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Path Diversity over Packet Switched Networks: Performance Analysis and Rate Allocation

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Abstract

Path diversity works by setting up multiple parallel connections between the end points using the topological path redundancy of the network. In this paper, *Forward Error Correction* (FEC) is applied across multiple independent paths to enhance the end-to-end reliability. Network paths are modeled as erasure Gilbert-Elliot channels [1]–[5]. It is known that over any erasure channel, *Maximum Distance Separable* (MDS) codes achieve the minimum probability of irrecoverable loss among all block codes of the same size [6], [7]. Based on the adopted model for the error behavior, we prove that the probability of irrecoverable loss for MDS codes decays exponentially for an asymptotically large number of paths. Then, optimal rate allocation problem is solved for the asymptotic case where the number of paths is large. Moreover, it is shown that in such asymptotically optimal rate allocation, each path is assigned a positive rate *iff* its quality is above a certain threshold. The quality of a path is defined as the percentage of the time it spends in the bad state. Finally, using dynamic programming, a heuristic suboptimal algorithm with polynomial runtime is proposed for rate allocation over a finite number of paths. This algorithm converges to the asymptotically optimal rate allocation when the number of paths is large. The simulation results show that the proposed algorithm approximates the optimal rate allocation (found by exhaustive search) very closely for practical number of paths, and provides significant performance improvement compared to the alternative schemes of rate allocation.[1]

Index Terms

Path diversity, Internet, MDS codes, erasure, forward error correction, rate allocation, complexity.

I. Introduction

In recent years, *path diversity* over the Internet has received significant attention. It has been shown that path diversity has the ability to simultaneously improve the end-to-end rate and reliability [3], [8]–[10]. In a dense network like the Internet, it is usually possible to find multiple independent paths between most pairs of nodes [11]–[16]. A set of paths are defined to be independent if their corresponding packet loss and delay characteristics are independent. Clearly, disjoint paths would be independent too [3],

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[4], [8], [11], [12], [17]–[19]. Even when the paths are not completely disjoint, their loss and delay patterns may show a high degree of independence as long as the nodes and links they share are not congestion points or bottlenecks [3], [11], [12], [14], [16]–[19]. In this paper, *Forward Error Correction* (FEC) is applied across multiple independent paths. Based on this model, we show that path diversity significantly enhances the performance of FEC.

In order to apply path diversity over any packet switched network, two problems need to be addressed: i) setting up multiple independent paths between the end-nodes, ii) utilizing the given independent paths to improve the end-to-end throughput and/or reliability. In this paper, we focus on the second problem only. However, it should be noted that the first problem has also received significant attention in the literature (see [8], [11], [12], [16], [19]–[26]). In case the end-points have enough control over the path selection process, the centralized and distributed algorithms in references [27] and [28] can be used to find multiple disjoint paths over a large connected graph. However, applying such algorithms over the Internet requires modification of IP routing protocol and extra signaling between the nodes (routers). Of course, modifying the traditional IP network is extremely costly. To avoid such an expense, overlay networks are introduced [16], [19], [29]. The basic idea of overlay networks is to equip very few nodes (smart nodes) with the desired new functionalities while the rest remain unchanged. The smart nodes form a virtual network connected through virtual or logical links on top of the actual network. Thus, overlay nodes can be used as relays to set up independent paths between the end nodes [22], [24]–[26], [30]. Han et. al have experimentally studied the number of available disjoint paths in the Internet using overlay networks [11]. They have also discussed the impact of network path diversity on the performance of overlay networks [12], [21]. Reference [20] addresses the problem of distributed overlay network design based on a game theoretical approach. Many other researchers have tried to optimize the design of overlay networks such that they offer the maximum degree of path diversity [22], [25], [26], [30]. Moreover, the idea of *multihoming* is proposed to set up extra independent paths between the end-points [23], [24]. In this technique, the end users are connected to more than one *Internet Service Providers* (ISP's) simultaneously. It is shown that combining multihoming with overlay assisted routing can improve the end-to-end performance considerably [24]. In the cases where the backbone network partially consists of optical links between the nodes, each optical fiber conveys tens of independent channels (tones). There has been efforts to take advantage of this inherent physical layer diversity in optical networks [30].

Recently, path diversity is utilized in many applications (see [4], [31]–[34]). Reference [32] combines multiple description coding and path diversity to improve quality of service (QoS) in video streaming. Packet scheduling over multiple paths is addressed in [35] to optimize the rate-distortion function of a video stream. Reference [34] utilizes path diversity to improve the quality of Voice over IP streams.

According to [34], sending some redundant voice packets through an extra path helps the receiver buffer and the scheduler optimize the trade-off between the maximum tolerable delay and the packet loss ratio [34]. In [8], multipath routing of TCP packets is applied to control the congestion with minimum signaling overhead. *Content Distribution Networks* (CDN's) can also take advantage of path diversity for performance improvement. CDN's are a special type of overlay networks consisting of *Edge Servers* (nodes) responsible for delivery of the contents from an original server to the end users [29], [36]. Current commercial CDN's like *Akamai* use path diversity based techniques like *SureRoute* to ensure that the edge servers maintain reliable connections to the original server. Video server selection schemes are discussed in [22] to maximize path diversity in CDN's.

Moreover, references [9] and [3] study the problem of rate allocation over multiple paths. Assuming each path follows the leaky bucket model, reference [9] shows that a water-filling scheme provides the minimum end-to-end delay. On the other hand, reference [3] considers a scenario of multiple senders and a single receiver, assuming all the senders share the same source of data. The connection between each sender and the receiver is assumed to follow the Gilbert-Elliot model. They propose a receiver-driven protocol for packet partitioning and rate allocation. The packet partitioning algorithm ensures no sender sends the same packet, while the rate allocation algorithm minimizes the probability of irrecoverable loss in the FEC scheme [3]. They only address the rate allocation problem for the case of two paths. A brute-force search algorithm is proposed in [3] to solve the problem. Generalization of this algorithm over multiple paths results in an exponential complexity in terms of the number of paths. Moreover, it should be noted that the scenario of [3] is equivalent, without any loss of generality, to the case in which multiple independent paths connect a pair of end-nodes as they assume the senders share the same data.

Maximum Distance Separable (MDS) codes have been shown to be optimum in the sense that they achieve the maximum possible minimum distance (d_{min}) among all the block codes of the same size [37]. Indeed, any $[N, K]$ MDS code (with block length N and K information symbols) can be successfully recovered from any subset of its entries of length K or more. This property makes MDS codes favorable FEC schemes over the erasure channels like the Internet [38]–[40]. However, the simple and practical encoding-decoding algorithms for such codes have quadratic time complexity in terms of the code size [41]. Theoretically, more efficient ($O\left(N\log^2(N)\right)$) MDS codes can be constructed based on evaluating and interpolating polynomials over specially chosen finite fields using Discrete Fourier Transform [42], but these methods are not competitive in practice with the simpler quadratic methods except for extremely large block sizes. Recently, a family of almost-MDS codes with low encoding-decoding time complexity (linear in term of the code length) is proposed and shown to be practical over the erasure channels like the Internet [43], [44]. In these codes, any subset of symbols of size $K(1+\epsilon)$ is sufficient to recover the original K symbols with high probability [44].

MDS codes also require alphabets of a large size. Indeed, all the known MDS codes have alphabet sizes growing at least linearly with the block length N. There is a conjecture stating that all the $[N, K]$ MDS codes over the Galois field $\mathbb{F}_q$ with $1 < K < N-1$ have the property that $N \leq q+1$ with two exceptions [37]. However, this is not an issue in the practical networking applications since the alphabet size is $q = 2^r$ where r is the packet size, i.e. the block size is much smaller than the alphabet size. Algebraic computation over Galois fields ($\mathbb{F}_q$) of such cardinalities is now practically possible with the increasing processing power of electronic circuits. Note that network coding schemes, recently proposed and applied for content distribution over large networks, have a comparable computational complexity [45]–[47].

In this work, we utilize path diversity to improve the performance of FEC between two end-nodes over a general packet switched network like the Internet. The details of path setup process is not discussed here. More precisely, it is assumed that L independent paths are set up by a smart overlay network or any other means [8], [11], [12], [16], [18]–[26]. Each path is modeled by a two-state continuous time Markov process called Gilbert-Elliot channel [1]–[5]. Probability of irrecoverable loss (P_E) is defined as the measure of FEC performance. It is known that MDS block codes have the minimum probability of error over our *End-to-End Channel* model, and over any other erasure channel with or without memory [6], [7]. Applying MDS codes, our analysis shows an exponential decay of P_E with respect to L for the asymptotic case where the number of paths is large. Of course, in many practical cases, the number of *disjoint* or *independent* paths between the end nodes is limitted. However, in our asymptotic analysis, we have assumed that it is possible to find L independent paths between the end points even when L is large. Moreover, the optimal rate allocation problem is solved in the asymptotic case. It is seen that in the asymptotically optimal rate allocation, each path is assigned a positive rate *iff* its quality is above a certain threshold. Quality of a path is defined as the percentage of the time it spends in the bad state. Furthermore, using dynamic programming, a heuristic suboptimal algorithm is proposed for rate allocation over a finite number of paths (limitted L). Unlike the brute-force search, this algorithm has a polynomial complexity, in terms of the number of paths. It is shown that the result of this algorithm converges to the asymptotically optimal solution for large number of paths. Finally, the proposed algorithm is simulated and compared with the optimal rate allocation found by exhaustive search for practical number of paths. Simulation results verify the near-optimal performance of the proposed suboptimal algorithm in practical scenarios.

The rest of this paper is organized as follows. Section II describes the system model. Probability distribution of the bad burst duration is discussed in section III. Performance of FEC in three cases of a single path, multiple identical paths, and non-identical paths are analyzed in section IV. Section V

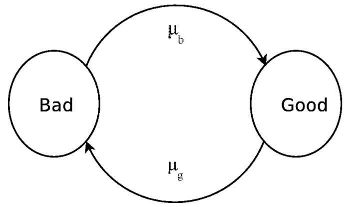


Fig. 1. Continuous-time two-state Markov model of the end-to-end channel

studies the rate allocation problem, and proposes a suboptimal rate allocation algorithm. Finally, section VI concludes the paper.

II. System Modeling and Formulation

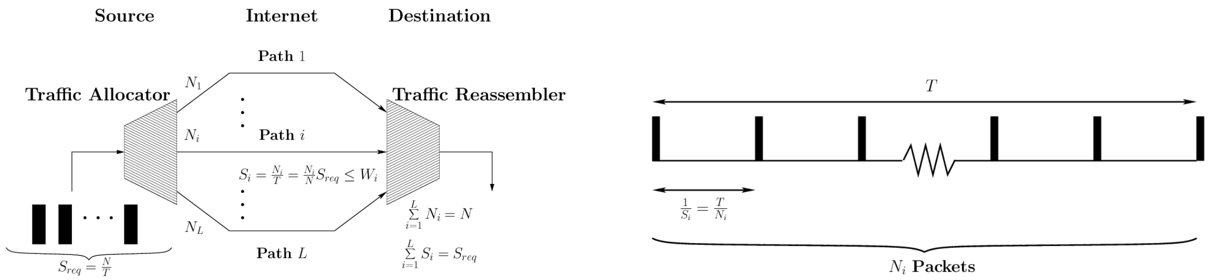
A. End-to-End Channel Model

From an end to end protocol's perspective, performance of the lower layers in the protocol stack can be modeled as a random *channel* called the *end-to-end channel*. Since each packet usually includes an internal error detection coding (for instance a Cyclic Redundancy Check), the end-to-end channel is satisfactorily modeled as an erasure channel. Delay of the end-to-end channel is strongly dependent on its packet loss pattern, and affects the QoS considerably [48], [49].

In this work, the model assumed for the end-to-end channel is a two-state Markov model called Gilbert-Elliot cell, depicted in Fig. 1. The channel spends an exponentially distributed random amount of time with the mean $\frac{1}{\mu_g}$ in the *Good* state. Then, it alternates to the *Bad* state and stays in that state for another random duration exponentially distributed with the mean $\frac{1}{\mu_b}$. It is assumed that the channel state does not change during the transmission of a given packet [4], [50], [51]. Hence, if a packet is transmitted from the source at anytime during the good state, it will be received correctly. Otherwise, if it is transmitted during the bad state, it will eventually be lost before reaching the destination. Therefore, the average probability of error is equal to the steady state probability of being in the bad state, $\pi_b = \frac{\mu_g}{\mu_g+\mu_b}$. To have a reasonably low probability of error, μ_g must be much smaller than μ_b. This model is widely used in the literature for theoretical analysis where delay is not a significant factor [1]–[5], [50]–[52]. Despite its simplicity, this model satisfactorily captures the bursty error characteristic of the end-to-end channel. More comprehensive models like the hidden Markov model are introduced in [49], [53]. Although analytically cumbersome, such models express the dependency of loss and delay more accurately.

B. Typical FEC Model

A concatenated coding is used for packet transmission. The coding inside each packet can be a simple Cyclic Redundancy Check (CRC) which enables the receiver to detect an error inside each packet. Then,



(a) (b)

Fig. 2. Rate allocation problem: a block of N packets is being sent from the source to the destination through L independent paths over the network during the time interval T with the required rate $S_{req} = \frac{N}{T}$. The block is distributed over the paths according to the vector $\mathbf{N} = (N_1, \ldots, N_L)$ which corresponds to the rate allocation vector $\mathbf{S} = (S_1, \ldots, S_L)$

the receiver can consider the end-to-end channel as an erasure channel. Other than the coding inside each packet, a *Forward Error Correction* (FEC) scheme is applied between packets. Every K packets are encoded to a *Block* of N packets where $N > K$ to create some redundancy. The N packets of each block are distributed across the L available independent paths, and are received at the destination with some loss (erasure). The ratio of $\alpha = \frac{N-K}{N}$ defines the FEC overhead. A *Maximum Distance Separable* (MDS) $[N, K]$ code, such as the Reed-Solomon code, can reconstruct the original K data packets at the receiver side if K or more of the N packets are received correctly [54]. According to the following theorem, an MDS code is the optimum block code we can design over any erasure channel. Although FEC imposes some bandwidth overhead, it might be the only option when feedback and retransmission are not feasible or fast enough to provide the desirable QoS.

Definition I. An erasure channel is defined as the one which maps every input symbol to either itself or to an erasure symbol ξ. More accurately, an arbitrary channel (memoryless or with memory) with the input vector $\mathbf{x} \in \mathcal{X}^N$, $|\mathcal{X}| = q$, the output vector $\mathbf{y} \in (\mathcal{X} \cup \{\xi\})^N$, and the transition probability $p(\mathbf{y}|\mathbf{x})$ is defined to be erasure *iff* it satisfies the following conditions:

1) $p\left(y_j \notin \{x_j, \xi\} \middle| x_j\right) = 0, \ \forall \ j.$
2) Defining the erasure identifier vector $\mathbf{e}$ as

$$e_j = \begin{cases} 1 & y_j = \xi \\ 0 & \text{otherwise} \end{cases}$$

$p(\mathbf{e}|\mathbf{x})$ is independent of $\mathbf{x}$.

Theorem I. A block code of size $[N, K]$ with equiprobable codewords over an arbitrary erasure channel (memoryless or with memory) has the minimum probability of error (assuming optimum, i.e., maximum likelihood decoding) among all block codes of the same size *if* that code is *Maximum Distance Separable* (MDS). The proof is given in [6], [7].

C. Rate Allocation Problem

The network is modeled as follows. L independent paths, $1, 2, \ldots, L$, connect the source to the destination, as indicated in Fig. 2(a). Information bits are transmitted as packets, each of a constant length r. Furthermore, there is a constraint on the maximum rate for each path, meaning that the i'th path can support a maximum rate of W_i packets per second. This constraint can be considered as an upperbound imposed by the physical characteristics of the path. As an example, [55] introduces the concept of the *maximum TCP-friendly bandwidth* for the maximum capacity of an Internet path. W_i's are assumed to be known at the transmitter side. For a specific application and FEC scheme, we require a rate of S_{req} packets per second from the source to the destination. Obviously, we should have $S_{req} \leq \sum_{i=1}^{L} W_i$ to have a feasible solution. The information packets are assumed to be coded in blocks of length N packets. Hence, it takes $T = \frac{N}{S_{req}}$ seconds to transmit a block of packets. In practical scenarios with finite number of paths, the end-to-end required rate (S_{req}) is given, and the values of N and T have to be chosen based on the feasible complexity of the MDS decoder and the delay constraint of the application, respectively.

According to the FEC model, we can send N_i packets through the path i as long as $\sum_{i=1}^{L} N_i = N$ and $\frac{N_i}{T} \leq W_i$. The rate assigned to path i can be expressed as $S_i = \frac{N_i}{T} = \frac{N_i}{N} S_{req}$, since the transmission instants of the N_i packets are distributed evenly over the block duration T (see Fig. 2(b)). Obviously, we have $\sum_{i=1}^{L} S_i = S_{req}$. The objective of rate allocation problem is to find the optimal rate allocation vector or the vector $\mathbf{N} = (N_1, \cdots, N_L)$ which minimizes the probability of irrecoverable loss (P_E).

The above formulation of rate allocation problem is valid for any finite number of paths and any chosen values of N and T. However, in section IV where the performance of path diversity is studied for a large number of paths, and also in Theorem III where the optimality of the proposed suboptimal algorithm is proved for the asymptotic case, we assume that N grows linearly in terms of the number of paths, i.e. $N = n_0 L$, for a fixed n_0. The reason behind this assumption is that when L grows asymptotically large, the number of paths eventually exceeds the block length, if N stays fixed. Thus, $L - N$ paths become useless for the values of N larger than N. At the same time, it is assumed that the delay imposed by FEC, T, stays fixed with respect to L. This model results in a linearly increasing rate as the number of paths grows. We will later show that utilizing multiple paths, it is possible to simultaneously achieve an exponential decay in P_E and a linear increase in rate, while the delay stays constant.

In this work, an irrecoverable loss is defined as the event where more than $N - K$ packets are lost in a block of N packets. P_E denotes the probability of this event. It should be noted that this probability is different from the decoding error probability of a maximum likelihood decoder performed on an MDS $[N, K]$ code, denoted by $\mathbb{P}\{\mathcal{E}\}$. Theoretically, an optimum maximum likelihood decoder of an MDS code may still decode the original codeword correctly with a positive, but very small probability, if it receives less than K symbols (packets). More precisely, such a decoder is able to correctly decode an MDS code over $\mathbb{F}_q$ with the probability of $\frac{1}{q^i}$ after receiving $K-i$ correct symbols (see the proof of Theorem I in [6], [7] for more details). Of course, for Galois fields with a large cardinality, this probability is usually negligible. The relationship between P_E and $\mathbb{P}\{\mathcal{E}\}$ can be summarized as follows:

$$\begin{aligned}
\mathbb{P}\{\mathcal{E}\} &= P_E - \sum_{i=1}^{K} \frac{\mathbb{P}\{K-i \text{ Packets received correctly}\}}{q^i} \\
&\geq P_E - \frac{1}{q}\sum_{i=1}^{K} \mathbb{P}\{K-i \text{ Packets received correctly}\} \\
&= P_E\left(1-\frac{1}{q}\right). \qquad (1)
\end{aligned}$$

Hence, $\mathbb{P}\{\mathcal{E}\}$ is bounded as

$$P_E\left(1-\frac{1}{q}\right) \leq \mathbb{P}\{\mathcal{E}\} \leq P_E. \qquad (2)$$

The reason P_E is used as the measure of system performance is that while many practical low-complexity decoders for MDS codes work perfectly if the number of correctly received symbols is at least K, their probability of correct decoding is much less than that of maximum likelihood decoders when the number of correctly received symbols is less than K [54]. Thus, in the rest of this paper, P_E is used as a close approximation of decoding error.

III. Probability Distribution of Bad Bursts

The continuous random variable B_i is defined as the duration of time that the path i spends in the bad state in a block duration, T. We denote the values of B_i with parameter t to emphasize that they are expressed in the unit of time. In this section, we focus on one path, for example path 1. Therefore, the index i can be temporarily dropped in analyzing the probability distribution function (pdf) of B_i.

We define the events g and b, respectively, as the channel being in the good or bad states at the start of a block. Then, the distribution of B can be written as

$$f_B(t) = f_{B|b}(t)\pi_b + f_{B|g}\pi_g. \qquad (3)$$

To proceed further, two assumptions are made. First, it is assumed that $\pi_g \gg \pi_b$ or equivalently $\frac{1}{\mu_g} \gg \frac{1}{\mu_b}$. This condition is valid for a channel with a reasonable quality. Besides, the block time T is assumed to be much shorter than the average good state duration $\frac{1}{\mu_g}$, i.e. $1 \gg \mu_g T$, such that T can contain either none or a single interval of bad burst (see [1], [3], [4] for justification). More precisely, the probability of having at least two bad bursts is negligible compared to the probability of having exactly one bad burst. However, it should be noted that all the results of this paper except subsection IV-A remain valid

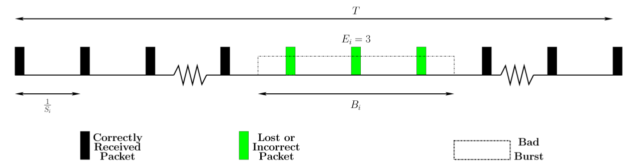


Fig. 3. A bad burst of duration B_i happens in a block of length T. $E_i = 3$ packets are corrupted or lost during the interval B_i. Packets are transmitted every $\frac{1}{S_i}$ seconds, where S_i is the rate of path i in pkt/sec.

regardless of these two assumptions. Of course, in that case, the exact probability distribution function of B_i should be used instead of the approximation used here (refer to Remark I in subsection IV-B).

Hence, the pdf of B conditioned on the event b can be approximated as

$$f_{B|b}(t) = \mu_b e^{-\mu_b t} + \delta(t-T)e^{-\mu_b T} \tag{4}$$

where $\delta(u)$ is the Dirac delta function. (4) follows from the memoryless nature of the exponential distribution, the assumption that T contains at most one bad burst, and the fact that any bad burst longer than T has to be truncated at $B = T$.

To compute $f_{B|g}(t)$, we have

$$f_{B|g}(t) = \mathbb{P}\{B = 0|g\}\delta(t) - \frac{\partial}{\partial t}\mathbb{P}\{B > t|g\} \tag{5}$$

where

$$\mathbb{P}\{B = 0|g\} = e^{-\mu_g T} \approx 1 - \mu_g T \tag{6}$$

and

$$\mathbb{P}\{B > t|g\} \stackrel{(a)}{=} (1 - e^{-\mu_g (T-t)})e^{-\mu_b t} \approx \mu_g (T-t) e^{-\mu_b t} \tag{7}$$

where (a) results from the fact that $\{B > t|g\}$ is equivalent to the initial good burst being shorter than $T - t$, and the following bad burst larger than t, and the duration T containing at most one bad burst. Now, combining (4), (5), (6), and (7), $f_B(t)$ can be computed.

A. Discrete to Continuous Approximation

To compute the probability of irrecoverable loss (P_E), we have to find the probability of k_i packets being lost out of the N_i packets transmitted through the path i, for i from 1 to L and k_i from 0 to N_i. Let us denote the number of erroneous or lost packets over the path i with the random variable E_i. Any two subsequent packets transmitted over the path i are $\frac{1}{S_i}$ seconds apart in time, where S_i is the transmission rate over the i'th path. We observe that the probability $\mathbb{P}\{E_i \geq k_i\}$ can be approximated with the continuous counterpart $\mathbb{P}\{B_i \geq \frac{k_i}{S_i}\}$ when the inter-packet interval is much shorter than the typical

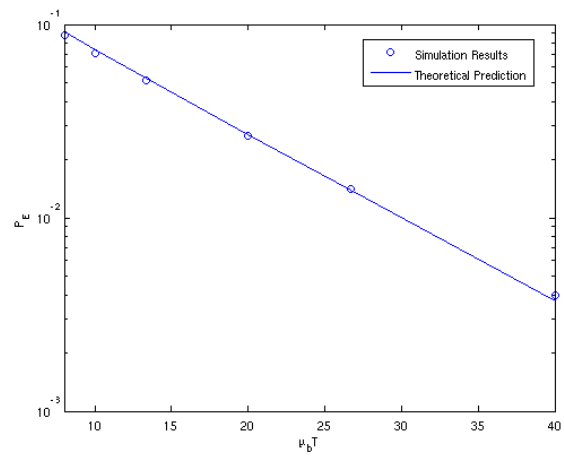


Fig. 4. Probability of irrecoverable loss versus $\mu_b T$ for one path with fixed μ_g, T and α.

bad burst ($\frac{1}{S_i} \ll \frac{1}{\mu_b}$, or equivalently $\mu_b \ll S_i$). The necessity of this condition can be intuitively justified as follows. In case this condition does not hold, any two consecutive packets have to be transmitted on two independent states of the channel. Thus, no gain would be achieved by applying diversity over multiple independent paths. Figure 3 shows an example of this approximation in detail. The continuous approximation simplifies the mathematical analysis as discussed in section IV.

IV. Performance Analysis of FEC on Multiple Paths

Assume that a rate allocation algorithm assigns N_i packets to the path i. According to the discrete to continuous approximation in subsection III-A, when the N_i packets of the FEC block are sent over path i, the loss count can be written as $\frac{B_i}{T} N_i$. Hence, the total ratio of lost packets is equal to

$$\sum_{i=1}^{L} \frac{B_i N_i}{TN} = \sum_{i=1}^{L} \frac{B_i \rho_i}{T}$$

where $\rho_i = \frac{S_i}{S_{req}}$, $0 \leq \rho_i \leq 1$, denotes the portion of the bandwidth assigned to path i. $x_i = \frac{B_i}{T}$ is defined as the portion of time that path i has been in the bad state ($0 \leq x_i \leq 1$). Hence, the probability of irrecoverable loss for an MDS code is equal to

$$P_E = \mathbb{P}\left\{ \sum_{i=1}^{L} \rho_i x_i > \alpha \right\} \tag{8}$$

where $\alpha = \frac{N-K}{N}$. In order to find the optimum rate allocation, P_E has to be minimized with respect to the allocation vector (ρ_i's), subject to the following constraints:

$$0 \leq \rho_i \leq \min\left\{1, \frac{W_i}{S_{req}}\right\}, \quad \sum_{i=1}^{L} \rho_i = 1 \tag{9}$$

where W_i is the bandwidth constraint on path i defined in subsection II-C. Note that since x_i's are proportional to B_i's, their pdf can be easily computed based on the pdf of B_i's.

A. *Performance of FEC on a Single Path*

Probability of irrecoverable loss for one path is equal to

$$P_E = \mathbb{P}\{B > \alpha T\} = \mathbb{P}\{B > \alpha T | b\}\pi_b + \mathbb{P}\{B > \alpha T | g\}\pi_g$$

where $\mathbb{P}\{B > \alpha T | b\}$ and $\mathbb{P}\{B > \alpha T | g\}$ can be computed as

$$\begin{aligned} \mathbb{P}\{B > \alpha T | b\} &= \int_{\alpha T}^{T} f_{B|b}(t)dt &= e^{-\mu_b \alpha T}, \\ \mathbb{P}\{B > \alpha T | g\} &= \int_{\alpha T}^{T} f_{B|g}(t)dt &= \mu_g(1-\alpha)Te^{-\mu_b \alpha T} \end{aligned}$$

when the assumptions in section III and equations (4) and (7) are used. Thus, we have

$$\begin{aligned} P_E &= \pi_b e^{-\mu_b \alpha T}(1 + \mu_b(1-\alpha)T) \\ &\stackrel{(a)}{\approx} \left[\frac{1}{\mu_b} + (1-\alpha)\,T\right]\mu_g e^{-\mu_b \alpha T} \end{aligned} \tag{10}$$

where (a) follows from the assumption that the end-to-end channel has a low probability of error ($\frac{1}{\mu_g} \gg \frac{1}{\mu_b}$).

As we observe, for large values of $\mu_b T$, P_E decays exponentially with $\mu_b T$. Figure 4 shows the results of simulating a typical scenario of streaming data between two end-points with the rate $S_{req} = 1000\frac{pkt}{sec}$, the block length $N = 200$, and the number of information packets $K = 180$. These values result in a block transmission time of $T = 200ms$. The average good burst of the end-to-end channel, μ_g, is selected such that $\mu_g T = \frac{1}{5}$. However, the average bad burst, μ_b, varies such that $\mu_b T$ varies from 8 to 40, in accordance with the values in [3], [4]. The slope of the best linear fit (in semilog scale) to the simulation points is 0.097 which is in accordance with the value of 0.100, resulted from the theoretical approximation in (10).

B. *Identical Paths*

When the paths are identical and have equal bandwidth constraints[2] ($W_i = W$ for $\forall\ 1 \le i \le L$), due to the symmetry of the problem, the uniform rate allocation ($\rho_i = \frac{1}{L}$) is obviously the optimum solution. Of course, the solution is feasible only when we have $\frac{1}{L} \le \frac{W}{S_{req}}$. Then, the probability of irrecoverable loss can be simplified as

$$P_E = \mathbb{P}\left\{\frac{1}{L}\sum_{i=1}^{L} x_i > \alpha\right\}. \tag{11}$$

Let us define $Q(x)$ as the probability distribution function of x. Since x is defined as $x = \frac{B}{T}$, clearly we have $Q(x) = Tf_B(xT)$. Defining $\mathbb{E}\{\}$ as the expected value operator throughout this paper, $\mathbb{E}\{x\}$ can be

[2] The case where W_i's are different is discussed in Remark V of subsection IV-C

computed based on $Q(x)$. We observe that in (11), the random variable x_i's are bounded and independent. Hence, the following well-known upperbound in large deviation theory [56] can be applied

$$P_E \leq e^{-u(\alpha)L}$$
$$u(\alpha) = \begin{cases} 0 & \text{for } \alpha \leq \mathbb{E}\{x\} \\ \lambda\alpha - \log(\mathbb{E}\{e^{\lambda x}\}) & \text{otherwise} \end{cases} \tag{12}$$

where the $\log$ function is computed in Neperian base, and λ is the solution of the following non-linear equation, which is shown to be unique by Lemma I.

$$\alpha = \frac{\mathbb{E}\{xe^{\lambda x}\}}{\mathbb{E}\{e^{\lambda x}\}}. \tag{13}$$

Since λ is unique, we can define $l(\alpha) = \lambda$. Even though being an upperbound, inequality (12) is exponentially tight for large values of L [56]. More precisely

$$P_E \doteq e^{-u(\alpha)L} \tag{14}$$

where the notation $\doteq$ means $\lim\limits_{L->\infty} -\dfrac{\log P_E}{L} = u(\alpha)$. Now, we state two useful lemmas whose proofs can be found in the appendices A and B.

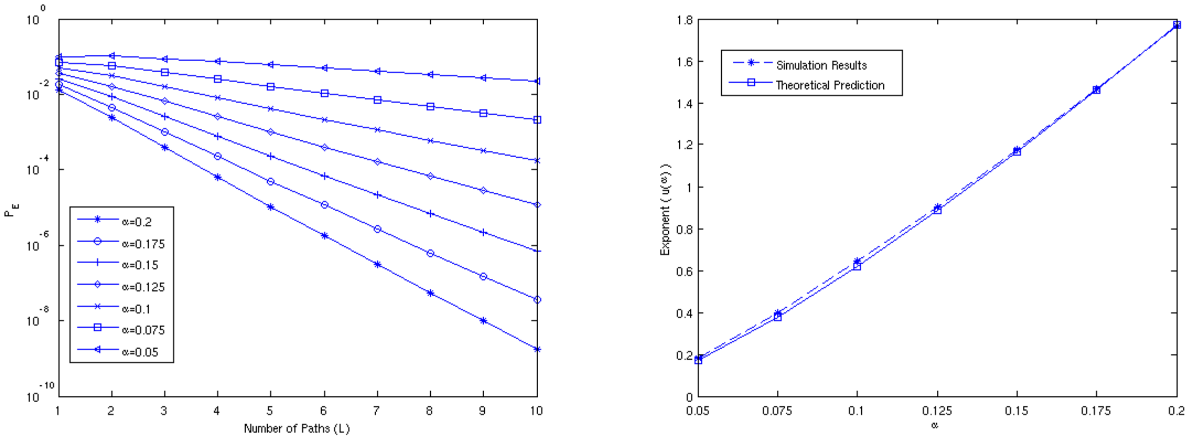
Lemma I. $u(\alpha)$ and $l(\alpha)$ have the following properties:

1) $\frac{\partial}{\partial\alpha} l(\alpha) > 0$
2) $l\,(\alpha = 0) = -\infty$
3) $l\,(\alpha = \mathbb{E}\{x\}) = 0$
4) $l\,(\alpha = 1) = +\infty$
5) $\frac{\partial}{\partial\alpha} u(\alpha) = l(\alpha) > 0$ for $\alpha > \mathbb{E}\{x\}$

Lemma II. Defining $y = \frac{1}{L}\sum_{i=1}^{L} x_i$, where x_i's are i.i.d. random variables as already defined, the probability density function of y satisfies $f_y(\alpha) \doteq e^{-u(\alpha)L}$, for all $\alpha > \mathbb{E}\{x\}$.

Figure 5 compares the theoretical and simulation results. We assume the block transmission time is $T = 200ms$. The block length is proportional to the number of paths as $N = 20L$. The average good burst of the end-to-end channel, μ_g, is selected such that $\mu_g T = \frac{1}{5}$. The end-to-end channel has the error probability of $\pi_b = 0.015$. Coding overhead is changed from $\alpha = 0.05$ to $\alpha = 0.2$. The probability of irrecoverable loss is plotted versus the number of paths, L, in semilogarithmic scale in Fig. 5(a) for different values of α. We observe that as L increases, $\log P_E$ decays linearly which is expected noting equation (12). Also, Fig. 5(b) compares the slope of each plot in Fig. 5(a) with $u(\alpha)$. Figure 5 shows a good agreement between the theory and the simulation results, and also verifies the fact that the stronger the FEC code is (larger α), the higher is the gain we achieve through path diversity (larger exponent).

Remark I. Equation (14) is a direct result of the discrete to continuous approximation in subsection III-A. Therefore, it remains valid even if the other approximations in section III do not hold. For



(a) (b)

Fig. 5. (a) P_E vs. L for different values of α. (b) The exponent (slope) of plot (a) for different values of α: experimental versus theoretical values.

example, if the block time contains more than one bad burst, equations (4) and (7) are no longer valid. However, equation (14) is still valid as long as the discrete to continuous approximation is used. Of course, in this case, the exact distributions of B and x should be used to compute $u(\alpha)$ and λ instead of their simplified versions.

Remark II. A special case is when the block code uses all the bandwidth of the paths. In this case, we have $N = LWT$, where W is the maximum bandwidth of each path, and T is the block duration. Assuming $\alpha > \mathbb{E}\{x\}$ is a constant independent of L, we observe that the information packet rate is equal to $\frac{K}{T} = (1-\alpha)WL$, and the error probability is $P_E \doteq e^{-u(\alpha)L}$. This shows using MDS codes over multiple independent paths provides an exponential decay in the irrecoverable loss probability and a linearly growing end-to-end rate in terms of the number of paths, simultaneously.

C. Non-Identical Paths

Now, let us assume there are J types of paths between the source and the destination, consisting of L_j identical paths of type j ($\sum_{j=1}^{J} L_j = L$). Without loss of generality, we assume that the paths are ordered according to their associated type, i.e. the paths from $1 + \sum_{k=1}^{j-1} L_k$ to $\sum_{k=1}^{j} L_k$ are of type j. We denote $\gamma_j = \frac{L_j}{L}$. According to the i.i.d. assumption, it is obvious that ρ_i has to be the same for all paths of the same type. η_j and y_j are defined as

$$\begin{aligned} \eta_j &= \sum_{\sum_{k=1}^{j-1} L_k < i \leq \sum_{k=1}^{j} L_k} \rho_i \\ y_j &= \frac{\eta_j}{L\gamma_j} \sum_{\sum_{k=1}^{j-1} L_k < i \leq \sum_{k=1}^{j} L_k} x_i. \end{aligned} \tag{15}$$

Following Lemma II, we observe that $f_{y_j}(\beta_j) \doteq e^{-\gamma_j u_j(\frac{\beta_j}{\eta_j})L}$. We define the sets $\mathcal{S}_I$, $\mathcal{S}_O$ and $\mathcal{S}_T$ as

$$\begin{aligned}
\mathcal{S}_I &= \left\{ (\beta_1, \beta_2, \cdots, \beta_J) \,|0 \le \beta_j \le 1, \ \sum_{j=1}^{J} \beta_j > \alpha \right\} \\
\mathcal{S}_O &= \left\{ (\beta_1, \beta_2, \cdots, \beta_J) \,|0 \le \beta_j \le 1, \ \sum_{j=1}^{J} \beta_j = \alpha \right\} \\
\mathcal{S}_T &= \left\{ (\beta_1, \beta_2, \cdots, \beta_J) \,|\eta_j \mathbb{E}\left\{x_j\right\} \le \beta_j, \sum_{j=1}^{J} \beta_j = \alpha \right\}
\end{aligned}$$

respectively. Hence, P_E can be written as

$$\begin{aligned}
P_E &= \mathbb{P}\left\{ \sum_{j=1}^{J} y_j > \alpha \right\} \\
&= \int_{\mathcal{S}_I} \prod_{j=1}^{J} f_{y_j}(\beta_j) d\beta_j \\
&\doteq \int_{\mathcal{S}_I} e^{-L \sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j}{\eta_j})} d\beta_j \\
&\overset{(a)}{\doteq} e^{-L \min_{\boldsymbol{\beta} \in \mathcal{S}_I \cup \mathcal{S}_O} \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j}{\eta_j} \right)} \\
&\overset{(b)}{\doteq} e^{-L \min_{\boldsymbol{\beta} \in \mathcal{S}_O} \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j}{\eta_j} \right)} \\
&\overset{(c)}{\doteq} e^{-L \min_{\boldsymbol{\beta} \in \mathcal{S}_T} \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j}{\eta_j} \right)} \\
&\overset{(d)}{\doteq} e^{-L \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j^\star}{\eta_j} \right)}
\end{aligned} \tag{16}$$

where (a) follows from Lemma III, (b) follows from the fact that $u_j(\alpha)$ is a strictly increasing function of α, for $\alpha > \mathbb{E}\{x_j\}$, and (c) can be proved as follows. Let us denote the vector which minimizes the exponent over the set $\mathcal{S}_O$ as $\hat{\boldsymbol{\beta}}^\star$. Since $\mathcal{S}_T$ is a subset of $\mathcal{S}_O$, $\hat{\boldsymbol{\beta}}^\star$ is either in $\mathcal{S}_T$ or in $\mathcal{S}_O - \mathcal{S}_T$. In the former case, (c) is obviously valid. When $\hat{\boldsymbol{\beta}}^\star \in \mathcal{S}_O - \mathcal{S}_T$, we can prove that $0 \le \hat{\beta}_j^\star \le \eta_j \mathbb{E}\{x_j\}$, for all $1 \le j \le J$, by contradiction. Let us assume the opposite is true, i.e., there is at least one index $1 \le j \le J$ such that $0 \le \hat{\beta}_j^\star \le \eta_j \mathbb{E}\{x_j\}$, and at least one other index $1 \le k \le J$ such that $\eta_k \mathbb{E}\{x_k\} < \hat{\beta}_k^\star$. Then, knowing that the derivative of of $u_j(\alpha)$ is zero for $\alpha = \mathbb{E}\{x_j\}$ and strictly positive for $\alpha > \mathbb{E}\{x_j\}$, a small increase in $\hat{\beta}_j^\star$ and an equal decrease in $\hat{\beta}_k^\star$ reduces the objective function, $\sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j}{\eta_j} \right)$, which contradicts the assumption that $\hat{\boldsymbol{\beta}}^\star$ is a minimum point. Knowing that $0 \le \hat{\beta}_j^\star < \eta_j \mathbb{E}\{x_j\}$, for all $1 \le j \le J$, it is easy to show that the minimum value of the objective function is zero over $\mathcal{S}_O$, and $\mathcal{S}_T$ has to be an empty set. Defining the minimum value of the positive objective function as zero over an empty set ($\mathcal{S}_T$) makes (c) valid for the latter case where $\hat{\boldsymbol{\beta}}^\star \in \mathcal{S}_O - \mathcal{S}_T$. Finally, applying Lemma IV results in (d) where $\boldsymbol{\beta}^\star$ is defined in the Lemma.

Lemma III. For any continuous positive function $h(\mathbf{x})$ over a convex set $\mathcal{S}$, and defining $H(L)$ as

$$H(L) = \int_{\mathcal{S}} e^{-h(\mathbf{x})L} d\mathbf{x}$$

we have

$$\lim_{L\to\infty} -\frac{\log(H(L))}{L} = \inf_{\mathcal{S}} h(\mathbf{x}) = \min_{cl(\mathcal{S})} h(\mathbf{x})$$

where $cl(\mathcal{S})$ denotes the closure of $\mathcal{S}$ (refer to [57] for the definition of the closure operator). Proof of Lemma III can be found in appendix C.

Lemma IV. There exists a unique vector $\boldsymbol{\beta}^\star$ with the elements $\beta_j^\star = \eta_j l_j^{-1}\left(\frac{\nu\eta_j}{\gamma_j}\right)$ which minimizes the convex function $\sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j}{\eta_j})$ over the convex set $\mathcal{S}_T$, where ν satisfies the following condition

$$\sum_{j=1}^{J} \eta_j l_j^{-1}\left(\frac{\nu\eta_j}{\gamma_j}\right) = \alpha. \tag{17}$$

$l^{-1}()$ denotes the inverse of the function $l()$ defined in subsection IV-B. Proof of Lemma IV can be found in appendix D.

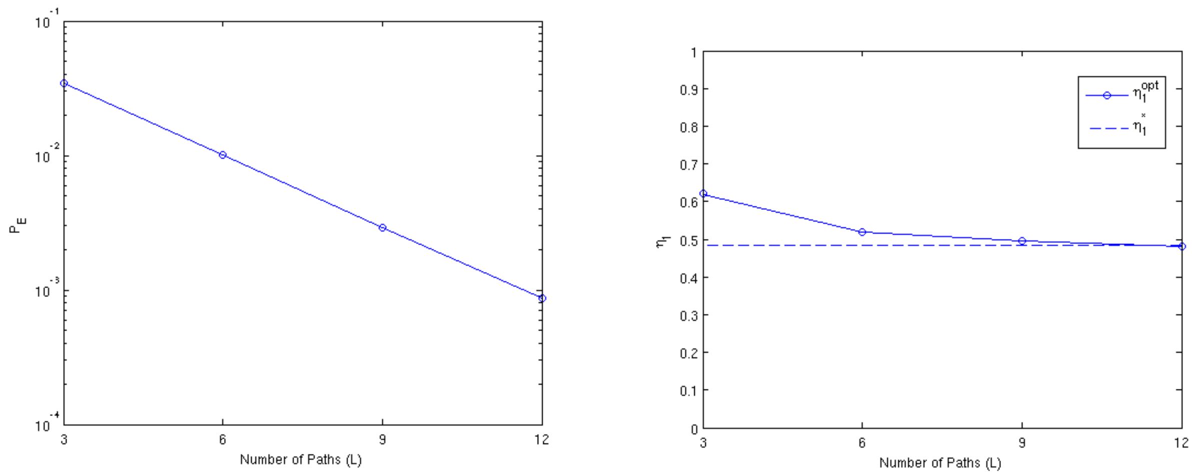
Equation (16) is valid for any fixed value of $\boldsymbol{\eta}$. To achieve the most rapid decay of P_E, the exponent must be maximized over $\boldsymbol{\eta}$.

$$\lim_{L\to\infty} -\frac{\log P_E}{L} = \max_{0\le\eta_j\le1} \sum_{j=1}^{J} \gamma_j u_j\left(\frac{\beta_j^\star}{\eta_j}\right) \tag{18}$$

where $\boldsymbol{\beta}^\star$ is defined for any value of the vector $\boldsymbol{\eta}$ in Lemma IV. Theorem II solves the maximization problem in (18) and identifies the asymptotically optimum rate allocation (for large number of paths).

Theorem II. Consider a point-to-point connection over the network with L independent paths from the source to the destination, each modeled as a Gilbert-Elliot cell, with a large enough bandwidth constraint[3]. The paths are from J different types, L_j paths from the type j. Assume a block FEC of size $[N, K]$ is sent during a time interval T. Let N_j denote the number of packets in a block of size N assigned to the paths of type j, such that $\sum_{j=1}^{J} N_j = N$. The rate allocation vector $\boldsymbol{\eta}$ is defined as $\eta_j = \frac{N_j}{N}$. For fixed values of $\gamma_j = \frac{L_j}{L},\ n_0 = \frac{N}{L},\ k_0 = \frac{K}{L},\ T$ and asymptotically large number of paths L, the optimum rate

[3]By the term 'large enough', we mean the bandwidth constraint on a path of type j, W_j, satisfies the condition $\frac{\eta_j n_0}{T\gamma_j} \le W_j$. The reason is that η_j must satisfy both conditions of $0 \le \eta_j \le 1$ and $\frac{N_j}{TL_j} = \frac{\eta_j n_0 L}{T\gamma_j L} \le W_j$, simultaneously. When W_j is large enough such that $\frac{\eta_j n_0}{T\gamma_j} \le W_j$, the latter condition is automatically satisfied, and the optimization problem can be solved.



(a) (b)

Fig. 6. (a) P_E versus L for the combination of two path types, one third from type I and the rest from type II. (b) The normalized aggregated weight of type I paths in the optimal rate allocation (η_1^{opt}), compared with the value of η_1 which maximizes the exponent of equation (18) ($\eta_1^{\star}$).

allocation vector $\boldsymbol{\eta}^{\star}$ can be found by solving the following optimization problem:

$$\max_{\boldsymbol{\eta}} g(\boldsymbol{\eta}),$$
$$\text{s.t. } \sum_{j=1}^{J} \eta_j = 1, \ 0 \leq \eta_j \leq 1$$

where $g(\boldsymbol{\eta}) = \sum_{j=1}^{J} \gamma_j u_j \left(\frac{\beta_j^{\star}}{\eta_j}\right)$, and $\boldsymbol{\beta}^{\star}$ is an implicit function of $\boldsymbol{\eta}$ defined in Lemma IV. The functions $u_j()$ and $l_j()$ are defined in subsections IV-B and IV-C. Solving the above optimization problem gives the unique solution $\boldsymbol{\eta}^{\star}$ as

$$\eta_j^{\star} = \begin{cases} 0 & \text{if } \alpha \leq \mathbb{E}\{x_j\} \\ \dfrac{\gamma_j l_j(\alpha)}{\displaystyle\sum_{i=1,\ \alpha > \mathbb{E}\{x_i\}}^{J} \gamma_i l_i(\alpha)} & \text{otherwise} \end{cases} \tag{19}$$

if there is at least one $1 \leq j \leq J$ for which $\alpha > \mathbb{E}\{x_j\}$. Otherwise, when $\alpha \leq \mathbb{E}\{x_j\}$ for all $1 \leq j \leq J$, the maximum value is zero for any arbitrary rate allocation vector, $\boldsymbol{\eta}$. In any case, the maximum value of the objective function is $g(\boldsymbol{\eta}^{\star}) = \sum_{j=1}^{J} \gamma_j u_j(\alpha)$ which is indeed the exponent of P_E versus L. The proof of the theorem can be found in appendix E.

Remark III. Theorem II can be interpreted as follows. For large values of L, adding a new type of path contributes to the path diversity *iff* the path satisfies the quality constraint $\alpha > \mathbb{E}\{x\}$, where x is the percentage of time that the path spends in the bad state in the time interval $[0, T]$. Only in this case, adding the new type of path exponentially improves the performance of the system in terms of the probability of irrecoverable loss.

Remark IV. Observing the exponent coefficient corresponding to the optimum allocation vector $\boldsymbol{\eta}^\star$, we can see that the typical error event occurs when the ratio of the lost packets on all types of paths is the same as the total fraction of the lost packets, α. However, this is not the case for any arbitrary rate allocation vector $\boldsymbol{\eta}$.

Remark V. An interesting extension of Theorem II is the case where all types have identical erasure patterns ($u_j(x) = u_k(x)$ for $\forall\ 1 \le j, k \le J$ and $\forall x$), but different bandwidth constraints. Adopting the notation of Theorem II, the bandwidth constraint on η_j can be written as $\frac{\eta_j n_0 L}{T \gamma_j L} \le W_j$, where W_j is the maximum bandwidth for a path of type j. Let us define $\tilde{\boldsymbol{\eta}}^\star$ as the allocation vector which maximizes the objective function of Theorem II ($g(\boldsymbol{\eta})$), and satisfies the bandwidth constraints too. $\boldsymbol{\eta}^\star$ is also defined as the maximizing vector for the unconstrained problem in Theorem II. According to equation (19), we have $\eta_j^\star = \gamma_j$ for $\forall 1 \le j \le J$. It is obvious that $\tilde{\boldsymbol{\eta}}^\star = \boldsymbol{\eta}^\star$ if $\eta_j^\star \le \frac{\gamma_j W_j T}{n_0}$ for all j. In case $\eta_j^\star$ does not satisfy the bandwidth constraint for some j, $\tilde{\boldsymbol{\eta}}^\star$ can be found by the water-filling algorithm. More accurately, we have

$$\tilde{\eta}_j^\star = \begin{cases} \dfrac{\gamma_j W_j T}{n_0} & \text{if } \tilde{\eta}_j^\star \le \gamma_j \Upsilon \\ \gamma_j \Upsilon & \text{if } \tilde{\eta}_j^\star < \dfrac{\gamma_j W_j T}{n_0} \end{cases} \tag{20}$$

where Υ can be found by imposing the condition $\sum_{j=1}^{J} \tilde{\eta}_j^\star = 1$. Figure 7 depicts water-filling among identical paths with four different bandwidth constraints. Proof of equation (20) can be found in appendix F.

Figure 6(a) shows P_E of the optimum rate allocation versus L for a system consisting of two types of path. The optimal rate allocation is found by exhaustive search among all possible allocation vectors. The block transmission time is $T = 200ms$. The block length is proportional to the number of paths as $N = 20L$. The average good burst, μ_g, is selected such that we have $\mu_g T = \frac{1}{5}$ for both types of paths. $\gamma_1 = \frac{1}{3}$ of the paths (of the first type) benefit from shorter bad bursts and lower error probability of $\pi_{b,1} = 0.015$, and the rest (the second type) suffer from longer congestion bursts resulting in a higher error probability of $\pi_{b,2} = 0.025$. The coding overhead is $\alpha = 0.1$. The figure depicts a linear behavior in semi-logarithmic scale with the exponent of 0.403, which is comparable to 0.389 resulted from (19).

In the scenario of Fig. 6(a), let us denote $\eta_1^\star$ as the value of of the first element of $\boldsymbol{\eta}$ in equation (19). Obviously, $\eta_1^\star$ does not depend on L. Moreover, η_1^{opt} is defined as the normalized aggregated weight of type I paths in the optimal rate allocation. Figure 6(b) compares η_1^{opt} with $\eta_1^\star$ for different number of paths. It is observed that η_1^{opt} converges rapidly to $\eta_1^\star$ as L grows. Figure 6(a) also verifies that the allocation vector candidate $\eta^\star$ proposed by Theorem II indeed meets the optimal allocation vector for large values of L.

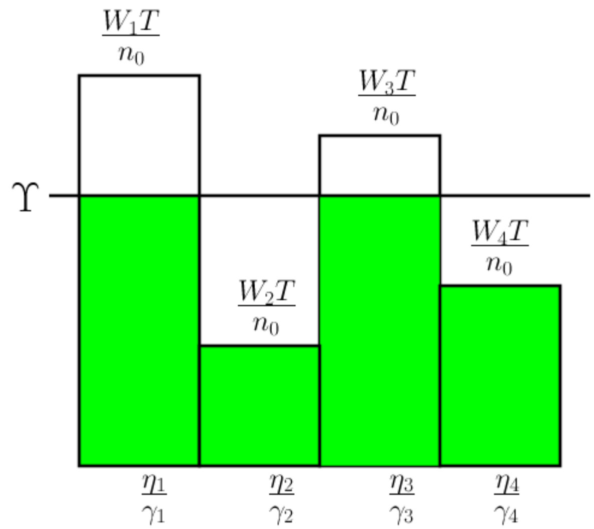


Fig. 7. WaterFilling algorithm over identical paths with four different bandwidth constraints.

V. Suboptimal Rate Allocation

In order to compute the complexity of the rate allocation problem, we focus our attention on the original discrete formulation in subsection II-C. According to the model of subsection IV-C, we assume the available paths are from J types, L_j paths from type j, such that $\sum_{j=1}^{J} L_j = L$. Obviously, all the paths from the same type should have equal rate. Therefore, the rate allocation problem is turned into finding the vector $\mathbf{N} = (N_1, \ldots, N_J)$ such that $\sum_{j=1}^{J} N_j = N$, and $0 \leq N_j \leq L_j W_j T$ for all j. N_j denotes the number of packets assigned to all the paths of type j. Let us temporarily assume that all paths have enough bandwidth such that N_j can vary from 0 to N for all j. There are $\binom{N+J-1}{J-1}$ L-dimensional non-negative vectors of the form $(N_1, \ldots, N_J)$ which satisfy the equation $\sum_{j=1}^{J} N_j = N$ each representing a distinct rate allocation. Hence, the number of candidates is exponential in terms of J.

First, we prove the problem of rate allocation is NP [58] in the sense that P_E can be computed in polynomial time for any candidate vector $\mathbf{N} = (N_1, \ldots, N_J)$. Let us define $P_e^{\mathbf{N}}(k, j)$ as the probability of having more than k errors over the paths of types 1 to j for a specific allocation vector $\mathbf{N}$. We also define $Q_j(n, k)$ as the probability of having exactly k errors out of the n packets sent over the paths of type j. $Q_j(n, k)$ can be computed and stored for all path types and values of n and k with polynomial complexity as explained in appendices G and H. Then, the following recursive formula holds for $P_e^{\mathbf{N}}(k,j)$

$$\begin{aligned} P_e^{\mathbf{N}}(k,j) &= \begin{cases} \sum_{i=0}^{N_j} Q_j(N_j,i) P_e^{\mathbf{N}}(k-i,j-1) & \text{if } k \geq 0 \\ 1 & \text{if } k < 0 \end{cases} \\ P_e^{\mathbf{N}}(k,1) &= \sum_{i=k+1}^{N_1} Q_1(N_1,i). \end{aligned} \tag{21}$$

To compute $P_e^{\mathbf{N}}(K,J)$ by the above recursive formula, we apply a well-known technique in the theory of algorithms called *memoization* [59]. Memoization works by storing the computed values of a recursive function in an array. By keeping this array in the memory, memoization avoids recomputing the function for the same arguments when it is called later. To compute $P_e^{\mathbf{N}}(K,J)$, an array of size $O(KJ)$ is required. This array should be filled with the values of $P_e^{\mathbf{N}}(k,j)$ for $0 < k \leq K$, and $1 \leq j \leq J$. Computing $P_e^{\mathbf{N}}(k,j)$ requires $O(K)$ operations assuming the values of $P_e^{\mathbf{N}}(i,j-1)$ and $Q_j(N_j,i)$ and $\sum_{i=k+1}^{N_j} Q_j(N_j,i)$ are already computed for $0 \leq i \leq k$. Thus, $P_e^{\mathbf{N}}(K,J)$ can be computed with the complexity of $O(K^2 J)$ if the values of $Q_j(N_j,k)$ are given for all N_j and $0 \leq k \leq K$. Following appendix H, we note that for each j, $Q_j(N_j,k)$ for $0 \leq k \leq K$ is computed offline with the complexity of $O(K^2 L_j) + O\left(\frac{N_j}{L_j} K\right)$. Hence, the total complexity of computing $P_e^{\mathbf{N}}(K,J)$ adds up to

$$\begin{aligned} & O(K^2 J) + \sum_{j=1}^{J} O\left(K^2 L_j + \frac{N_j}{L_j} K\right) \\ \stackrel{(a)}{=} \; & O(K^2 J) + \sum_{j=1}^{J} O\left(K^2 L_j + N_j K\right) \\ \stackrel{(b)}{=} \; & O\left(K^2 L + KN\right) \end{aligned} \tag{22}$$

where (a) follows from the fact that $\frac{N_j}{L_j} < N_j$, and the term $O(K^2 J)$ is omitted in (b) since we know that $J < L$.

Now, we propose a suboptimal polynomial time algorithm to estimate the best path allocation vector, $\mathbf{N}^{opt}$. Let us define $P_e^{opt}(n,k,j)$ as the probability of having more than k errors for a block of length n over the paths of types 1 to j minimized over all possible rate allocations ($\mathbf{N} = \mathbf{N}^{opt}$). First, we find a lowerbound $\hat{P}_e(n,k,j)$ for $P_e^{opt}(n,k,j)$ from the following recursive formula

$$\hat{P}_e(n,k,j) = \begin{cases} \min\limits_{0\le n_j\le \min\{n,\lfloor L_jW_jT\rfloor\}} \sum_{i=0}^{n_j} Q_j(n_j,i)\cdot \\ \qquad \hat{P}_e(n-n_j,k-i,j-1) & \text{if } k>0 \\ \\ 1 & \text{if } k\le 0 \end{cases}$$

$$\hat{P}_e(n,k,1) = \sum_{i=k+1}^{n} Q_1(n,i). \tag{23}$$

Using memoization technique, we need an array of size $O(NKJ)$ to store the values of $\hat{P}_e(n,k,j)$ for $0<n\le N$, $0<k\le K$, and $1\le j\le J$. According to the recursive definition above, computing $\hat{P}_e(n,k,j)$ requires $O(NK)$ operations assuming the values of $Q_j(n_j,i)$ and $\hat{P}_e(n-n_j,k-i,j-1)$ and $\sum_{i=k+1}^{n_j} Q_j(n_j,i)$ are already computed for all i and n_j. Thus, it is easy to verify that $\hat{P}_e(N,K,J)$ can be computed with the complexity of $O(N^2K^2J)$ when the values of $Q_j(n_j,i)$ are given for all $0<n_j\le n$ and $0\le i\le K$. According to appendix H, for each $1\le j\le J$, and for each $0<n_j\le N$, $Q_j(n_j,i)$ for all $0\le i\le n_j$ is computed offline with the complexity of $O(n_j^2L_j)+O\left(\frac{n_j}{L_j}n_j\right)=O(n_j^2L_j)$. Thus, computing $Q_j(n_j,i)$ for all $1\le j\le J$, and $0<n_j\le N$, and $0\le i\le n_j$, has the complexity of $\sum_{j=1}^{J}\sum_{n_j=1}^{N}O(n_j^2L_j)=O(N^3L)$. Finally, $\hat{P}_e(N,K,J)$ can be computed with the total complexity of $O(N^2K^2J+N^3L)$.

The following lemma guarantees that $\hat{P}_e(n,k,j)$ is in fact a lowerbound for $P_e^{opt}(n,k,j)$.

Lemma V. $P_e^{opt}(n,k,j)\ge \hat{P}_e(n,k,j)$. The proof is given in appendix I.

The following algorithm recursively finds a suboptimum allocation vector $\hat{\mathbf{N}}$ based on the lowerbound of Lemma V.

(1): *Initialize* $j\leftarrow J$, $n\leftarrow N$, $k\leftarrow K$.

(2): *Set*

$$\begin{aligned} \hat{N}_j &= \operatorname*{argmin}_{0\le n_j\le \min\{n,\lfloor L_jW_jT\rfloor\}} \sum_{i=0}^{n_j} Q_j(n_j,i)\cdot \\ &\qquad \hat{P}_e(n-n_j,k-i,j-1) \\ K_j &= \operatorname*{argmax}_{0\le i\le \hat{N}_j} Q_j(\hat{N}_j,i)\hat{P}_e(n-\hat{N}_j,k-i,j-1) \end{aligned}$$

(3): *Update* $n\leftarrow n-\hat{N}_j$, $k\leftarrow k-K_j$, $j\leftarrow j-1$.

(4): *If* $j>1$ *and* $k\ge 0$, *goto (2).*

(5): *For* $m=1$ *to* j, *set* $\hat{N}_m\leftarrow \lfloor\frac{n}{j}\rfloor$.

(6): $\hat{N}_j\leftarrow \hat{N}_j+$ *Rem*(n,j) *where Rem*(a,b) *denotes the remainder of dividing* a *by* b.

Intuitively speaking, the above algorithm tries to recursively find the typical error event (K_j's) which has the maximum contribution to the error probability, and assigns the rate allocations ($\hat{N}_j$'s) such that the estimated typical error probability ($\hat{P}_e$) is minimized. Indeed, Lemma V shows that the estimate used in the algorithm ($\hat{P}_e$) is a lower-bound for the minimum achievable error probability (P_e^{opt}). Comparing (23) and the step (2) of our algorithm, we observe that the values of $\hat{N}_j$ and K_j can be found in $O(1)$ during the computation of $\hat{P}_e(N, K, J)$. Hence, complexity of the proposed algorithm is the same as that of computing $\hat{P}_e(N, K, J)$, $O(N^2K^2J + N^3L)$.

The following theorem guarantees that the output of the above algorithm converges to the asymptotically optimal rate allocation introduced in Theorem II of section IV-C, and accordingly, it performs optimally for large number of paths.

Theorem III. Consider a point-to-point connection over the network with L independent paths from the source to the destination, each modeled as a Gilbert-Elliot cell with a large enough bandwidth constraint. The paths are from J different types, L_j paths from the type j. Assume a block FEC of the size $[N, K]$ is sent during an interval time T. For fixed values of $\gamma_j = \frac{L_j}{L},\ n_0 = \frac{N}{L},\ k_0 = \frac{K}{L},\ T$ and asymptotically large number of paths (L) we have

1) $\hat{P}_e(N, K, J) \doteq P_e^{opt}(N, K, J) \doteq e^{-L\sum_{j=1}^{J}\gamma_j u_j(\alpha)}$
2) $\dfrac{\hat{N}_j}{N} = \eta_j^\star + o(1)$
3) $\frac{K_j}{\hat{N}_j} = \alpha + o(1)$ for $\alpha > \mathbb{E}\{x_j\}$.

where $\alpha = \frac{k_0}{n_0}$ and $u_j()$ are defined in subsections IV-B and IV-C. $\hat{P}_e(N, K, J)$ is the lowerbound for $P_e^{opt}(n, k, j)$ defined in equation (23). $\hat{N}_j$ is the total number of packets assigned to the paths of type j by the suboptimal rate allocation algorithm. $\eta_j^\star$ is the asymptotically optimal rate allocation given in equation (19). K_j is also defined in the step (2) of the algorithm. The notation $f(L) = o(g(L))$ means $\lim_{L\to\infty}\frac{f(L)}{g(L)} = 0$. The proof can be found in appendix J.

The proposed algorithm is compared with four other allocation schemes over $L = 6$ paths in Fig. 8. The optimal method uses exhaustive search over all possible allocations. '*Best Path Allocation*' assigns everything to the best path only, ignoring the rest. '*Equal Distribution*' scheme distributes the packets among all paths equally. Finally, the '*Asymptotically Optimal*' allocation assigns the rates based on equation (19). The block length and the number of information packets are assumed to be $N = 100$ and $K = 90$, respectively. The overall rate is $S_{req} = 1000pkt/sec$ which results in $T = 100ms$. The average good burst, μ_g, is selected such that we have $\mu_g T = \frac{1}{5}$. However, quality of the paths are different as they have different average bad burst durations. Packet error probability of the paths are listed as $[0.0175 \pm \frac{\Delta}{2}, 0.0175 \pm \frac{3\Delta}{2}, 0.0175 \pm \frac{5\Delta}{2}]$, such that the median is fixed at 0.0175. Δ is also defined as a measure of deviation from this median. $\Delta = 0$ represents the case where all the paths are identical. The

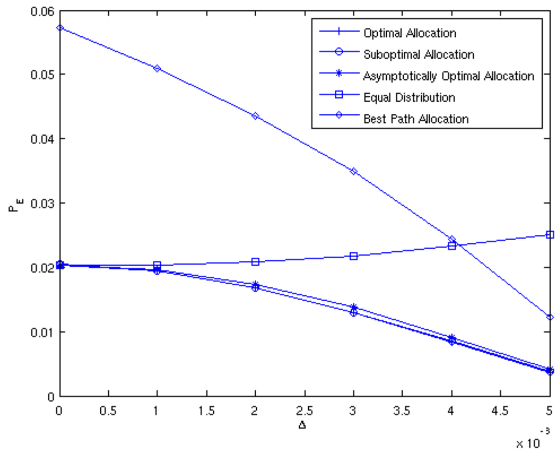


Fig. 8. Optimal and suboptimal rate allocations are compared with equal distribution and best path allocation schemes for different values of Δ

larger is Δ, the more variety we have among the paths and the more diversity gain might be achieved using a judicious rate allocation.

As seen, our suboptimal algorithm tracks the optimal algorithm so closely that the corresponding curves are not easily distinguishable over a wide range. However, the '*Asymptotically Optimal*' rate allocation results in lower performance since there is only one path from each type which makes the asymptotic analysis assumptions invalid. When $\Delta = 0$, '*Equal Distribution*' scheme obviously coincides with the optimal allocation. This scheme eventually diverges from the optimal algorithm as Δ grows. However, it still outperforms the best path allocation method as long as Δ is not too large. For very large values of Δ, the best path dominates all the other ones, and we can ignore the rest of the paths. Hence, the best path allocation eventually converges to the optimal scheme when Δ increases.

VI. Conclusion

In this work, we have studied the performance of forward error correction over a block of packets sent through multiple independent paths. It is known that *Maximum Distance Separable* (MDS) block codes are optimum over our *End-to-End Channel* model, and any other erasure channel with or without memory, in the sense that their probability of error is minimum among all block codes of the same size [6], [7]. Adopting MDS codes, the probability of irrecoverable loss, P_E, is analyzed for the cases of a single path, multiple identical, and multiple non-identical paths based on the discrete to continuous relaxation. When there are L identical paths, P_E is upperbounded using large deviation theory. This bound is shown to be exponentially tight in terms of L. The asymptotic analysis shows that the exponential decay of P_E with L is still valid in the case of non-identical paths. Furthermore, the optimal rate allocation problem is solved in the asymptotic case where L is very large. It is seen that for the optimal rate allocation, each path is assigned a positive rate *iff* its quality is above certain threshold. The quality of a path is defined as the percentage of the time it spends in the bad state. Finally, we focus on the problem of optimum rate allocation when L is not necessarily large. A heuristic suboptimal algorithm is proposed which computes a near-optimal allocation in polynomial time. For large values of L, the result of this algorithm converges to the optimal solution. Moreover, simulation results are provided which verify the validity of our theoretical analyses in several practical scenarios, and also show that the proposed suboptimal algorithm approximates the optimal allocation very closely.

APPENDIX A
PROOF OF LEMMA I

1) We define the function $v(\lambda)$ as

$$v(\lambda) = \frac{\mathbb{E}\{xe^{\lambda x}\}}{\mathbb{E}\{e^{\lambda x}\}}. \tag{24}$$

Then, the first derivative of $v(\lambda)$ will be

$$\frac{\partial}{\partial \lambda} v(\lambda) = \frac{\mathbb{E}\{x^2 e^{\lambda x}\}\mathbb{E}\{e^{\lambda x}\} - [\mathbb{E}\{xe^{\lambda x}\}]^2}{[\mathbb{E}\{e^{\lambda x}\}]^2}. \tag{25}$$

According to Cauchy-Schwarz inequality, the following statement is always true for any two functions of $f()$ and $g()$

$$\left(\int_x f(x)g(x)dx\right)^2 < \int_x f^2(x)dx \int_x g^2(x)dx \tag{26}$$

unless $f(x) = Kg(x)$ for a constant K and all values of x. If we choose $f(x) = \sqrt{x^2 Q(x) e^{x\lambda}}$ and $g(x) = \sqrt{Q(x)e^{x\lambda}}$, they can not be proportional to each other for all values of x. Therefore, the numerator of equation (25) has to be strictly positive for all λ. Since the function $v(\lambda)$ is strictly increasing, it has an inverse $v^{-1}(\alpha)$ which is also strictly increasing. Moreover, the non-linear equation $v(\lambda) = \alpha$ has a unique solution of the form $\lambda = v^{-1}(\alpha) = l(\alpha)$.

2) To show that $l(\alpha = 0) = -\infty$, we prove an equivalent statement of the form $\lim_{\lambda \to -\infty} v(\lambda) = 0$. Since x is a random variable in the range $[0, 1]$ with the probability density function $Q(x)$, for any $0 < \epsilon < 1$, we can write

$$\begin{aligned}
\lim_{\lambda \to -\infty} v(\lambda) &= \lim_{\lambda \to -\infty} \frac{\int_0^\epsilon xQ(x)e^{x\lambda}dx + \int_\epsilon^1 xQ(x)e^{x\lambda}dx}{\int_0^1 Q(x)e^{x\lambda}dx} \\
&\leq \lim_{\lambda \to -\infty} \frac{\int_0^\epsilon xQ(x)e^{x\lambda}dx}{\int_0^\epsilon Q(x)e^{x\lambda}dx} + \frac{\int_\epsilon^1 xQ(x)dx}{\int_0^\epsilon Q(x)e^{(x-\epsilon)\lambda}dx} \\
&\overset{(a)}{=} \lim_{\lambda \to -\infty} \frac{\int_0^\epsilon xQ(x)e^{x\lambda}dx}{\int_0^\epsilon Q(x)e^{x\lambda}dx} \\
&\overset{(b)}{=} \lim_{\lambda \to -\infty} \frac{x_1 Q(x_1)e^{\lambda x_1}}{Q(x_2)e^{\lambda x_2}}
\end{aligned} \tag{27}$$

for some $x_1, x_2 \in [0, \epsilon]$. (a) follows from the fact that for $x \in [0, \epsilon]$, $(x - \epsilon)\lambda \to +\infty$ when $\lambda \to -\infty$, and (b) is a result of the mean value theorem for integration [60]. This theorem states that for every continuous function $f(x)$ in the interval $[a, b]$, we have

$$\exists\ x_0 \in [a, b] \quad s.t. \quad \int_a^b f(x)dx = f(x_0)[b - a]. \tag{28}$$

Equation (27) is valid for any arbitrary $0 < \epsilon < 1$. If we choose $\epsilon \to 0$, x_1 and x_2 are both squeezed in the interval $[0, \epsilon]$. Thus, we have

$$\lim_{\lambda \to -\infty} v(\lambda) \leq \lim_{\lambda \to -\infty} \lim_{\epsilon \to 0} \frac{x_1 Q(x_1) e^{\lambda x_1}}{Q(x_2) e^{\lambda x_2}} = \lim_{\epsilon \to 0} x_1 = 0 \tag{29}$$

Based on the distribution of x, $v(\lambda)$ is obviously non-negative for any λ. Hence, the inequality in (29) can be replaced by equality.

3) By observing that $v(\lambda = 0) = \mathbb{E}\{x\}$, it is obvious that $l(\alpha = \mathbb{E}\{x\}) = 0$.

4) To show that $l(\alpha = 1) = +\infty$, we prove the equivalent statement of the form $\lim_{\lambda \to +\infty} v(\lambda) = 1$. For any $0 < \epsilon < 1$ and $x \in [1 - \epsilon, 1]$, $(x - 1 + \epsilon)\lambda \to +\infty$ when $\lambda \to +\infty$. Then, defining $\zeta = 1 - \epsilon$, we have

$$\lim_{\lambda \to +\infty} \frac{\int_0^\zeta x Q(x) e^{x\lambda} dx}{\int_0^1 Q(x) e^{x\lambda} dx} \leq \lim_{\lambda \to +\infty} \frac{\int_0^\zeta x Q(x) dx}{\int_\zeta^1 Q(x) e^{(x - \zeta)\lambda} dx} = 0. \tag{30}$$

Since the fraction in (30) is obviously non-negative for all λ, this inequality can be replaced by an equality. Similarly, we have

$$\lim_{\lambda \to +\infty} \frac{\int_0^\zeta Q(x) e^{x\lambda} dx}{\int_\zeta^1 x Q(x) e^{x\lambda} dx} \leq \lim_{\lambda \to +\infty} \frac{\int_0^\zeta Q(x) dx}{\int_\zeta^1 x Q(x) e^{(x - \zeta)\lambda} dx} = 0. \tag{31}$$

which can also be replaced by equality. Now, the limit of $v(\lambda)$ is written as

$$\begin{aligned}
\lim_{\lambda \to +\infty} v(\lambda) &= \lim_{\lambda \to +\infty} \frac{\int_0^\zeta x Q(x) e^{x\lambda} dx + \int_\zeta^1 x Q(x) e^{x\lambda} dx}{\int_0^1 Q(x) e^{x\lambda} dx} \\
&\stackrel{(a)}{=} \lim_{\lambda \to +\infty} \frac{\int_\zeta^1 x Q(x) e^{x\lambda} dx}{\int_0^1 Q(x) e^{x\lambda} dx} \\
&\stackrel{(b)}{=} \left(\lim_{\lambda \to +\infty} \frac{\int_0^\zeta Q(x) e^{x\lambda} dx + \int_\zeta^1 Q(x) e^{x\lambda} dx}{\int_\zeta^1 x Q(x) e^{x\lambda} dx} \right)^{-1} \\
&\stackrel{(c)}{=} \left(\lim_{\lambda \to +\infty} \frac{\int_\zeta^1 Q(x) e^{x\lambda} dx}{\int_\zeta^1 x Q(x) e^{x\lambda} dx} \right)^{-1} \\
&\stackrel{(d)}{=} \left(\lim_{\lambda \to +\infty} \frac{Q(x_1) e^{x_1 \lambda}}{x_2 Q(x_2) e^{x_2 \lambda}} \right)^{-1}
\end{aligned} \tag{32}$$

for some $x_1, x_2 \in [1 - \epsilon, 1]$. (a) follows from equation (30), and (b) is valid since the final result shows that $\lim_{\lambda \to +\infty} v(\lambda)$ is finite and non-zero [60]. (c) follows from equation (31), and (d) is a result of the mean value theorem for integration. If we choose $\epsilon \to 0$, x_1 and x_2 are both squeezed in the interval $[1-\epsilon, 1]$. Then, equation (32) turns into

$$\lim_{\lambda\to+\infty} v(\lambda) = \left(\lim_{\lambda\to+\infty}\lim_{\epsilon\to 0}\frac{Q(x_1)e^{x_1\lambda}}{x_2 Q(x_2)e^{x_2\lambda}}\right)^{-1} = \left(\lim_{\epsilon\to 0}\frac{1}{x_2}\right)^{-1} = 1.$$

5) According to equations (12) and (13), the first derivative of $u(\alpha)$ is

$$\frac{\partial u(\alpha)}{\partial \alpha} = l(\alpha) + \alpha\frac{\partial l(\alpha)}{\partial \alpha} - \frac{\mathbb{E}\{xe^{\lambda x}\}}{\mathbb{E}\{e^{\lambda x}\}}\frac{\partial l(\alpha)}{\partial \alpha} = l(\alpha).$$

Appendix B
Proof of Lemma II

Based on the definition of probability density function, we have

$$\begin{aligned}
& \lim_{L\to\infty} -\frac{1}{L}\log\left(f_y(\alpha)\right) \\
= \ & \lim_{L\to\infty} -\frac{1}{L}\log\left(\lim_{\delta\to\ 0}\frac{\mathbb{P}\{y>\alpha\}-\mathbb{P}\{y>\alpha+\delta\}}{\delta}\right) \\
\overset{(a)}{=} \ & \lim_{\delta\to\ 0}\lim_{L\to\infty} -\frac{1}{L}\log\left(\frac{\mathbb{P}\{y>\alpha\}-\mathbb{P}\{y>\alpha+\delta\}}{\delta}\right) \\
\geq \ & \lim_{\delta\to\ 0}\lim_{L\to\infty} \frac{1}{L}\left(-\log\left(\mathbb{P}\{y>\alpha\}\right)+\log\delta\right) \\
\overset{(b)}{=} \ & u(\alpha)
\end{aligned} \tag{33}$$

where (a) is valid since $\log$ is a continuous function, and both limitations do exist and are interchangeable. (b) follows from equation (14). The exponent of $f_y(\alpha)$ can be upper-bounded as

$$\begin{aligned}
& \lim_{L\to\infty} -\frac{1}{L}\log\left(f_y(\alpha)\right) \\
\overset{(a)}{=} & \lim_{\delta\to\ 0}\lim_{L\to\infty} \frac{-\log\left(\mathbb{P}\{y>\alpha\}-\mathbb{P}\{y>\alpha+\delta\}\right)+\log\delta}{L} \\
\overset{(b)}{\leq} & \lim_{\delta\to\ 0}\lim_{L\to\infty} \frac{-\log\left(e^{-L(u(\alpha)+\epsilon)}-e^{-L(u(\alpha+\delta)-\epsilon)}\right)+\log\delta}{L} \\
= & \lim_{\delta\to\ 0}\lim_{L\to\infty} u(\alpha)+\epsilon-\frac{\log\left(1-e^{-L\chi}\right)}{L} \\
\overset{(c)}{=} & u(\alpha)+\epsilon
\end{aligned} \tag{34}$$

where $\chi = u(\alpha+\delta)-u(\alpha)-2\epsilon$. Since $u(\alpha)$ is a strictly increasing function (Lemma I), we can make χ positive by choosing ϵ small enough. (a) is valid since $\log$ is a continuous function, and both limits do exist and are interchangeable. (b) follows from the definition of limit if L is sufficiently large, and (c) is a result of χ being positive. Selecting ϵ arbitrarily small, results (33) and (34) prove the lemma.

Appendix C
Proof of Lemma III

According to the definition of infimum, we have

$$\begin{aligned} & \lim_{L\to\infty} -\frac{\log(H(L))}{L} \\ \geq & \lim_{L\to\infty} -\frac{1}{L}\log\left(e^{-L\inf_{\mathcal{S}} h(\mathbf{x})}\int_{\mathcal{S}} d\mathbf{x}\right) \\ \overset{(a)}{=} & \inf_{\mathcal{S}} h(\mathbf{x}). \end{aligned} \tag{35}$$

where (a) follows from the fact that $\mathcal{S}$ is a bounded region. Since $h(\mathbf{x})$ is a continuous function, it has a minimum in the bounded closed set $cl(\mathcal{S})$ which is denoted by $\mathbf{x}^\star$. Due to the continuity of $h(\mathbf{x})$ at $\mathbf{x}^\star$, for any $\epsilon > 0$, there is a neighborhood $\mathcal{B}(\epsilon)$ centered at $\mathbf{x}^\star$ such that any $\mathbf{x} \in \mathcal{B}(\epsilon)$ has the property of $|h(\mathbf{x}) - h(\mathbf{x}^\star)| < \epsilon$. Moreover, since $\mathcal{S}$ is a convex set, we have $\mathrm{vol}\left(\mathcal{B}(\epsilon)\cap\mathcal{S}\right) > 0$. Now, we can write

$$\begin{aligned} & \lim_{L\to\infty} -\frac{\log(H(L))}{L} \\ \leq & \lim_{L\to\infty} -\frac{1}{L}\log\left(\int_{\mathcal{S}\cap\mathcal{B}(\epsilon)} e^{-Lh(\mathbf{x})}d\mathbf{x}\right) \\ \leq & \lim_{L\to\infty} -\frac{1}{L}\log\left(e^{-L(h(\mathbf{x}^\star)+\epsilon)}\int_{\mathcal{S}\cap\mathcal{B}(\epsilon)} d\mathbf{x}\right) \\ = & h(\mathbf{x}^\star) + \epsilon. \end{aligned} \tag{36}$$

Selecting ϵ to be arbitrarily small, (35) and (36) prove the lemma.

Appendix D
Proof of Lemma IV

According to Lemma I, $u_j(x)$ is increasing and convex for $\forall 1 \leq j \leq J$. Thus, the objective function $f(\boldsymbol{\beta}) = \sum_{j=1}^{J}\gamma_j u_j(\frac{\beta_j}{\eta_j})$ is also convex, and the region $\mathcal{S}_T$ is determined by J convex inequality constraints and one affine equality constraint. Hence, in this case, KKT conditions are both necessary and sufficient for optimality [61]. In other words, if there exist constants ϕ_j and ν such that

$$\frac{\gamma_j}{\eta_j} l_j(\frac{\beta_j^\star}{\eta_j}) - \phi_j - \nu = 0 \quad \forall 1 \leq j \leq J \tag{37}$$

$$\phi_j\left[\eta\mathbb{E}\{x_j\} - \beta_j^\star\right] = 0 \quad \forall 1 \leq j \leq J \tag{38}$$

then the point $\boldsymbol{\beta}^\star$ is a global minimum.

Now, we prove that either $\beta_j^\star = \eta_j\mathbb{E}\{x_j\}$ for all $1 \leq j \leq J$, or $\beta_j^\star > \eta_j\mathbb{E}\{x_j\}$ for all $1 \leq j \leq J$. Let us assume the opposite is true, and there are at least two elements of the vector $\boldsymbol{\beta}^\star$, indexed with k and

m, which have the values of $\beta_k^\star = \eta_k \mathbb{E}\{x_k\}$ and $\beta_m^\star > \eta_m \mathbb{E}\{x_m\}$, respectively. For any arbitrary $\epsilon > 0$, the vector $\boldsymbol{\beta}^{\star\star}$ can be defined as below

$$\beta_j^{\star\star} = \begin{cases} \beta_j^\star + \epsilon & \text{if } j = k \\ \beta_j^\star - \epsilon & \text{if } j = m \\ \beta_j^\star & \text{otherwise.} \end{cases} \tag{39}$$

Then, we have

$$\begin{aligned} & \lim_{\epsilon \to 0} \frac{f(\boldsymbol{\beta}^{\star\star}) - f(\boldsymbol{\beta}^\star)}{\epsilon} \\ = & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \gamma_k u_k \left(\frac{\beta_k^\star + \epsilon}{\eta_k} \right) + \gamma_m u_m \left(\frac{\beta_m^\star - \epsilon}{\eta_m} \right) \right. \\ & \qquad \left. - \gamma_m u_m \left(\frac{\beta_m^\star}{\eta_k} \right) \right\} \\ \overset{(a)}{=} & \lim_{\epsilon \to 0} \frac{\gamma_k}{\eta_k} l_k \left(\frac{\beta_k^\star + \epsilon'}{\eta_k} \right) - \frac{\gamma_m}{\eta_m} l_m \left(\frac{\beta_m^\star + \epsilon''}{\eta_m} \right) \\ = & - \frac{\gamma_m}{\eta_m} l_m \left(\frac{\beta_m^\star}{\eta_m} \right) < 0 \end{aligned} \tag{40}$$

where $\epsilon', \epsilon'' \in [0, \epsilon]$, and (a) follows from the Taylor's theorem. Thus, moving from $\boldsymbol{\beta}^\star$ to $\boldsymbol{\beta}^{\star\star}$ decreases the function which contradicts the assumption of $\boldsymbol{\beta}^\star$ being the global minimum.

Out of the remaining possibilities, the case where $\beta_j^\star = \eta_j \mathbb{E}\{x_j\}$ $(\forall 1 \leq j \leq J)$ obviously agrees with Lemma IV for the special case of $\nu = 0$. Therefore, the lemma can be proved assuming $\beta_j^\star > \eta_j \mathbb{E}\{x_j\}$ $(\forall 1 \leq j \leq J)$. Then, equation (38) turns into $\phi_j = 0$ $(\forall 1 \leq j \leq J)$. By rearranging equation (37) and using the condition $\sum_{j=1}^{J} \beta_j = \alpha$, Lemma IV is proved.

APPENDIX E
PROOF OF THEOREM II

Sketch of the proof: First, it is proved that $\eta_j^\star > 0$ if $\mathbb{E}\{x_j\} < \alpha$. At the second step, we prove that $\eta_j^\star = 0$, if $\mathbb{E}\{x_j\} \geq \alpha$. Then, KKT conditions [61] are applied for the indices $1 \leq k \leq J$ where $\mathbb{E}\{x_k\} < \alpha$ to find the maximizing allocation vector, $\boldsymbol{\eta}^\star$.

Proof: The parameter ν is obviously a function of the vector $\boldsymbol{\eta}$. Differentiating equation (17) with respect to η_k results in

$$\frac{\partial \nu}{\partial \eta_k} = - \frac{v_k \left(\frac{\nu \eta_k}{\gamma_k} \right) + \frac{\nu \eta_k}{\gamma_k} v_k' \left(\frac{\nu \eta_k}{\gamma_k} \right)}{\sum_{j=1}^{J} \frac{\eta_j^2}{\gamma_j} v_j' \left(\frac{\nu \eta_j}{\gamma_j} \right)} \tag{41}$$

where $v_j(x) = l_j^{-1}(x)$, and $v'_j(x)$ denotes its derivative with respect to its argument. The objective function can be simplified as

$$g(\boldsymbol{\eta}) = \sum_{j=1}^{J} \gamma_j u_j(\frac{\beta_j^\star}{\eta_j}) = \sum_{j=1}^{J} \gamma_j u_j \left(v_j(\frac{\nu \eta_j}{\gamma_j}) \right). \tag{42}$$

$\nu^\star$ is defined as the value of ν corresponding to $\boldsymbol{\eta}^\star$. Next, we show that $\nu^\star > 0$. Let us assume the opposite is true, i.e., $\nu^\star \leq 0$. Then, according to Lemma I, we have $v_j(\frac{\nu^\star \eta_j}{\gamma_j}) \leq \mathbb{E}\{x_j\}$ for all j which results in $g(\boldsymbol{\eta}^\star) = 0$. However, it is possible to achieve a positive value of $g(\boldsymbol{\eta})$ by setting $\eta_j = 1$ for the one vector which has the property of $\mathbb{E}\{x_j\} < \alpha$, and setting $\eta_j = 0$ for the rest. Thus, $\boldsymbol{\eta}^\star$ can not be the maximal point. This contradiction proves the fact that $\nu^\star > 0$.

At the first step, we prove that $\eta_j^\star > 0$ if $\mathbb{E}\{x_j\} < \alpha$. Assume the opposite is true for an index $1 \leq k \leq J$. Since $\sum_{j=1}^{J} \eta_j^\star = 1$, there should be at least one index m such that $\eta_m^\star > 0$. For any arbitrary $\epsilon > 0$, the vector $\boldsymbol{\eta}^{\star\star}$ can be defined as below

$$\eta_j^{\star\star} = \begin{cases} \epsilon & \text{if } j = k \\ \eta_j^\star - \epsilon & \text{if } j = m \\ \eta_j^\star & \text{otherwise.} \end{cases} \tag{43}$$

$\nu^{\star\star}$ is defined as the corresponding value of ν for the vector $\boldsymbol{\eta}^{\star\star}$. Based on equation (41), we can write

$$\begin{aligned} &\Delta\nu = \\ &\nu^{\star\star} - \nu^\star = \\ &\frac{v_m\left(\frac{\nu^\star \eta_m^\star}{\gamma_m}\right) + \frac{\nu^\star \eta_m^\star}{\gamma_m} v'_m\left(\frac{\nu^\star \eta_m^\star}{\gamma_m}\right) - \mathbb{E}\{x_k\}}{\sum_{j=1}^{J} \frac{\eta_j^{\star 2}}{\gamma_j} v'_j\left(\frac{\nu^\star \eta_j^\star}{\gamma_j}\right)} \epsilon + O(\epsilon^2). \end{aligned} \tag{44}$$

Then, we have

$$\begin{aligned} &\lim_{\epsilon \to 0} \frac{g(\boldsymbol{\eta}^{\star\star}) - g(\boldsymbol{\eta}^\star)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \frac{\nu^{\star 2} \eta_k^\star}{\gamma_k} v'_k\left(\frac{\nu^\star \eta_k^\star}{\gamma_k}\right) \epsilon - \frac{\nu^{\star 2} \eta_m^\star}{\gamma_m} v'_m\left(\frac{\nu^\star \eta_m^\star}{\gamma_m}\right) \epsilon \right. \\ &\qquad \left. + \nu^\star \Delta\nu \sum_{j=1}^{J} \frac{\eta_j^{\star 2}}{\gamma_j} v'_j\left(\frac{\nu^\star \eta_j^\star}{\gamma_j}\right) + O(\epsilon^2) \right\} \\ &\stackrel{(a)}{=} \nu^\star \left\{ v_m\left(\frac{\nu^\star \eta_m^\star}{\gamma_m}\right) - \mathbb{E}\{x_k\} \right\} \end{aligned} \tag{45}$$

where (a) follows from (44). If the value of (45) is positive for an index m, moving in that direction increases the objective function which contradicts with the assumption of $\boldsymbol{\eta}^\star$ being a maximal point. If the value of (45) is non-positive for all indexes m whose $\eta_m^\star > 0$, we can write

$$\mathbb{E}\{x_k\} \geq \sum_{m=1}^{J} \eta_m^\star v_m\left(\frac{\nu^\star \eta_m^\star}{\gamma_m}\right) = \alpha \tag{46}$$

which obviously contradicts the assumption of $\mathbb{E}\{x_k\} < \alpha$.

At the second step, we prove that $\eta_j^\star = 0$ if $\mathbb{E}\{x_j\} \geq \alpha$. Assume the opposite is true for an index $1 \leq r \leq J$. Since $\sum_{j=1}^{J} \eta_j^\star = 1$, we should have $\eta_s^\star < 1$ for all other indices s. For any arbitrary $\epsilon > 0$, the vector $\boldsymbol{\eta}^{\star\star\star}$ can be defined as

$$\eta_j^{\star\star\star} = \begin{cases} \eta_j^\star - \epsilon & \text{if } j = r \\ \eta_j^\star + \epsilon & \text{if } j = s \\ \eta_j^\star & \text{otherwise.} \end{cases} \tag{47}$$

$\nu^{\star\star\star}$ is defined as the corresponding value of ν for the vector $\boldsymbol{\eta}^{\star\star\star}$. Based on equation (41), we can write

$$\begin{aligned} \Delta\nu &= \nu^{\star\star\star} - \nu^\star \\ &= \frac{\epsilon}{\sum_{j=1}^{J} \frac{\eta_j^{\star 2}}{\gamma_j} v_j'\left(\frac{\nu^\star \eta_j^\star}{\gamma_j}\right)} \left\{ v_r\left(\frac{\nu^\star \eta_r^\star}{\gamma_r}\right) + \frac{\nu^\star \eta_r^\star}{\gamma_r} v_r'\left(\frac{\nu^\star \eta_r^\star}{\gamma_r}\right) \right. \\ &\quad \left. - v_s\left(\frac{\nu^\star \eta_s^\star}{\gamma_s}\right) - \frac{\nu^\star \eta_s^\star}{\gamma_s} v_s'\left(\frac{\nu^\star \eta_s^\star}{\gamma_s}\right) \right\} + O(\epsilon^2). \end{aligned} \tag{48}$$

Then, we have

$$\begin{aligned} & \lim_{\epsilon \to 0} \frac{g(\boldsymbol{\eta}^{\star\star\star}) - g(\boldsymbol{\eta}^\star)}{\epsilon} \\ = \; & \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \frac{\nu^{\star 2} \eta_s^\star}{\gamma_s} v_s'\left(\frac{\nu^\star \eta_s^\star}{\gamma_s}\right) \epsilon - \frac{\nu^{\star 2} \eta_r^\star}{\gamma_r} v_r'\left(\frac{\nu^\star \eta_r^\star}{\gamma_r}\right) \epsilon \right. \\ & \left. + \nu^\star \Delta\nu \sum_{j=1}^{J} \frac{\eta_j^{\star 2}}{\gamma_j} v_j'\left(\frac{\nu^\star \eta_j^\star}{\gamma_j}\right) + O(\epsilon^2) \right\} \\ \stackrel{(a)}{=} \; & \nu^\star \left\{ v_r\left(\frac{\nu^\star \eta_r^\star}{\gamma_r}\right) - v_s\left(\frac{\nu^\star \eta_s^\star}{\gamma_s}\right) \right\} \end{aligned} \tag{49}$$

where (a) follows from (48). If the value of (49) is positive for an index s, moving in that direction increases the objective function which contradicts with the assumption of $\boldsymbol{\eta}^\star$ being a maximal point . If the value of (49) is non-positive for all indices s whose $\eta_s^\star > 0$, we can write

$$\mathbb{E}\{x_r\} < v_r\left(\frac{\nu^\star \eta_r^\star}{\gamma_r}\right) \leq \sum_{s=1}^{J} \eta_s^\star v_s\left(\frac{\nu^\star \eta_s^\star}{\gamma_s}\right) = \alpha \tag{50}$$

which obviously contradicts the assumption of $\mathbb{E}\{x_r\} \geq \alpha$.

Now that the boundary points are checked, we can safely use the KKT conditions [61] for all $1 \leq k \leq J$, where $\mathbb{E}\{x_k\} < \alpha$, to find the maximizing allocation vector, $\boldsymbol{\eta}^\star$.

$$\begin{aligned} \zeta &= \frac{\nu^{\star 2} \eta_k^\star}{\gamma_k} v_k'\left(\frac{\nu^\star \eta_k^\star}{\gamma_k}\right) + \nu^\star \sum_{j=1}^{J} \frac{\eta_j^{\star 2}}{\gamma_j} v_j'\left(\frac{\nu^\star \eta_j^{\star 2}}{\gamma_j}\right) \frac{\partial \nu}{\partial \eta_k}\Big|_{\nu=\nu^\star} \\ &\stackrel{(a)}{=} -\nu^\star v_k\left(\frac{\nu^\star \eta_k^\star}{\gamma_k}\right) \end{aligned} \tag{51}$$

where ζ is a constant independent of k, and (a) follows from (41). Using the fact that $\sum_{j=1}^{J}\eta_j = 1$ together with equations (17) and (51) results in

$$\begin{aligned}\zeta &= -\alpha\nu^\star\\ \nu^\star &= \sum_{\mathbb{E}\{x_j\}<\alpha} \gamma_j l_j(\alpha).\end{aligned} \tag{52}$$

Combining equations (51) and (52) results in equation (19) and $g(\boldsymbol{\eta}^\star) = \sum_{j=1}^{J}\gamma_j u_j(\alpha)$.

Appendix F
Proof of Remark V

Based on the arguments similar to the ones in appendix E, it can be shown that $\tilde{\eta}_j^\star = 0$ *iff* $\mathbb{E}\{x_j\} \geq \alpha$. Since all the types are identical here, this means $\tilde{\eta}_j^\star > 0$ for all j. Similar to equation (51), applying KKT conditions [61], gives us

$$v_j\left(\frac{\tilde{\nu}^\star\tilde{\eta}_j^\star}{\gamma_j}\right) = \begin{cases} -\zeta & \text{if } \tilde{\eta}_j^\star < \dfrac{\gamma_j W_j T}{n_0} \\ \\ -\zeta - \sigma_j & \text{if } \tilde{\eta}_j^\star = \dfrac{\gamma_j W_j T}{n_0} \end{cases} \tag{53}$$

where σ_j's are non-negative parameters [61]. Putting $\Upsilon = \frac{l_j(-\zeta)}{\tilde{\nu}^\star}$ proves equation (20).

Appendix G
Discrete Analysis of One Path

$Q(n,k,l)$ is defined as the probability of having exactly k errors out of the n packets sent over the path l. Depending on the initial state of the path l, $P_g(n,k,l)$ and $P_b(n,k,l)$ are defined as the probabilities of having k errors out of the n packets sent over this path when we start the transmission in the good or in the bad state, respectively. It is easy to see that

$$Q(n,k,l) = \pi_g P_g(n,k,l) + \pi_b P_b(n,k,l). \tag{54}$$

$P_g(n,k,l)$ and $P_b(n,k,l)$ can be computed from the following recursive equations

$$\begin{aligned}P_b(n,k,l) &= \pi_{b|b}P_b(n-1,k-1,l) + \pi_{g|b}P_g(n-1,k-1,l)\\ P_g(n,k,l) &= \pi_{b|g}P_b(n-1,k,l) + \pi_{g|g}P_g(n-1,k,l)\end{aligned} \tag{55}$$

with the initial conditions

$$\begin{aligned}P_g(n,k,l) &= 0 \qquad \text{for } k \geq n\\ P_b(n,k,l) &= 0 \qquad \text{for } k > n\\ P_g(n,k,l) &= 0 \qquad \text{for } k < 0\\ P_b(n,k,l) &= 0 \qquad \text{for } k \leq 0\end{aligned} \tag{56}$$

where $\pi_{s_2|s_1}$ is the probability of the channel being in the state $s_2 \in \{g, b\}$ provided that it has been in the state $s_1 \in \{g, b\}$ when the last packet was transmitted. $\pi_{s_2|s_1}$ has the following values for different combinations of s_1 and s_2 [1]

$$\begin{aligned}
\pi_{g|g} &= \pi_g + \pi_b \, e^{-\dfrac{\mu_g + \mu_b}{S_l}} \\
\pi_{b|g} &= 1 - \pi_{g|g} \\
\pi_{b|b} &= \pi_b + \pi_g \, e^{-\dfrac{\mu_g + \mu_b}{S_l}} \\
\pi_{g|b} &= 1 - \pi_{b|b}
\end{aligned} \tag{57}$$

where S_l denotes the transmission rate on the path l, i.e., the packets are transmitted on the path l every $\frac{1}{S_l}$ seconds.

According to the recursive equations in (55), to compute $P_b(n, k, l)$ and $P_g(n, k, l)$ by memoization technique, the functions $P_b()$ and $P_g()$ should be calculated at the following set of points denoted as $\mathcal{S}(n, k)$

$$\mathcal{S}(n, k) = \{(n', k') \mid 0 \le k' \le k, \; n' - n + k \le k' \le n'\} .$$

Cardinality of the set $\mathcal{S}(n, k)$ is of the order $|\mathcal{S}(n, k)| = O\left(k\left(n - k\right)\right)$. Since three operations are needed to compute the recursive functions $P_b()$ and $P_g()$ at each point, $P_b(n, k, l)$ and $P_g(n, k, l)$ are computable with the complexity of $O\left(k\left(n - k\right)\right)$ which give us $Q(n, k, l)$ according to equation (54).

Appendix H
Discrete Analysis of One Type

When there are n packets to be distributed over L_j identical paths of type j, uniform distribution is obviously the optimum. However, since the integer n may be indivisible by L_j, the L_j dimensional vector $\mathbf{N}$ is selected as

$$N_l = \begin{cases} \lfloor \dfrac{n}{L_j} \rfloor + 1 & \text{for } 1 \le l \le \text{Rem}(n, L_j) \\ \\ \lfloor \dfrac{n}{L_j} \rfloor & \text{for Rem}(n, L_j) < l \le L_j \end{cases} \tag{58}$$

where $\text{Rem}(a, b)$ denotes the remainder of dividing a by b. $\mathbf{N}$ represents the closest integer vector to a uniform distribution.

$E^{\mathbf{N}}(k, l)$ is defined as the probability of having exactly k erasures among the n packets transmitted over the identical paths 1 to l with the allocation vector $\mathbf{N}$. According to the definitions of $Q_j(n, k)$ and

$E^{\mathbf{N}}(k,l)$, it is obvious that $Q_j(n,k) = E^{\mathbf{N}}(k,L_j)$. $E^{\mathbf{N}}(k,l)$ can be computed recursively as

$$\begin{aligned} E^{\mathbf{N}}(k,l) &= \sum_{i=0}^{k} E^{\mathbf{N}}(k-i,l-1)Q(N_l,i,l) \\ E^{\mathbf{N}}(k,1) &= Q(N_1,k,1) \end{aligned} \tag{59}$$

where $Q(N_l,i,l)$ is given in appendix G. Since all the paths are assumed to be identical here, $Q(N_l,k,l)$ is the same for all path indices, l. According to the recursive equations in (55), the values of $Q(N_l,i,l)$ for all $0 \leq i \leq k$ and $1 \leq l \leq L_j$ can be calculated with the complexity of $O(N_l k) = O\left(\frac{n}{L_j}k\right)$. According to the recursive equations in (59), computing $E^{\mathbf{N}}(k,l)$ requires memoization over an array of size $O(kl)$ whose entries can be calculated with $O(k)$ operations each. Thus, $E^{\mathbf{N}}(k,l)$ is computable with the complexity of $O(k^2 l)$ if $Q(N_l,i,l)$'s are already given. Finally, noting that $Q_j(n,k) = E^{\mathbf{N}}(k,L_j)$, we can compute $Q_j(n,k)$ with the overall complexity of $O(k^2 L_j) + O\left(\frac{n}{L_j}k\right)$.

APPENDIX I
PROOF OF LEMMA V

The lemma is proved by induction on j. The case of $j = 1$ is obviously true as $\hat{P}_e(n,k,1) = P_e^{opt}(n,k,1)$. Let us assume this statement is true for $j = 1$ to $J-1$. Then, for $j = J$, we have

$$\begin{aligned} & \hat{P}_e(n,k,J) \\ \overset{(a)}{\leq} & \sum_{i=0}^{N_J} Q_J(N_J^{opt},i)\hat{P}_e(n-N_J^{opt},k-i,J-1) \\ \overset{(b)}{\leq} & \sum_{i=0}^{N_J} Q_J(N_J^{opt},i)P_e^{opt}(n-N_J^{opt},k-i,J-1) \\ \overset{(c)}{\leq} & \sum_{i=0}^{N_J} Q_J(N_J^{opt},i)P_e^{\mathbf{N}^{opt}}(k-i,J-1) \\ \overset{(d)}{=} & P_e^{\mathbf{N}^{opt}}(k,J) = P_e^{opt}(n,k,J) \end{aligned}$$

where $\mathbf{N}^{opt}$ denotes the optimum allocation of n packets among the J types of paths such that the probability of having more than k lost packets is minimized. (a) follows from the recursive equation (21), and (b) is the induction assumption. (c) comes from the definition of $P_e^{opt}(n,k,l)$, and (d) is a result of equation (23).

APPENDIX J
PROOF OF THEOREM III

Sketch of the proof: First, the asymptotic behavior of $Q_j(n,k)$ is analyzed, and it is shown that for large values of L_j (or equivalently L), equation (63) computes the exponent of $Q_j(n,k)$ versus L. Next, we prove the first part of the theorem by induction on J. The proof of this part is divided to two different cases, depending on whether $\frac{K}{N}$ is larger than $\mathbb{E}\{x_J\}$ or vice versa. Finally, the second and the third parts of the theorem are proved by induction on j while the total number of path types, J, is fixed. Again, the proof is divided into two different cases, depending on whether $\frac{K}{N}$ is larger than $\mathbb{E}\{x_j\}$ or vice versa.

Proof: First, we compute the asymptotic behavior of $Q_j(n,k)$ for $k > n\mathbb{E}\{x_j\}$, and n growing proportionally to L_j, i.e. $n = n'L_j$. Here, we can apply Sanov's Theorem [56], [62] as n and k are discrete variables and n' is a constant.

Sanov's Theorem. Let $X_1, X_2, \dots, X_n$ be i.i.d. discrete random variables from an alphabet set $\mathcal{X}$ with the size $|\mathcal{X}|$ and probability mass function (pmf) $Q(x)$. Let $\mathcal{P}$ denote the set of pmf's in $\mathbb{R}^{|\mathcal{X}|}$, i.e. $\mathcal{P} = \left\{\mathbf{P} \in \mathbb{R}^{|\mathcal{X}|} \middle| \; P(i) \geq 0, \;\; \sum_{i=1}^{|\mathcal{X}|} P(i) = 1\right\}$. Also, let $\mathcal{P}_L$ denote the subset of $\mathcal{P}$ corresponding to all possible empirical distributions of $\mathcal{X}$ in L observations [62], i.e. $\mathcal{P}_L = \{\mathbf{P} \in \mathcal{P} | \; \forall i, \; LP(i) \in \mathbb{Z}\}$. For any dense and closed set [57] of pmf's $E \subseteq \mathcal{P}$, the probability that the empirical distribution of L observations belongs to the set E is equal to

$$\mathbb{P}\left\{E\right\} = \mathbb{P}\left\{E \cap \mathcal{P}_L\right\} \doteq e^{-LD(\mathbf{P}^\star || \mathbf{Q})} \tag{60}$$

where $\mathbf{P}^\star = \underset{\mathbf{P}\in E}{\operatorname{argmin}} \, D(\mathbf{P}||\mathbf{Q})$ and $D(\mathbf{P}||\mathbf{Q}) = \sum_{i=1}^{|\mathcal{X}|} P(i) \log \frac{P(i)}{Q(i)}$.

Focusing our attention on the main problem, assume that $\mathbf{P}$ is defined as the empirical distribution of the number of errors in each path, i.e. for $\forall i, \; 1 \leq i \leq n', \; P(i)$ shows the ratio of the total paths which contain exactly i lost packets. Similarly, for $\forall i, \; 1 \leq i \leq n', \; Q(i)$ denotes the probability of exactly i packets being lost out of the n' packets transmitted on a path of type j. The sets E and E_{out} are defined as follows

$$\begin{aligned} E &= \{\mathbf{P} \in \mathcal{P} | \sum_{i=0}^{n'} iP(i) \geq \beta\} \\ E_{out} &= \{\mathbf{P} \in \mathcal{P} | \sum_{i=0}^{n'} iP(i) = \beta\} \end{aligned} \tag{61}$$

where $\beta = \dfrac{k}{n}$. Noting E and E_{out} are dense sets, we can compute $Q_j(n,k)$ as

$$Q_j(n,k) \overset{(a)}{=} \mathbb{P}\left\{E_{out}\right\} \overset{(b)}{\doteq} e^{-L_j \underset{\mathbf{P}\in E_{out}}{\min} D\left(\mathbf{P}||\mathbf{Q}\right)} \tag{62}$$

where (a) follows from the definition of $Q_j(n,k)$ as the probability of having exactly k errors out of the n packets sent over the paths of type j given in section V, and (b) results from Sanov's Theorem.

Knowing the fact that the Kullback Leibler distance, $D(\mathbf{P}||\mathbf{Q})$, is a convex function of $\mathbf{P}$ and $\mathbf{Q}$ [63], we conclude that its minimum over the convex set E either lies on an interior point which is a global minimum of the function over the whole set $\mathcal{P}$ or is located on the boundary of E. However, we know that the global minimum of Kullback Leibler distance occurs at $\mathbf{P} = \mathbf{Q} \notin E$. Thus, the minimum of $D(\mathbf{P}||\mathbf{Q})$ is located on the boundary of E. This results in

$$\begin{aligned} Q_j(n,k) &\overset{(a)}{\doteq} e^{-L_j \min_{\mathbf{P}\in E_{out}} D(\mathbf{P}||\mathbf{Q})} \\ &= e^{-L_j \min_{\mathbf{P}\in E} D(\mathbf{P}||\mathbf{Q})} \overset{(b)}{\doteq} e^{-\gamma_j L u_j(\frac{k}{n})} \end{aligned} \tag{63}$$

where (a) and (b) follow from equations (62) and (14), respectively.

1) We prove the first part of the theorem by induction on J. When $J = 1$, the statement is correct for both cases of $\frac{K}{N} > \mathbb{E}\{x_1\}$ and $\frac{K}{N} \leq \mathbb{E}\{x_1\}$, recalling the fact that $\hat{P}_e(n,k,1) = P_e^{opt}(n,k,1)$ and $u_1(x) = 0$ for $x \leq \mathbb{E}\{x_1\}$. Now, let us assume the first part of the theorem is true for $j = 1$ to $J-1$. We prove the same statement for J as well. The proof can be divided into two different cases, depending on whether $\frac{K}{N}$ is larger than $\mathbb{E}\{x_J\}$ or vice versa.

1.1) $\frac{K}{N} > \mathbb{E}\{x_J\}$

According to the definition, the value of $\hat{P}_e(N,K,J)$ is computed by minimizing $\sum_{i=0}^{n_J} Q_J(n_J,i)\hat{P}_e(N-n_J, K-i, J-1)$ over n_J (see equation (23)). Now, we show that for any value of n_J, the corresponding term in the minimization is asymptotically at least equal to $P_e^{opt}(N,K,J)$. n_J can take integer values in the range $0 \leq n_J \leq N$. We split this range into three non-overlapping intervals of $0 \leq n_J \leq \epsilon L$, $\epsilon L \leq n_J \leq N(1-\epsilon)$, and $N(1-\epsilon) < n_J \leq N$ for any arbitrary constant $\epsilon \leq \min\left\{\gamma_j, 1-\frac{K}{N}\right\}$. The reason is that equation (63) is valid in the second interval only, and we need separate analyses for the first and last intervals.

First, we show the statement for $\epsilon L \leq n_J \leq N(1-\epsilon)$. Defining $i_J = \lfloor n_J \frac{K}{N} \rfloor$, we have

$$\begin{aligned} \frac{i_J}{n_J} &= \frac{K}{N} + O(\frac{1}{L}), \\ \frac{K-i_J}{N-n_J} &= \frac{K}{N} + O(\frac{1}{L}) \end{aligned} \tag{64}$$

as ϵ is constant, and $K = O(L)$, $N = O(L)$. Hence, we have

$$\begin{aligned} &\sum_{i=0}^{n_J} Q_J(n_J,i)\hat{P}_e(N-n_J, K-i, J-1) \\ \geq\ & Q_J(n_J,i_J)\hat{P}_e(N-n_J, K-i_J, J-1) \\ \overset{(a)}{\doteq}\ & e^{-L\sum_{j=1}^{J}\gamma_j u_j\left(\frac{K}{N}+O\left(\frac{1}{L}\right)\right)} \\ \overset{(b)}{\doteq}\ & e^{-L\sum_{j=1}^{J}\gamma_j u_j\left(\frac{K}{N}\right)} \end{aligned} \tag{65}$$

where (a) follows from (63) and the induction assumption, and (b) follows from the fact that $u_j()$'s are differentiable functions according to Lemma I in subsection IV-B.

For $0 \leq n_J \leq \epsilon L$, since $\epsilon < \gamma_j$, the number of packets assigned to the paths of type J is less than the number of such paths. Thus, one packet is allocated to n_J of the paths, and the rest of the paths of type J are not used. Defining $\pi_{b,J}$ as the probability of a path of type J being in the bad state, we can write

$$Q_J(n_J, n_J) = \pi_{b,J}^{n_J} = e^{-n_J \log\left(\frac{1}{\pi_{b,J}}\right)}. \tag{66}$$

Therefore, for $0 \leq n_J \leq \epsilon L$, we have

$$\begin{aligned} & \sum_{i=0}^{n_J} Q_J(n_J, i)\hat{P}_e(N-n_J, K-i, J-1) \\ \geq \quad & Q_J(n_J, n_J)\hat{P}_e(N-n_J, K-n_J, J-1) \\ \doteq \quad & e^{-L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K-n_J}{N-n_J}\right) - n_J \log\left(\frac{1}{\pi_{b,J}}\right)} \\ \overset{(a)}{\geq} \quad & e^{-L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K}{N}\right) - L\epsilon \log\left(\frac{1}{\pi_{b,J}}\right)} \\ \overset{(b)}{\doteq} \quad & e^{-L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K}{N}\right)} \geq e^{-L\sum_{j=1}^{J}\gamma_j u_j\left(\frac{K}{N}\right)} \end{aligned} \tag{67}$$

where (a) follows from the fact that $\frac{K-n_J}{N-n_J} \leq \frac{K}{N}$, and (b) results from the fact that we can select ϵ arbitrarily small.

Finally, we prove the statement for the case $n_J > N(1-\epsilon)$. In this case, we have

$$\begin{aligned} & \sum_{i=0}^{n_J} Q_J(n_J, i)\hat{P}_e(N-n_J, K-i, J-1) \\ \geq \quad & Q_J(n_J, K)\hat{P}_e(N-n_J, 0, J-1) \\ \overset{(a)}{\dot{\geq}} \quad & e^{-L\gamma_J u_J\left(\frac{K}{N(1-\epsilon)}\right)} \\ \overset{(b)}{\dot{\geq}} \quad & e^{-L\sum_{j=1}^{J}\gamma_j u_j\left(\frac{K}{N}\right)} \end{aligned} \tag{68}$$

where (a) follows from the fact that $\epsilon < 1 - \frac{K}{N}$ and $\hat{P}_e(n, 0, j) = 1$, for all n and j. Setting ϵ small enough results in (b).

Inequalities (65), (67), and (68) result in

$$\hat{P}_e(N, K, J) \quad \dot{\geq} \quad e^{-L\sum_{j=1}^{J}\gamma_j u_j(\alpha)} \tag{69}$$

Combining (69) with Lemma V proves the first part of Theorem III for the case when $\frac{K}{N} > \mathbb{E}\{x_J\}$.

1.2) $\frac{K}{N} \leq \mathbb{E}\{x_J\}$

Similar to the case of $\frac{K}{N} > \mathbb{E}\{x_J\}$ in subsection 1.1, we show that for any value of $0 \leq n_J \leq N$, the corresponding term of the minimization in equation (23) is asymptotically at least equal to $P_e^{opt}(N, K, J)$. Again, the range of n_J is partitioned into three non-overlapping intervals.

For any arbitrary $0 < \epsilon < \min\left\{\gamma_J, 1 - \frac{K}{N}, \frac{1}{K}\right\}$, and for all n_J in the range of $\epsilon L < n_J \leq N(1-\epsilon)$, we define i_J as $i_J = \lceil n_J \mathbb{E}\{x_J\} \rceil$. We have

$$\begin{aligned} \frac{i_J}{n_J} &= \mathbb{E}\{x_J\} + O\left(\frac{1}{L}\right) \geq \mathbb{E}\{x_J\} \\ \frac{K - i_J}{N - n_J} &< \frac{K}{N} + O\left(\frac{1}{L}\right) \end{aligned} \tag{70}$$

Hence,

$$\begin{aligned} & \sum_{i=0}^{n_J} Q_J(n_J, i) \hat{P}_e(N - n_J, K - i, J - 1) \\ \geq \; & Q_J(n_J, i_J) \hat{P}_e(N - n_J, K - i_J, J - 1) \\ \overset{(a)}{\doteq} \; & e^{-L\gamma_J u_J\left(\frac{i_J}{n_J}\right) - L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K - i_J}{N - n_J}\right)} \\ \overset{(b)}{\geq} \; & e^{-L\gamma_J u_J\left(\mathbb{E}\{x_J\} + O\left(\frac{1}{L}\right)\right)} \cdot \\ & e^{-L\sum_{j=1}^{J-1}\gamma_j u_j\left(\frac{K}{N} + O\left(\frac{1}{L}\right)\right)} \\ \overset{(c)}{\doteq} \; & e^{-L\sum_{j=1}^{J}\gamma_j u_j\left(\frac{K}{N}\right)} \end{aligned} \tag{71}$$

where (a) follows from (63) and the induction assumption, and (b) is based on (70). (c) results from the facts that $u_j()$'s are differentiable functions, and we have $u_J\left(\mathbb{E}\{x_J\}\right) = 0$, both according to Lemma I in subsection IV-B.

For $0 \leq n_J \leq \epsilon L$, the analysis of section 1.1 and inequality (67) are still valid. For $n_J > (1-\epsilon)N$, we set $i_J = \lceil \mathbb{E}\left\{x_J\right\} n_J \rceil$. Now, we have

$$i_J \geq n_J \mathbb{E}\{x_J\} > (1-\epsilon) N \mathbb{E}\{x_J\} \geq (1-\epsilon) K. \tag{72}$$

The above inequality can be written as

$$K - i_J < \epsilon K < 1 \tag{73}$$

since $\epsilon < \frac{1}{K}$. Noting that K and i_J are integer values, it is concluded that $K \leq i_J$. Now, we can write

$$\begin{aligned} & \sum_{i=0}^{n_J} Q_J(n_J, i)\hat{P}_e(N - n_J, K - i, J - 1) \\ \geq \;& Q_J(n_J, i_J)\hat{P}_e(N - n_J, K - i_J, J - 1) \\ \stackrel{(a)}{=} \;& Q_J(n_J, i_J) \\ \dot{\geq} \;& e^{-L\gamma_J u_J\left(\mathbb{E}\left\{x_J\right\} + \frac{1}{n_J}\right)} \\ \stackrel{(b)}{\dot{\geq}} \;& e^{-L\gamma_J u_J\left(\mathbb{E}\left\{x_J\right\} + \frac{1}{(1-\epsilon)N}\right)} \\ \doteq \;& e^{-L\gamma_J u_J\left(\mathbb{E}\left\{x_J\right\} + O\left(\frac{1}{L}\right)\right)} \stackrel{(c)}{\doteq} 1 \end{aligned} \tag{74}$$

where (a) follows from the fact that $K \leq i_J$, and $\hat{P}_e(n, k, j) = 1$, for $k \leq 0$. (b) and (c) result from $n_J > (1 - \epsilon)N$ and $u_J\left(\mathbb{E}\{x_J\}\right) = 0$, respectively.

Hence, inequalities (67), (71), and (74) result in

$$\hat{P}_e(N, K, J) \;\; \dot{\geq} \;\; e^{-L\sum_{j=1}^{J} \gamma_j u_j\left(\alpha\right)} \tag{75}$$

which proves the first part of Theorem III for the case of $\frac{K}{N} \leq \mathbb{E}\{x_J\}$ when combined with Lemma V.

2) We prove the second and the third parts of the theorem by induction on j while the total number of types, J, is fixed. The proof of the statements for the base of the induction, $j = J$, is similar to the proof of the induction step, from $j + 1$ to j. Hence, we just give the proof for the induction step. Assume the second and the third parts of the theorem are true for $m = J$ to $j + 1$. We prove the same statements for j. The proof is divided into two different cases, depending on whether $\frac{K}{N}$ is larger than $\mathbb{E}\{x_j\}$ or vice versa.

Before we proceed further, it is helpful to introduce two new parameters N' and K' as

$$\begin{aligned} N' &= N - \sum_{m=j+1}^{J} \hat{N}_j \\ K' &= K - \sum_{m=j+1}^{J} K_j. \end{aligned}$$

According to the above definitions and the induction assumptions, it is obvious that

$$\frac{K'}{N'} = \frac{K}{N} + o(1) = \alpha + o(1). \tag{76}$$

2.1) $\frac{K}{N} > \mathbb{E}\{x_j\}$

First, by contradiction, it will be shown that for small enough values of $\epsilon > 0$, we have $\hat{N}_j > \epsilon N'$. Let us assume the opposite is true, i.e. $\hat{N}_j \leq \epsilon N'$. Then, we can write

$$\begin{aligned}
& \hat{P}_e(N', K', j) \\
\overset{(a)}{=} & \sum_{i=0}^{\hat{N}_j} \hat{P}_e(N' - \hat{N}_j, K' - i, j-1) Q_j(\hat{N}_j, i) \\
\geq & \hat{P}_e(N' - \hat{N}_j, K' - \hat{N}_j, j-1) Q_j(\hat{N}_j, \hat{N}_j) \\
\overset{(b)}{\doteq} & Q_j(\hat{N}_j, \hat{N}_j) e^{-L \sum_{r=1}^{j-1} \gamma_r u_r \left(\frac{K' - \hat{N}_j}{N' - \hat{N}_j} \right)} \\
\overset{(c)}{\geq} & e^{-L n_0 \left(1 - \sum_{r=j+1}^{J} \eta_r \right) \epsilon \log \left(\frac{1}{\pi_{b,j}} \right)} \cdot \\
& e^{-L \sum_{r=1}^{j-1} \gamma_r u_r \left(\frac{K'}{N'} \right)} \\
\overset{(d)}{\dot{>}} & e^{-L \sum_{r=1}^{j} \gamma_r u_r (\alpha)}
\end{aligned} \tag{77}$$

where (a) follows from equation (23) and step (2) of our suboptimal algorithm, (b) results from the first part of Theorem III, and (c) can be justified using arguments similar to those of inequality (67). (d) is obtained assuming ϵ is small enough such that the corresponding term in the exponent is strictly less than $L\gamma_j u_j \left(\frac{K'}{N'}\right)$ and also the fact that $\frac{K'}{N'} = \alpha + o(1)$. The result in (77) is obviously in contradiction with the first part of Theorem III, proving that $\hat{N}_j > \epsilon N'$.

Now, we show that if $\hat{N}_j > (1-\epsilon)N'$ for arbitrarily small values of ϵ, we should have $\mathbb{E}\{x_r\} > \alpha$ for all $1 \leq r \leq j-1$. In such a case, we observe $\frac{\hat{N}_j}{N'} = 1 + o(1)$, proving the second statement of Theorem III. To show this, let us assume $\hat{N}_j > (1-\epsilon)N'$. Hence,

$$\begin{aligned}
\hat{P}_e(N', K', j) &= \sum_{i=0}^{\hat{N}_j} \hat{P}_e(N' - \hat{N}_j, K' - i, j-1) Q_j(\hat{N}_j, i) \\
&\dot{\geq} \hat{P}_e(N' - \hat{N}_j, 0, j-1) Q_j(\hat{N}_j, K') \\
&\overset{(a)}{\dot{\geq}} e^{-L\gamma_j u_j \left(\frac{K'}{(1-\epsilon)N'} \right)} \overset{(b)}{\doteq} e^{-L\gamma_j u_j (\alpha + o(1))}
\end{aligned} \tag{78}$$

where (a) follows from the fact that $\hat{P}_e(n, 0, j) = 1$, for all values of n and j, and the fact that $\hat{N}_j \geq (1-\epsilon)N'$. (b) is obtained by making ϵ arbitrarily small and using equation (76). Applying (78) and knowing the fact that $\hat{P}_e(N', K', j) \doteq e^{-L \sum_{r=1}^{j} \gamma_r u_r(\alpha)}$, we conclude that $\mathbb{E}\{x_r\} > \alpha$, for all values of $1 \leq r \leq j-1$.

$\hat{P}_e(N', K', j)$ can be written as

$$
\begin{aligned}
& \hat{P}_e(N', K', j) \\
= & \min_{0 \le N_j \le N'} \sum_{i=0}^{N_j} \hat{P}_e(N'-N_j, K'-i, j-1) Q_j(N_j, i) \\
\overset{(a)}{\doteq} & \min_{\epsilon N' \le N_j \le (1-\epsilon)N'} \max_{0 \le i \le N_j} \\
& \hat{P}_e(N'-N_j, K'-i, j-1) Q_j(N_j, i) \\
\overset{(b)}{\doteq} & \min_{\epsilon N' \le N_j \le (1-\epsilon)N'} \max_{\mathbb{E}\{x_j\}N_j < i \le N_j} \\
& e^{-L\gamma_j u_j\left(\frac{i}{N_j}\right) - L\sum_{r=1}^{j-1} \gamma_r u_r\left(\frac{K'-i}{N'-N_j}\right)} \\
\doteq & e^{-L \max_{\epsilon N' \le N_j \le (1-\epsilon)N'} \min_{\mathbb{E}\{x_j\}N_j < i \le N_j} M_d(i, N_j)} \\
\overset{(c)}{\doteq} & e^{-L \max_{\epsilon \le \lambda_j \le (1-\epsilon)} \min_{\mathbb{E}\{x_j\}\lambda_j < \beta_j \le \lambda_j} M_c(\beta_j, \lambda_j)} .
\end{aligned}
\tag{79}
$$

where $M_d(i, N_j)$ and $M_c(\beta_j, \lambda_j)$ are defined as

$$
\begin{aligned}
M_d(i, N_j) &= \gamma_j u_j\left(\frac{i}{N_j}\right) + \sum_{r=1}^{j-1} \gamma_r u_r\left(\frac{K'-i}{N'-N_j}\right) \\
M_c(\beta_j, \lambda_j) &= \gamma_j u_j\left(\frac{\beta_j}{\lambda_j}\right) + \sum_{r=1}^{j-1} \gamma_r u_r\left(\frac{\alpha-\beta_j}{1-\lambda_j}\right).
\end{aligned}
$$

In (79), (a) follows from the fact that $\hat{N}_j$ is bounded as $\epsilon N' \le \hat{N}_j \le (1-\epsilon)N'$. (b) results from equation (63), $\hat{P}_e(n, k, j)$ being a decreasing function of k, and the fact that we have $Q_j(N_j, i) \le 1 \doteq Q_j(N_j, \mathbb{E}\{x_j\} N_j)$ for $i < \mathbb{E}\{x_j\} N_j$. β_j and λ_j are defined as $\beta_j = \frac{i}{N'}$ and $\lambda_j = \frac{N_j}{N'}$. (c) is a result of having $M_c(\beta_j, \lambda_j) = M_d(i, N_j) + O\left(\frac{1}{L}\right)$. Hence, the discrete to continuous relaxation is valid.

Let us define $\left(\beta_j^*, \lambda_j^*\right)$ as the values of (β_j, λ_j) which solve the max-min problem in (79). Differentiating $M_c(\beta_j, \lambda_j)$ with respect to β_j and λ_j results in

$$
0 = \frac{\gamma_j}{\lambda_j^*} l_j\left(\frac{\beta_j^*}{\lambda_j^*}\right) - \sum_{\substack{r=1, \\ \mathbb{E}\{x_r\} < \zeta}}^{j-1} \frac{\gamma_r}{1-\lambda_j^*} l_r(\zeta)
$$

$$
\begin{aligned}
0 = \Bigg\{ & -\frac{\gamma_j \beta_j^*}{\lambda_j^{*2}} l_j\left(\frac{\beta_j^*}{\lambda_j^*}\right) + \sum_{\substack{r=1, \\ \mathbb{E}\{x_r\} < \zeta}}^{j-1} \frac{\gamma_r(\alpha - \beta_j^*)}{(1-\lambda_j^*)^2} l_r(\zeta) \\
& + \left(\frac{\gamma_j}{\lambda_j^*} l_j\left(\frac{\beta_j^*}{\lambda_j^*}\right) - \sum_{\substack{r=1, \\ \mathbb{E}\{x_r\} < \zeta}}^{j-1} \frac{\gamma_r}{1-\lambda_j^*} l_r(\zeta) \right) \frac{\partial \beta_j^*}{\partial \lambda_j}\Big|_{\lambda_j = \lambda_j^*} \Bigg\}
\end{aligned}
$$

where $\zeta = \dfrac{\alpha - \beta_j^*}{1-\lambda_j^*}$. Solving the above equations gives the unique optimum solution (β_j^*, λ_j^*) as

$$\begin{aligned} \beta_j^* &= \alpha \lambda_j^* \\ \lambda_j^* &= \frac{\gamma_j l_j(\alpha)}{\displaystyle\sum_{r=1,\alpha>\mathbb{E}\{x_r\}}^{j} l_r(\alpha)} \end{aligned} \tag{80}$$

Hence, the integer parameters $K_j, \hat{N}_j$ defined in the suboptimal algorithm have to satisfy $\frac{K_j}{N'} = \beta_j^* + o(1)$ and $\frac{\hat{N}_j}{N'} = \lambda_j^* + o(1)$, respectively. Based on the induction assumption, it is easy to show that

$$\frac{N'}{N} = \frac{\displaystyle\sum_{r=1,\mathbb{E}\{x_r\}<\alpha}^{j} \gamma_r u_r(\alpha)}{\displaystyle\sum_{r=1,\mathbb{E}\{x_r\}<\alpha}^{J} \gamma_r u_r(\alpha)} \tag{81}$$

which completes the proof for the case of $\mathbb{E}\{x_j\} < \frac{K}{N}$.

2.2) $\dfrac{K}{N} \leq \mathbb{E}\{x_j\}$

In this case, we show that $\frac{\hat{N}_j}{N} = o(1)$. Defining $i_j = \lceil \mathbb{E}\{x_j\}\hat{N}_j \rceil$, we have

$$\frac{K' - i_j}{N' - \hat{N}_j} = \alpha - (\mathbb{E}\{x_j\} - \alpha)\frac{\hat{N}_j}{N' - \hat{N}_j} + o(1) \tag{82}$$

using equation (76). Now, we have

$$\begin{aligned} & \hat{P}_e(N', K', j) \\ = & \sum_{i=0}^{\hat{N}_j} \hat{P}_e(N' - \hat{N}_j, K' - i, j-1) Q_j(\hat{N}_j, i) \\ \geq & \hat{P}_e(N' - \hat{N}_j, K' - i_j, j-1) Q_j(\hat{N}_j, i_j) \\ \overset{(a)}{\doteq} & e^{-L\gamma_j u_j\left(\mathbb{E}\{x_j\} + o(1)\right)} . \\ & e^{-L\sum_{r=1}^{j-1} \gamma_r u_r\left(\alpha - (\mathbb{E}\{x_j\} - \alpha)\frac{\hat{N}_j}{N' - \hat{N}_j}\right)} \\ \doteq & e^{-L\sum_{r=1}^{j-1} \gamma_r u_r\left(\alpha - (\mathbb{E}\{x_j\} - \alpha)\frac{\hat{N}_j}{N' - \hat{N}_j}\right)} \end{aligned} \tag{83}$$

where (a) follows from the first part of Theorem III and (63). On the other hand, according to the result of the first part of Theorem III, we know that

$$\hat{P}_e(N', K', j) \doteq e^{-L\sum_{r=1}^{j-1} \gamma_r u_r(\alpha)}. \tag{84}$$

According to Lemma I, $u_r(\beta)$ is an increasing function of β for all $1 \leq r \leq j-1$. Thus, $\sum_{r=1}^{j-1} \gamma_r u_r(\beta)$ is also a one-to-one increasing function of β. Noting this fact and comparing (83) and (84), we conclude that $\frac{\hat{N}_j}{N'} = o(1)$ as $\mathbb{E}\{x_j\} - \alpha$ is strictly positive. Noting (81), we have $\frac{\hat{N}_j}{N} = o(1)$ which proves the second part of Theorem III for the case of $\frac{K}{N} \leq \mathbb{E}\{x_j\}$.

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