\sin\omega_m t) \\ &= A\cos\omega_c t\cos(\beta\sin\omega_m t) - A\sin\omega_c t\sin(\beta\sin\omega_m t)\end{aligned} \tag{413}$$

For small values of β, we obtain,

$$\begin{aligned}\cos(\beta\sin\omega_m t) &\simeq 1 \\ \sin(\beta\sin\omega_m t) &\simeq \beta\sin\omega_m t\end{aligned} \tag{414}$$

This is called narrowband FM (NBFM). Parameter β ($\beta = \Delta\omega/\omega_m$) is called the *modulation index* of the FM signal. In practice, $\beta < 0.2$ is enough for NBFM.

Note that NBFM has some similarities with an AM signal. The main difference is that in AM the modulation is added in phase with the carrier while in FM this addition is achieved in quadrature phase. This means that NBFM and NBPM are closer to linear as compared to FM and PM.

The advantages of NBFM over AM is the following: (i) NBFM is more immune to additive noise. (ii) the response of NBFM extends up to zero frequency. This means that with NBFM, one can transmit signals with very low frequency components as well.

In NBFM and NBPM, the resulting bandwidth in modulating a signal of bandwidth ω_m is equal to $W = 2\omega_m$.

The NBFM results in phase variation with very little amplitude changes while the AM signal results in amplitude changes with no phase deviation.

The addition of the modulated signal in quadrature with the carrier in NBFM suggests the configurations shown in Fig. 46 for the generation of NBFM and NBPM.

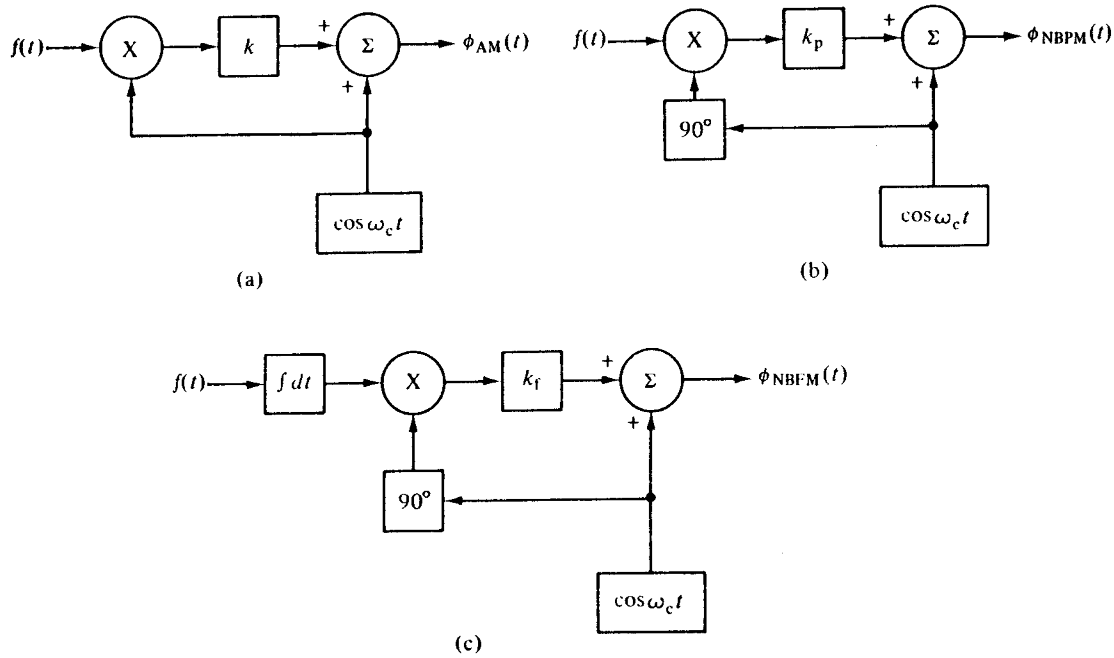

Figure 46: Generation of AM, NBFM and NBPM.

7.2 Wideband FM

For large values of β, the fourier transform of the FM/PM signal can not be directly computed. In this case, as the exact computation is usually impossible, one relies on some bounds on the performance.

Definition: Peak Frequency Deviation $\equiv$ maximum amount that the instantaneous frequency deviates from the carrier.

In general, the spectrum of an FM signal is affected by the following two mechanism: (i) changes in the modulating signal, (ii) the fact that the instantaneous frequency of the FM signal changes in proportion to the amplitude of the modulating signal. Note that due to the second effect, modulation of a single frequency sine wave results in a band of frequencies. In NBFM approximation, (small value of β, $\Delta\omega \ll \omega_m$), the second effect was neglected in the favor of the first one. On the opposite, for large values of β, the amplitude-to-frequency conversion predominates and the bandwidth is on the order of $2\Delta\omega$.

7.3 General approximation for the spectral density of WBFM

Basic idea: From the concept of a spectral density, we expect that the spectral magnitude to be in proportion to the fractional time spent at each frequency.

Assume a sinusoid modulating signal. For a large value of β, the frequency deviation about the carrier frequency ($\omega_i' = \omega_i - \omega_c$) is equal to:

$$\omega_i' = \Delta\omega \cos \omega_m t \tag{415}$$

or,

$$t = \frac{1}{\omega_m} \cos^{-1}\left(\frac{\omega_i'}{\Delta\omega}\right) \quad \text{for} \quad |\omega_i'| \leq \Delta\omega \tag{416}$$

The fractional amount of time per unit of frequency is:

$$\frac{1}{T}\left|\frac{dt}{d\omega_i'}\right| = \frac{1/(2\pi)}{\Delta\omega\sqrt{1-(\omega_i'/\Delta\omega)^2}} \quad \text{for} \quad |\omega_i'| \leq \Delta\omega \tag{417}$$

This results in the following envelope for the spectral density (in a frequency band of $2\Delta\omega$ around ω_c):

$$\frac{1/(2\pi)}{\Delta\omega\sqrt{1-(\omega_i'/\Delta\omega)^2}} \quad \text{for} \quad |\omega_i'| \leq \Delta\omega \tag{418}$$

This is shown in Fig. 47.

7.4 Fourier analysis of FM with sinusoid modulating signal

Modulating signal: $f(t) = a \cos \omega_m t$.

Instantaneous Frequency: $\omega_i(t) = \omega_c + ak_f \cos \omega_m t = \omega_c + \Delta\omega \cos \omega_m t$

Instantaneous Phase:

$$\int_0^t \omega_i(\tau)d\tau = \omega_c t + \frac{\Delta\omega}{\omega_m} \sin \omega_m t = \omega_c t + \beta \sin \omega_m t \tag{419}$$

Using complex notation:

$$\phi_{FM}(t) = \mathcal{R}\{Ae^{j\omega_c t}e^{j\beta \sin \omega_m t}\} \tag{420}$$

Using Fourier series, we obtain,

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_m t} \tag{421}$$

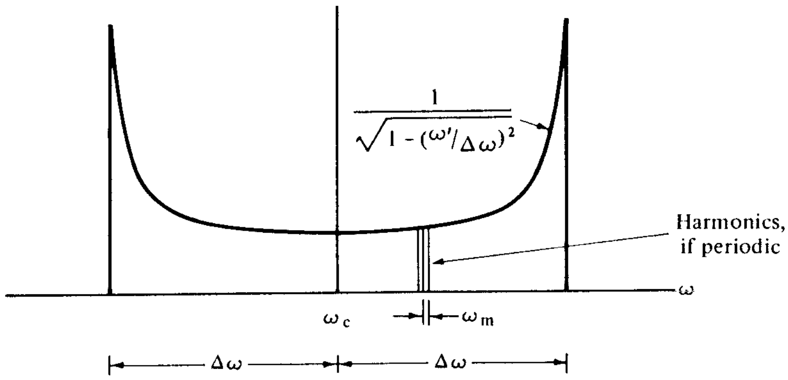

Figure 47: Approximation to the magnitude FM spectral density as $\beta \to \infty$, sinusoidal case.

where

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt \tag{422}$$

Assuming $\psi = \omega_m t = (2\pi/T)t$, we obtain,

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin \psi - n\psi)} d\psi \tag{423}$$

The integral in (423) is known as the bessel function of the first kind, of order n and argument β, and is denoted by $J_n(\beta)$.

We know that:

1. $J_n(\beta)$ are real valued
2. $J_n(\beta) = J_{-n}(\beta)$, for n even
3. $J_n(\beta) = -J_{-n}(\beta)$, for n odd
4. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

Combining (421), (423) we obtain,

$$e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \tag{424}$$

resulting in:

$$\phi_{FM}(t) = \mathcal{R}\left\{ Ae^{j\omega_c t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t} \right\} \tag{425}$$

$$\phi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t \tag{426}$$

Figure 48 shows a plot of different $J_n(\beta)$. Figure 49 shows the corresponding spectral density for different values of β.

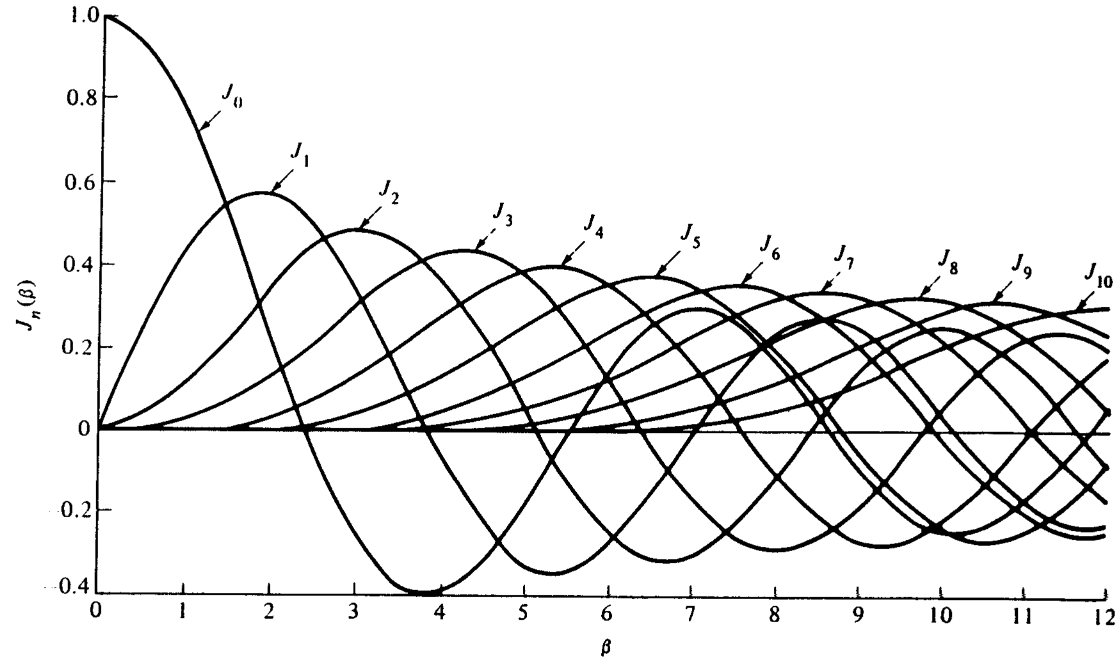

Figure 48: Plot of Bessel function of the first kind, $J_n(\beta)$.

Referring to (426), we see that an FM signal has an infinite number of sidebands, however the magnitudes of the spectral components of the higher-order sidebands become negligible. As a common rule, a sideband is realized to be *significant* if $|J_n(\beta)| \geq 0.01$. For large values of β, the bandwidth is in the order of $2\Delta\omega$. For small values of β, the bandwidth is in the order of $2\omega_m$.

The following rule (proposed by J. R. Carson) is used for the intermediate cases:

$$W \simeq 2(\Delta\omega + \omega_m) = 2\omega_m(1 + \beta) \tag{427}$$

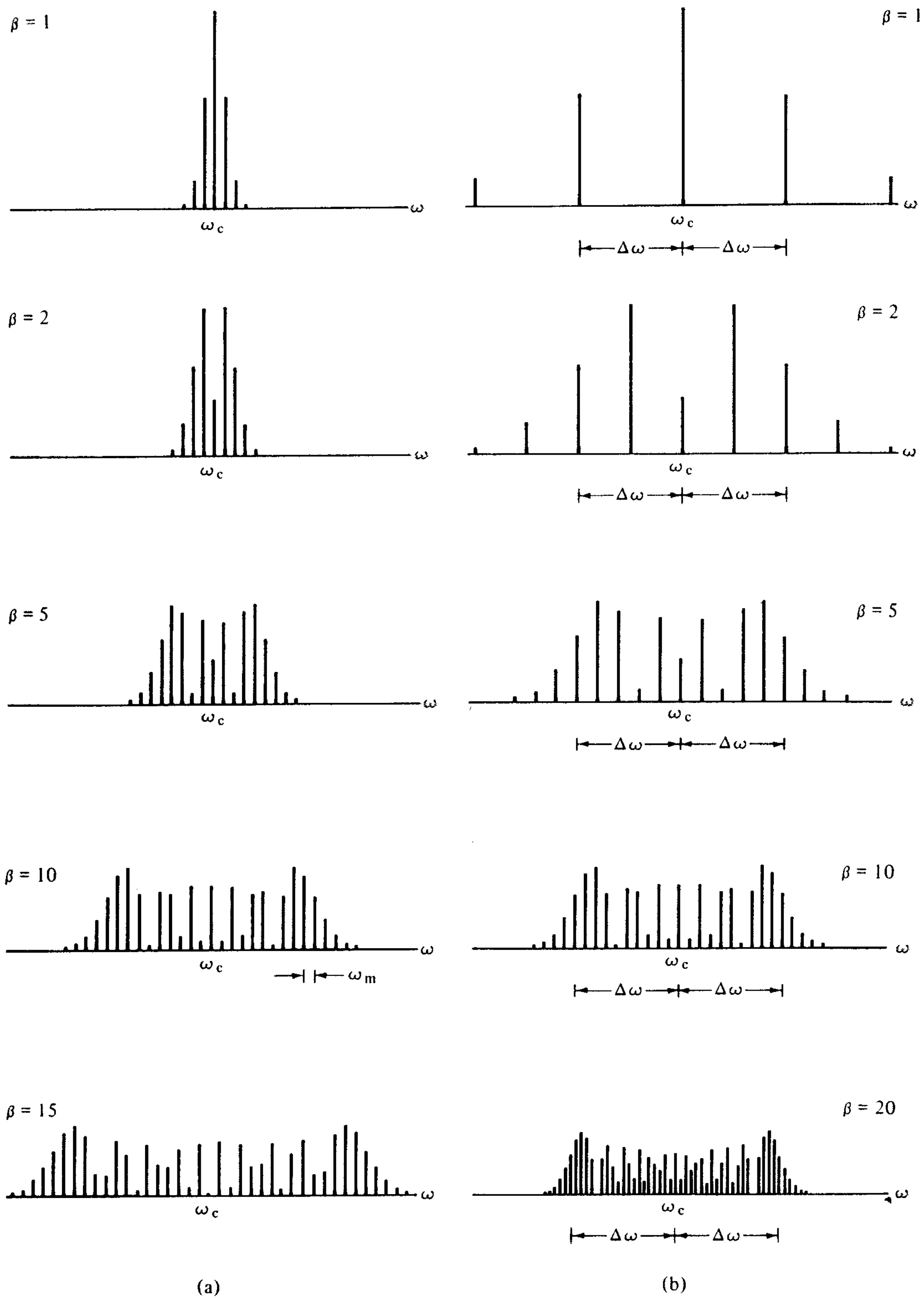

Figure 49: Magnitude line spectra for FM waveforms with sinusoidal modulation: (a) for constant ω_m; (b) for constant $\Delta\omega$.

7.5 Average power in angle modulation

For modulation of a sinusoid, we have,

$$\phi_{FM}(t) = A\cos(\omega_c t + \beta \sin \omega_m t) \tag{428}$$

resulting in an average power (mean-square value) of,

$$\overline{\phi_{FM}^2(t)} = \frac{1}{2}A^2 \tag{429}$$

This means the the average power is constant regardless of the value of the modulation index. This is in contrast to AM where the total average power was proportional to the modulation index.

Relationship (429) can be also derived using the series expansion of the FM signal. We known that,

$$\phi_{FM}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos(\omega_c + n\omega_m)t \tag{430}$$

Considering the orthogonality of the cosine terms, the mean-square value of the summation is equal to the summation of the mean square values, i.e.,

$$\overline{\phi_{FM}^2(t)} = \frac{1}{2}A^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta) \tag{431}$$

Replacing $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$ in (431), we obtain (429).

7.6 Phase Modulation

The FM and PM modulations are very similar.

For the FM modulation of $f(t) = a\cos\omega_m t$, we have:

1. Instantaneous frequency: $\omega_i(t) = \omega_c + ak_f \cos\omega_m t = \omega_c + \Delta\omega\cos\omega_m t$ where

   - k_f: Frequency-modulator constant in radian per second per volt
   - $\Delta\omega$: Peak frequency deviation (in radian per second)

2. Modulation index: $\beta = \Delta\omega/\omega_m$ is a dimensionless number determining the behavior of the carrier and sidebands.

For the PM modulation of $f(t) = a\cos\omega_m t$, we have:

1. Instantaneous phase: $\theta(t) = \omega_c t + ak_p \cos\omega_m t + \theta_0 = \omega_c t + \Delta\theta\cos\omega_m t + \theta_0$ where

   - k_p: Phase-modulator constant in radian per volt
   - $\Delta\theta$: Peak phase deviation (in radian)

   Instantaneous frequency: $\omega_i(t) = d\theta/dt = \omega_c - ak_p\omega_m \sin\omega_m t = \omega_c - \Delta\omega\sin\omega_m t$

2. Modulation index: same as in FM. We can compute $\Delta\omega = ak_p\omega_m = \omega_m\Delta\theta$ and then compute the frequency spectrum as in FM.

The peak frequency deviation in PM is proportional not only to the amplitude of the modulating waveform but also to its frequency. This make PM less desirable when $\Delta\omega$ is fixed (as in commercial FM). However, as we will see later, there are some advantages in the demodulation of PM.

7.7 Generation of Wideband FM signals

7.7.1 Indirect method

This is based on generating a narrowband FM and then using a frequency multiplier to increase the modulation index.

Input-output characteristics of a square-law device: $e_o(t) = ae_i^2(t)$

Input an FM signal: $e_i(t) = A\cos(\omega_c t + \beta\sin\omega_m t)$

Output:

$$e_o(t) = aA^2\cos^2(\omega_c t + \beta\sin\omega_m t) = (1/2)aA^2[1 + \cos(2\omega_c t + 2\beta\sin\omega_m t)] \qquad (432)$$

We can remove the dc-value by a filter. It is seen that both the carrier frequency and the modulation index are doubled in this process. Similarly, an nth order devise results in increasing the carrier frequency and the modulation index by a factor of n. In practice, an increase by a factor of 1000 is achievable. If the resulting carrier frequency is too high for our purpose, we can use a frequency converter (as in AM) to translate the spectrum to lower frequencies (refer to Fig. 50). Note that the frequency conversion does not change the spectral contents of the signal.