Fairness in Multiuser Systems with Polymatroid Capacity Region

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Abstract

For a wide class of multi-user systems, a subset of capacity region which includes the corner points and the sum-capacity facet has a special structure known as polymatroid. Multiaccess channels with fixed input distributions and multiple-antenna broadcast channels are examples of such systems. Any interior point of the sum-capacity facet can be achieved by time-sharing among corner points or by an alternative method known as *rate-splitting*. The main purpose of this paper is to find a point on the sum-capacity facet which satisfies a notion of fairness among active users. This problem is addressed in two cases: (i) where the complexity of achieving interior points is not feasible, and (ii) where the complexity of achieving interior points is feasible. For the first case, the corner point for which the minimum rate of the active users is maximized (max-min corner point) is desired for signaling. A simple greedy algorithm is introduced to find the optimum max-min corner point. For the second case, the polymatroid properties are exploited to locate a rate-vector on the sum-capacity facet which is optimally fair in the sense that the minimum rate among all users is maximized (max-min rate). In the case that the rate of some users can not increase further (attain the max-min value), the algorithm recursively maximizes the minimum rate among the rest of the users. It is shown that the problems of deriving the time-sharing coefficients or rate-spitting scheme can be solved by decomposing the problem to some lower-dimensional subproblems. In addition, a fast algorithm to compute the time-sharing coefficients to attain a general point on the sum-capacity facet is proposed.

Index Terms

Polymatroid Structure, Multiuser Systems, Multiaccess Channels, Broadcast Channels, Fairness, Successive Decoding, Time-Sharing, Rate-Splitting.

I. Introduction

In the multi-user scenarios, multiple transmitters/receivers share a common communication medium, and therefore, there is an inherent competition in accessing the channel. Information theoretic results for such systems imply that in order to achieve a high spectral efficiency, the users with stronger channel should have a higher portion of the resources. The drawback to this is the loss of the fairness among the users. Providing fairness, while achieving high-spectral efficiency, is thus a challenging problem.

A lot of research has addressed this problem and suggested different criteria to design a fair system. One of the first criteria is known as *max-min* measure. In this method, the main effort is to maximize the minimum rate of the users, by giving the highest priority to the user with the worst channel. In other words, this method penalizes the users with better channel and sacrifices overall efficiency.

By relaxing the strict condition on fairness, the spectral efficiency can be increased. By compromising between fairness and throughput, proportional fairness is proposed in [1]. Based on this criterion, the rates of users with a stronger channel can be increased with the cost of decreasing the rates of users with a weaker channel. Any change in the rates is acceptable if the total proportional increase in the rates of some users is larger than the total proportional decrease in the rates of the rest. In fact, by relaxing the strict condition on fairness, the spectral efficiency increases. In [2], a criterion based on Nash Bargaining solution in the context of Game Theory is proposed. This method generalizes the proportional fairness and increases the efficiency of the system.

All of the aforementioned methods deal with a general multi-user system. However, for a wide class of multi-user systems, the capacity region has a special structure that we can exploit to provide fairness. Particularly in some multiuser systems, the boundary of the capacity region includes a facet on which the sum-rate is maximum (sum-capacity facet). In such systems, one can benefit from the available degrees of freedom, and determine the fairest rate-vector on the sum-capacity facet.

As a special case, we consider a class of multi-user systems, in which the whole or a subset of the capacity region which includes the corner points and the sum-capacity facet forms a structure known as polymatroid. For this class of multi-user systems, the sum-capacity facet has $a!$ corner points, where a is the number of users with non-zero power (active users). The sum-capacity facet is the convex hull of these corner points. This means that the interior points of the sum-capacity facet can be attained by time-sharing among such corner points. As an example of such systems, it is shown that the capacity region of multiaccess channels (MAC) with fixed and independent input distributions forms a polymatroid [3]. In MAC, the sum-capacity is achieved by successive decoding. Applying different orders for the users in successive decoding results in different rate-vectors, all with the sum-rate equal to the sum-capacity. The resulting rate-vectors correspond to the corner points of the sum-capacity facet. Any point in the convex hull of these corner points is on the boundary. In [4], it is proven that the Marton inner bound (see [5]) for capacity region of the broadcast channel under fixed joint probability of the auxiliary and input variables, with some conditions, has a polymatroid structure[1]. As another example, we will show that a subset of the capacity region for multiple-input multiple-output (MIMO) broadcast channel which includes the corner points forms a polymatroid.

In [3], the optimal dynamic power allocation strategy for time-varying single-antenna multiple-access channel is established. To this end, the polymatroid properties of the capacity region for time-invariant multiple-access channel with fixed input distributions have been exploited. In [6], the polymatroid properties have been used to find a fair power allocation strategy. This problem is formulated by representing a point on the face of the contra-polymatroid (see [3], [7]) as a convex combination of its extreme points.

This article aims at finding a point on the sum-capacity facet which satisfies a notion of fairness among active users by exploiting the properties of polymatroids. In order to provide fairness, the minimum rate among all users is maximized (max-min rate). In the case that the rate of some users can not increase further (attain the max-min value), the algorithm recursively maximizes the minimum rate among the rest of the users. Since this rate-vector is in the face of the polymatroid, it can be achieved by time sharing among the corner points. It is shown that the problem of deriving the time-sharing coefficients to attain this point can be decomposed to some lower-dimensional subproblems. An alternative approach to attain an interior point for multiple access channels is *rate splitting* [8], [9]. This method is based on splitting all input sources except one into two parts and treating each spilt input as two virtual inputs (or two virtual users). By splitting the sources appropriately and successive decoding of virtual users in a suitable order, any point on the sum-capacity facet can be attained [8], [9]. Similar to the time-sharing procedure, we show that the problem of rate-splitting can be decomposed to some lower dimensional subproblems.

There are cases that the complexity of achieving interior points is not feasible. This motivates us to compute the corner point for which the minimum rate of the active users is maximized (max-min corner point). A simple greedy algorithm is introduced to find the max-min corner point.

The rest of the paper is organized as follows. In Section II, the structure of the polymatroid

[1]Throughout the paper, we deal with the systems where the underlying capacity region or a its subset which included sum-capacity facet forms a polymatroid. Apparently, the proposed method can be applied over any achievable region which has the similar geometrical structure. In this case, the sum-capacity facet is replaced with maximum-sum-rate facet.

is presented. In addition, the relationship between the capacity region of some channels and the polymatroid structure is described. Section III discusses the case in which the optimal fair corner point is computed. In Section IV, the optimal fair rate-vector on the sum-capacity facet is computed by exploiting polymatroid structures. In addition, it is shown that the problem of deriving the time-sharing coefficients and rate-splitting can be solved by decomposing the problem into some lower-dimensional subproblems.

Notation: All boldface letters indicate vectors (lower case) or matrices (upper case). $\det(\mathbf{H})$ and $\mathbf{H}^{\dagger}$ denote the determinant and the transpose conjugate of the matrix $\mathbf{H}$, respectively. $\mathbf{M} \succeq 0$ represents that the matrix $\mathbf{M}$ is positive semi-definite. $\mathbf{1}_n$ represents an n dimensional vector with all entries equal to one. E is a set of integers $E = \{1, \cdots, |E|\}$, where $|E|$ denotes the cardinality of the set E. The set function $f : 2^E \longrightarrow \mathcal{R}_+$ is a mapping from all subsets of E (there are a total of $2^{|E|}$ subsets) to the positive real numbers. A permutation of the set E is denoted by π and $\pi(i)$, $1 \leq i \leq |E|$, represents the element of the set E located in the i^{th} position after the permutation. For an a-dimensional vector $\mathbf{x} = \{x_1, x_2, \ldots, x_a\} \in \mathcal{R}^a$ and $S \subset E$, $\mathbf{x}(S)$ denotes $\sum_{i \in S} x_i$. Also, for a set of positive semi-definite matrices $\mathbf{D}_i$, $\mathbf{D}(S)$ represents $\sum_{i \in S} \mathbf{D}_i$.

II. Preliminaries

A. Polymatroid Structure

Definition [10, Ch. 18]: Let $E = \{1, 2, \ldots, a\}$ and $f : \; 2^E \longrightarrow \mathcal{R}_+$ be a set function. The polyhedron

$$\mathcal{B}(f, E) = \{(x_1, \ldots, x_a) : \mathbf{x}(S) \leq f(S), \forall S \subset E, \forall x_i \geq 0\} \tag{1}$$

is a polymatroid, if the set function f satisfies

$$(normalized) \qquad f(\emptyset) = 0 \tag{2}$$

$$(increasing) \qquad f(S) \leq f(T) \text{ if } S \subset T \tag{3}$$

$$(submodular) \qquad f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \tag{4}$$

Any function f that satisfies the above properties is termed as *rank function*. Note that (1) imposes $2^{|E|}$ constraints on any given vector $(x_1, \ldots, x_a) \in \mathcal{B}(f, E)$.

Corresponding to each permutation π of the set E, the polymatroid $\mathcal{B}(f,E)$ has a corner point $\mathbf{v}(\pi) \in \mathcal{R}_+^a$ which is equal to:

$$v_{\pi(i)}(\pi) = \begin{cases} f(\{\pi(i)\}) & i = 1 \\ \\ f(\{\pi(1),\ldots,\pi(i)\}) & \\ \quad -f(\{\pi(1),\ldots,\pi(i-1)\}) & i = 2,\ldots,a \end{cases} \tag{5}$$

Consequently, the polymatroid $\mathcal{B}(f,E)$ has $a!$ corner points corresponding to different permutations of the set E. All the corner points are on the facet $\mathbf{x}(E) = f(E)$. In addition, any point in the polymatroid on the facet $\mathbf{x}(E) = f(E)$ is in the convex hull of these corner points. The hyperplane $\mathbf{x}(E) = f(E)$ is called as dominant face, or simply face of the polymatroid. In this paper, we use the term *sum-capacity facet* to denote the face of the polymatroid.

B. Capacity Region and Polymatroid Structure

For a wide class of multi-user systems, the whole or a subset of the capacity region forms a polymatroid structure. As the first example, consider a multiaccess system with a users, where the distribution of inputs are independent and equal to $p(x_1),\ldots,p(x_M)$. Then, the capacity region of such a system is characterized by [11], [12]

$$\left\{\mathbf{r} \in \mathcal{R}_+^a | \mathbf{r}(S) \leq I\left(y;\{x_i, i \in S\} | \{x_i, i \in S^c\}\right) \quad \forall S \subset E\right\}, \tag{6}$$

where y is the received signal, $\mathbf{r}$ represents rate vector, I denotes the mutual information, and S^c is equal to $E - S$. It has been shown that the above polyhedron forms a polymatroid [3].

As the second example, we consider the capacity region of a multiple-antenna broadcast system. In the sequel, we show that a subset of the capacity region which includes the corner points and sum-capacity facet forms a polymatroid.

Consider a MIMO Broadcast Channel (MIMO-BC) with M transmit antennas and K users, where the k^{th} user is equipped with N_k receive antennas. In a flat fading environment, the baseband model of this system is given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{s} + \mathbf{w}_k, \quad 1 \leq k \leq K, \tag{7}$$

where $\mathbf{H}_k \in \mathcal{C}^{N_k \times M}$ denotes the channel matrix from the base station to user k, $\mathbf{s} \in \mathcal{C}^{M \times 1}$ represents the transmitted vector, and $\mathbf{y}_k \in \mathcal{C}^{N_k \times 1}$ signifies the received vector by user k. The vector $\mathbf{w}_k \in \mathcal{C}^{N_k \times 1}$ is a white Gaussian noise with zero-mean and identity-matrix covariance. Consider an order of the users $(\pi(1), \pi(2), \ldots, \pi(K))$. By assuming that user $\pi(i)$ knows the codewords selected for the users $\pi(j)$, $j = 1, \ldots, i-1$, the interference of the users $\pi(j)$, $j = 1, \ldots, i-1$, over user $\pi(i)$ can be effectively canceled based on dirty-paper-coding theorem [13]. Therefore, the rate of user $\pi(i), i = 1, \ldots, K$, is equal to

$$r_{\pi(i)} = \log \frac{\det\left(\mathbf{I}_{N_k,N_k} + \mathbf{H}_{\pi(i)}\left(\sum_{j\geq i}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}^{\dagger}_{\pi(i)}\right)}{\det\left(\mathbf{I}_{N_k,N_k} + \mathbf{H}_{\pi(i)}\left(\sum_{j> i}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}^{\dagger}_{\pi(i)}\right)}, \tag{8}$$

where $\mathbf{Q}_{\pi(j)}$ is the covariance of the signal vector to user $\pi(j)$. The capacity region is characterized as the convex hull of the union of such rate-vectors over all permutations $(\pi(1), \pi(2), \ldots, \pi(K))$ and over all positive semi-definite covariance matrices $\mathbf{Q}_i$, $i = 1, \ldots, K$ such that $\mathrm{Tr}\left(\sum_{i=k}^{K}\mathbf{Q}_i\right) \leq P_T$, where P_T denotes the total transmit power [14]. In [15]–[17], a duality between the MIMO-BC and the MIMO-MAC is established. In the dual MIMO-MAC, the channel between user k and the base station is $\mathbf{H}_k^{\dagger}$ and the covariance of the power allocated to user k is $\mathbf{P}_k$. The relationship between $\mathbf{P}_k$ and $\mathbf{Q}_k$, $k = 1, \ldots, K$, has been derived [16]. The duality is used to characterize the sum-capacity of the MIMO-BC as follows

$$\begin{aligned} r_{\text{SC}} \quad &= \quad \max_{\mathbf{P}_1,\ldots,\mathbf{P}_K} \log\det\left(\mathbf{I}_{M,M} + \sum_{k=1}^{K}\mathbf{H}_k^{\dagger}\mathbf{P}_k\mathbf{H}_k\right). \\ s.t. \quad & \sum_{k=1}^{K}\mathrm{Tr}(\mathbf{P}_k) \leq P_T, \\ & \mathbf{P}_k \succeq 0 \end{aligned} \tag{9}$$

The above optimization problem determines the power allocated to each user in the dual MIMO-MAC, and consequently, the power of each user in the MIMO-BC. Note that only a subset of users is active and the power allocated to the rest is zero. Equation (9) determines the so-called sum-capacity facet. If the cardinality of the set of active users is a, i.e. $E = \{1, \cdots, a\}$, the sum-capacity facet has $a!$ corner points corresponding to different permutations of the active users. Note that the rates of the non-active users remain zero regardless of the permutation. The corner point corresponding to a permutation can be computed using (8). Assuming the active users are indexed by $i = 1, \ldots, a$, we define

$$\mathbf{D}_i = \mathbf{H}_i^{\dagger}\mathbf{P}_i^{*}\mathbf{H}_i, i = 1, \ldots, a, \tag{10}$$

where $\mathbf{P}_i^*$, $i = 1, \ldots, a$, correspond to optimizing matrices in (9). It is shown that the corner point in (8) can be reformulated as [16]

$$r_{\pi(i)} = \log \frac{\det\left(\mathbf{I}_{M,M} + \sum_{j\leq i} \mathbf{D}_{\pi(j)}\right)}{\det\left(\mathbf{I}_{M,M} + \sum_{j<i} \mathbf{D}_{\pi(j)}\right)}, \quad i = 1, \ldots, a, \tag{11}$$

which is the corner point of the dual MAC.

Regarding the polymatroid structure of the multiaccess channels and considering the duality of the MIMO-MAC and MIMO-BC, we can observe the polymatroid structure of a subset of MIMO-BC capacity region which includes the sum-capacity facet. However, to provide a better insight about the problem, we introduce a special polymatroid and establish its relationship with the capacity region of the MIMO-BC. For a set of positive semi-definite matrices $\mathbf{D}_i$, we define the set function g as,

$$g(S) = \log\det\left(\mathbf{I} + \mathbf{D}(S)\right) \quad \text{for} \quad S \subset E. \tag{12}$$

Lemma 1 *Given $g(S)$ defined in (12), the polyhedron $\mathcal{B}(g, E)$ defined as follows is a polymatroid.*

$$\mathcal{B}(g, E) = \{(x_1, \ldots, x_a) \in \mathcal{R}_+^a : \mathbf{x}(S) \leq g(S), \ \forall S \subset E\}. \tag{13}$$

Proof: Clearly, $g(\emptyset) = 0$. Assume $\mathbf{B} \succeq 0$ and $\mathbf{C} \succeq 0$ are two Hermitian matrices. If $\mathbf{B} - \mathbf{C} \succeq 0$, then $\det(\mathbf{B}) \geq \det(\mathbf{C})$ [14, Proposition I.2]. Furthermore, if $\mathbf{\Delta} \succeq 0$, then [14, Proposition I.3]

$$\frac{\det(\mathbf{\Delta} + \mathbf{B} + \mathbf{C})}{\det(\mathbf{\Delta} + \mathbf{B})} \leq \frac{\det(\mathbf{B} + \mathbf{C})}{\det(\mathbf{B})}. \tag{14}$$

Using above properties, it is straight-forward to prove (3) and (4) for the set function $g(.)$. ■

In the set function $g(S)$, define $\mathbf{D}_i$ as defined in (10). It is easy to verify that the polymatroid $\mathcal{B}(g, E)$ is a subset of the capacity region of the MIMO-BC. The hyperplane $\mathbf{x}(E) = g(E)$ and its corner points (11) are the same as the sum-capacity facet and its corner points. Due to this property, we focus on the polymatroid $\mathcal{B}(g, E)$ (see Fig. 1).

III. The Fairest Corner Point

As mentioned, in some cases, the complexity of computing and implementing an appropriate time-sharing or rate-splitting algorithm is not feasible. This motivates us to compute the corner point for which the minimum rate of the active users is maximized (max-min corner point).

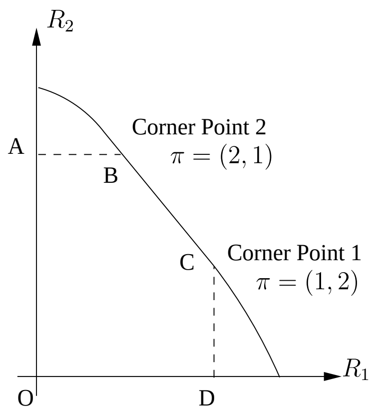

Fig. 1. Capacity region of the MIMO-BC and its corner Points. The region OABCD is a polymatroid. The line BC is the sum-capacity facet.

In the following, we present a simple greedy algorithm to find the max-min corner point of a general polymatroid $\mathcal{B}(f, E)$.

Algorithm I

1) Set $\alpha = a$, $S = \varnothing$.
2) Set $\pi^*(\alpha)$ as

$$\pi^*(\alpha) = \arg \min_{z \in E, z \notin \mathcal{S}} f\left(E - S - \{z\}\right). \tag{15}$$

3) If $\alpha > 1$, then $S \longleftarrow S \cup \{\pi^*(\alpha)\}$, $\alpha \longleftarrow \alpha - 1$, and go to Step 2; otherwise stop.

The following theorem proves the optimality of the above algorithm.

Theorem 1 *Let the vector* $\mathbf{v}(\pi^*)$ *be the corner point of the polymatroid* $\mathcal{B}(f, E)$ *corresponding to the permutation* $\pi^* = (\pi^*(1), \ldots, \pi^*(a))$. *For any other permutation* $\pi = (\pi(1), \ldots, \pi(a))$,

$$\min_i v_{\pi^*(i)}(\pi^*) \geq \min_i v_{\pi(i)}(\pi). \tag{16}$$

Proof: Assume that in the permutation π^*, the user θ which is located in position l in the permutation π^* (i.e. $\theta = \pi^*(l)$) has the minimum rate

$$v_{\pi^*(l)}(\pi^*) = \min_i v_{\pi^*(i)}(\pi^*). \tag{17}$$

Let us define two sets:

- The set of users located before $\pi^*(l)$ in π^*: $\Phi = \{\pi^*(1), \ldots, \pi^*(l-1)\}$.
- The set of users located after $\pi^*(l)$ in π^*: $\Psi = \{\pi^*(l+1), \ldots, \pi^*(a)\}$.

Using (5), we have

$$v_\theta(\pi^*) = f(\Phi \cup \{\theta\}) - f(\Phi). \tag{18}$$

In the following, we consider different scenarios which generate new permutations and prove that in all cases, (16) is valid.

Case 1. *Permutation in Φ and Ψ:* By considering (18), it is apparent that any permutation of the users in Φ and Ψ does not change the rate of the user $\pi^*(l)$ (see Fig. 2).

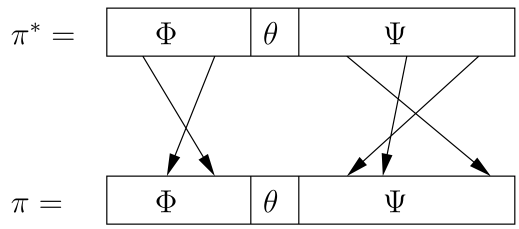

Fig. 2. Case 1. Permutation in Φ and Ψ.

Case 2. *Moving a set of users from Ψ to the set Φ:* Assume a set Υ of users, $\Upsilon \subset \Psi$, is moved from Ψ to the set Φ to generate a new permutation π (see Fig. 3). The rate of the user θ in the new permutation is equal to:

$$v_\theta(\pi) = f(\Phi \cup \Upsilon \cup \{\theta\}) - f(\Phi \cup \Upsilon). \tag{19}$$

From (4), we can show that

$$f(\Phi \cup \{\theta\}) + f(\Phi \cup \Upsilon) \geq f(\Phi \cup \Upsilon \cup \{\theta\}) + f(\Phi). \tag{20}$$

Using (18), (19), and (20), we conclude that $v_\theta(\pi) \leq v_\theta(\pi^*)$, and therefore, $\min_i v_{\pi(i)}(\pi) \leq \min_i v_{\pi^*(i)}(\pi^*)$.

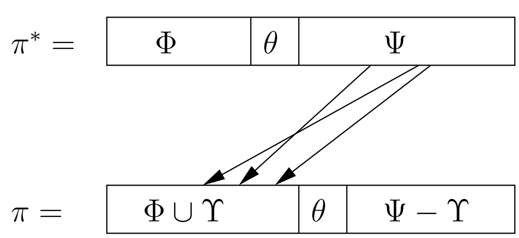

Fig. 3. Case 2. Moving a set of users from Ψ to the set Φ.

Case 3. *Moving one or more users from the set* Φ *to the set* Ψ *(with or without moving some users from the set* Ψ *to the set* Φ*):* Assume that one or more users move from Φ to Ψ (with or without moving some users from the set Ψ to the set Φ) to generate the new permutation π. As depicted in Fig. 4, assume that the user ν is positioned last in the permutation π among the users moved from Φ to Ψ (user $\pi(1)$ is positioned first and user $\pi(a)$ is positioned last in the permutation π).

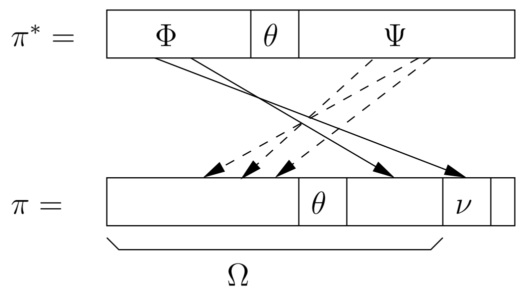

Fig. 4. Case 3. Moving one or more users from the set Φ to the set Ψ (with or without moving some users from the set Ψ to the set Φ).

Let Ω be the set of users located before the user ν in the permutation π. Using (5), we have,

$$v_\nu(\pi) = f(\Omega \cup \{\nu\}) - f(\Omega). \tag{21}$$

It is clear that,

$$\{\theta\} \cup \Phi - \{\nu\} \subset \Omega. \tag{22}$$

Using (4) with $S = \Phi \cup \{\theta\}$ and $T = \Omega$, and regarding (22), we have,

$$f(\Omega \cup \{\nu\}) - f(\Omega) \leq f(\Phi \cup \{\theta\}) - f(\Phi \cup \{\theta\} - \{\nu\}). \tag{23}$$

On the other hand, the user ν is in the set Φ in permutation π^*. It means that in Step 2 of the algorithm, this user has been compared with other users in the set $\Phi \cup \{\theta\}$ to be located in the position l, but the user θ has been chosen for the position, i.e. $f\left(\Phi \cup \{\theta\} - \{\theta\}\right) \leq f\left(\Phi \cup \{\theta\} - \{\nu\}\right)$, therefore,

$$f\left(\Phi\right) \leq f\left(\Phi \cup \{\theta\} - \{\nu\}\right). \tag{24}$$

Using (18), (21), (23), and (24), we conclude that $v_\nu(\pi) \leq v_\theta(\pi^*)$, and therefore, we have $\min_i v_{\pi(i)}(\pi) \leq \min_i v_{\pi^*(i)}(\pi^*)$. Note that the permutation of users located before (or after) the user ν in the permutation π does not increase $v_\nu(\pi)$. ■

Remark: For multiple access channels, the above algorithm suggests that to attain the fairest corner point with successive decoding, at each step, one should decode the strongest user (the user with the highest rate, while the signals of the remaining users are considered as interference). Note that in MAC, the corner point corresponding to the specific permutation π is obtained by the successive decoding in the reverse order of the permutation.

It is worth mentioning that by using a similar algorithm, one can find the corner point for which the maximum rate is minimum. The algorithm is as follows:

Algorithm II

1) Set $\alpha = 1$, $S = \varnothing$.
2) Set $\pi^*(\alpha)$ as

$$\pi^*(\alpha) = \arg \max_{z \in E, z \notin \mathcal{S}} f\left(S + \{z\}\right). \tag{25}$$

3) If $\alpha < a$, then $S \longleftarrow S \cup \{\pi^*(\alpha)\}$, $\alpha \longleftarrow \alpha + 1$, and go to Step 2; otherwise stop.

The optimality of the above algorithm can be proven by a similar method as used to prove Theorem 1.

IV. Optimal Rate-Vector on the Sum-Capacity Facet

A. Max-Min Operation over a Polymatroid

In the following, the polymatroid properties are exploited to locate an optimal fair point on the sum-capacity facet. For an optimal fair point, the minimum rate among all the users should be maximized (max-min rate). For a sum-capacity of r_{SC}, a fair rate allocation would ideally achieve an equal rate of $\frac{r_{\text{SC}}}{a}$ for the a active users. Although this rate-vector is feasible for some special cases (see Fig. 5), it is not attainable in the general case (see Fig. 6). The maximum possible value for the minimum entry of a vector $\mathbf{x}$, where $\mathbf{x} \in \mathcal{B}(f, E)$, can be computed using the following lemma.

Lemma 2 *In the polymatroid $\mathcal{B}(f, E)$, define*

$$\begin{aligned} \delta &= \max \min_{i \in E} x_i. \\ &\quad s.t. \quad (x_1, \ldots, x_a) \in \mathcal{B}(f, E). \end{aligned} \tag{26}$$

Then,

$$\delta = \min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}. \tag{27}$$

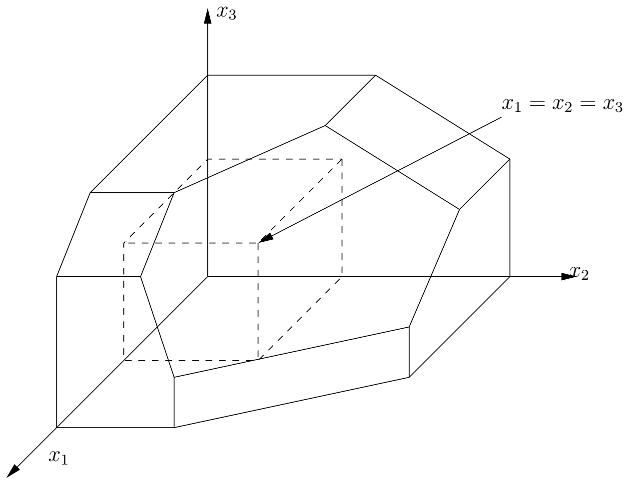

Fig. 5. All-Equal Rate-Vector Is on the Sum-Capacity Facet

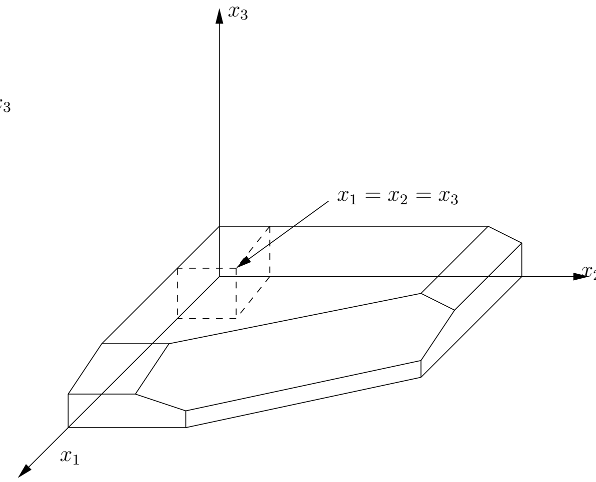

Fig. 6. All-Equal Rate-Vector Is NOT on the Sum-Capacity Facet

Proof: Consider $\mathbf{x} \in \mathcal{B}(f, E)$, and let $\sigma = \min_i x_i$. Therefore,

$$\forall S \subset E,\ \sigma|S| \leq \mathbf{x}(S). \tag{28}$$

Noting $\forall S \subset E,\ \mathbf{x}(S) \leq f(S)$ and using the above inequality, we have

$$\forall S \subset E,\ \sigma|S| \leq f(S). \tag{29}$$

Consequently, $\sigma \leq \min_{S \subset E,\ S \neq \emptyset} \frac{f(S)}{|S|}$. Therefore, $\min_{S \subset E,\ S \neq \emptyset} \frac{f(S)}{|S|}$ provides an upper bound on $\min_i x_i$. By selecting $\mathbf{x} = \delta \mathbf{1}_a \in \mathcal{B}(f, E)$, where $\delta = \min_{S \subset E,\ S \neq \emptyset} \frac{f(S)}{|S|}$, the upper bound is achieved, and the proof is completed. ■

In minimization (27), if the minimizer is not the set E, then δ (the optimal max-min value) is less than $\frac{r_{\text{SC}}}{a}$ ($r_{\text{SC}} = f(E)$ is the sum-capacity), and therefore, the ideal fairness is not feasible. For example, in the polymatroid depicted in Fig 6, the minimizing set in (27) is the set $\{3\}$, and therefore $\delta = f(\{3\})$.

In the following, a recursive algorithm is proposed to locate a rate vector $\mathbf{x}^*$ on the sum-capacity facet which not only attains the optimal max-min value δ, but also provides fairness among the users which have the rates higher than δ. The proposed algorithm partitions the set of active users into $t+1$ disjoint subsets, $S^{(0)}, ..., S^{(t)}$, such that in the i'th subset the rate of all users is equal to $m^{(i)}, i = 0, \cdots, t$, where $\delta = m^{(0)} < m^{(1)} < \cdots < m^{(t)}$. Starting from $m^{(0)}$, the algorithm maximizes $m^{(i)}$, $i = 1, \cdots, t$, given that $m^{(j)}$'s, $j = 0, \cdots, i-1$, are already at their maximum possible values. To simplify this procedure, we establish a chain of nested polymatroids, $\mathcal{B}(f^{(\alpha)}, E^{(\alpha)})$, $\alpha = 0, \ldots, t$, where

$$\mathcal{B}(f^{(t)}, E^{(t)}) \subset \mathcal{B}(f^{(t-1)}, E^{(t-1)}) \subset \ldots \subset \mathcal{B}(f^{(0)}, E^{(0)}) = \mathcal{B}(f, E). \tag{30}$$

In this algorithm, we use the result of the following lemma.

Lemma 3 *Let $E = \{1, \ldots, a\}$ and $A \subset E$, $A \neq E$. If the set function $f : 2^E \longrightarrow \mathcal{R}_+$ is a rank function, then $h : 2^{E-A} \longrightarrow \mathcal{R}_+$, defined as*

$$h(S) = f(S \cup A) - f(A), \quad S \subset E - A, \tag{31}$$

is a rank function.

Proof: By direct verification. ∎

Using the following algorithm, one can compute the rate-vector $\mathbf{x}^*$.

Algorithm III

1) Initialize the iteration index $\alpha = 0$, $E^{(0)} = E$, and $f^{(0)} = f$.
2) Find $m^{(\alpha)}$, where
$$m^{(\alpha)} = \min_{S \subset E^{(\alpha)}, S \neq \emptyset} \frac{f^{(\alpha)}(S)}{|S|}. \tag{32}$$
Set $S^{(\alpha)}$ equal to the optimizing subset.
3) For all $i \in S^{(\alpha)}$, set $x_i^* = m^{(\alpha)}$.
4) Define the polymatroid $\mathcal{B}(f^{(\alpha+1)}, E^{(\alpha+1)})$, where
$$E^{(\alpha+1)} = E^{(\alpha)} - S^{(\alpha)}, \tag{33}$$
and $\forall S \subset E^{(\alpha+1)}$,
$$f^{(\alpha+1)}(S) = f^{(\alpha)}(S \cup S^{(\alpha)}) - f^{(\alpha)}(S^{(\alpha)}). \tag{34}$$
5) If $E^{(\alpha+1)} \neq \emptyset$, set $\alpha \longleftarrow \alpha + 1$ and move to step 2, otherwise stop.

This algorithm computes the optimization sets $S^{(\alpha)}$, $\alpha = 0, \cdots, t$ and their corresponding $m^{(\alpha)}$, where $E = \bigcup_{j=0}^{t} S^{(j)}$ and $x_i^* \in \{m^{(0)}, \cdots, m^{(t)}\}, i = 1, \cdots, a$.

To provide better insight about the algorithm, let us apply it over the polymatroids depicted in figures 5 and 6. For the polymatroid in Fig. 5, the algorithm results in $\mathbf{x}^* = (m^{(0)}, m^{(0)}, m^{(0)})$ where $m^{(0)} = \frac{f(\{1,2,3\})}{3}$. For the polymatroid shown in Fig 6, the resulting point is $\mathbf{x}^* = (m^{(1)}, m^{(1)}, m^{(0)})$, where $m^{(0)} = \frac{f(\{3\})}{1}$ and $m^{(1)} = \frac{f^{(1)}(\{1,2\})}{2} = \frac{f(\{1,2,3\}) - f(\{3\})}{2}$ (see Fig. 7).

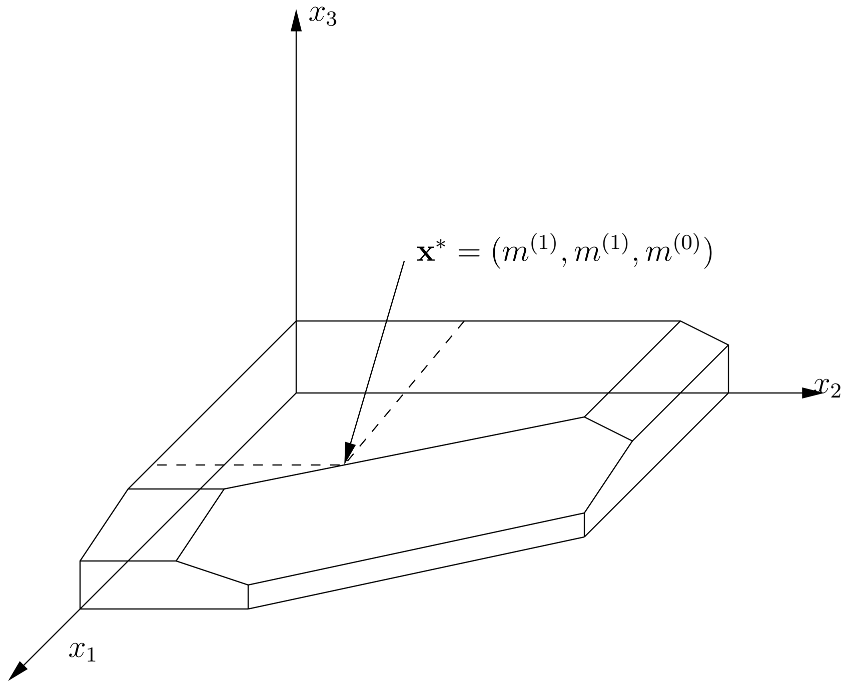

Fig. 7. The Fairest Rate Vector $\mathbf{x}^*$ on the Sum-Rate Facet of the Polymatroid

In the following, we prove some properties of the vector $\mathbf{x}^*$.

Theorem 2 *Assume that the algorithm III is applied over the polymatroid $\mathcal{B}(f, E)$, then*

(I) $\mathbf{x}^* \in \mathcal{B}(f, E)$ *and is located on the sum-capacity facet* $\mathbf{x}(E) = f(E)$.

(II) *The minimum entry of the vector* $\mathbf{x}^*$ *attains the optimum value determined by Lemma 2 and*

$$\delta = m^{(0)} < m^{(1)} < \cdots < m^{(t)}. \tag{35}$$

Proof:

Part (I): We show that $\mathbf{x}^* \in \mathcal{B}(f, E)$. According to the algorithm, we have $m^{(0)} = \min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}$, where $S^{(0)}$ is the minimizing set. In addition, $x_i^* = m^{(0)}$ for all $i \in S^{(0)}$. It is straightforward to check that the assigned values for $x_i^*, i \in S^{(0)}$, do not violate the constraints of the polymatroid $\mathcal{B}(f, E)$, expressed in (1). By substituting the assigned values for $x_i, i \in S^{(0)}$, in the constraints of the polymatroid $\mathcal{B}(f, E)$, the constraints over the coordinate $i,\ i \in E - S^{(0)}$, are updated as follows: from the definition of the polymatroid, we have a set of constraints on

$\mathbf{x}(S)$, $S \subset E - S^{(0)}$, which has the following format:

$$\forall A \subset S^{(0)}, \mathbf{x}(S \cup A) \leq f^{(0)}(S \cup A). \tag{36}$$

Since $S \cap A = \emptyset$, then $\mathbf{x}(S \cup A) = \mathbf{x}(S) + \mathbf{x}(A)$. Consequently, from (36), we have,

$$\forall A \subset S^{(0)}, \mathbf{x}(S) \leq f^{(0)}(S \cup A) - \mathbf{x}(A). \tag{37}$$

Consequently, $\forall\ S \subset E - S^{(0)}$,

$$\mathbf{x}(S) \leq \min_{A \subset S^{(0)}} \{f^{(0)}(S \cup A) - \mathbf{x}(A)\}. \tag{38}$$

We claim that $\min_{A \subset S^{(0)}} \{f^{(0)}(S \cup A) - \mathbf{x}(A)\}$ is equal to $f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)})$. The proof is as follows:

$$\forall A \subset S^{(0)}, \qquad f^{(0)}(S \cup A) - \mathbf{x}(A) \tag{39}$$

$$\geq f^{(0)}(S \cup A) - f^{(0)}(A) \tag{40}$$

$$\geq f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)}). \tag{41}$$

The first inequality relies on the fact that $\forall A,\ \mathbf{x}(A) \leq f^{(0)}(A)$. The second inequality is proven by using (4) and the fact that $A \subset S^{(0)}$ and $S \cap S^{(0)} = \emptyset$. It is easy to check that the above inequalities change to equalities for $A = S^{(0)}$.

Regarding the above statements, for the non-allocated entries of $\mathbf{x}$, we have the following set of constraints,

$$\forall S \subset E - S^{(0)},\ \mathbf{x}(S) \leq f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)}). \tag{42}$$

Let us define $E^{(1)} = E^{(0)} - S^{(0)}$, $f^{(1)}(S) = f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)})$, $\forall S \subset E^{(1)}$. By using Lemma 3, the set of constraints (42) on $E^{(1)}$ defines the polymatroid $\mathcal{B}(f^{(1)}, E^{(1)})$, which is a subset of $\mathcal{B}(f, E)$. Now, we use the same procedure that is applied for $\mathcal{B}(f^{(0)}, E^{(0)})$ over $\mathcal{B}(f^{(1)}, E^{(1)})$, and continue recursively. Therefore, in iteration indexed by α, $\alpha = 0, \ldots, t$, the rates of a subset of coordinates are determined such that the constraints of the polymatroid $\mathcal{B}(f^{(\alpha)}, E^{(\alpha)})$ are not violated. Since $\mathcal{B}(f^{(\alpha)}, E^{(\alpha)}) \subset \mathcal{B}(f, E)$, then $\mathbf{x}^* \in \mathcal{B}(f, E)$. Direct verification proves that $\mathbf{x}^*(E) = f(E)$.

Part (II): We must show that the smallest entries of $\mathbf{x}^*$ is equal to $\min_{S \subset E} \frac{f(S)}{|S|}$. According to the algorithm, for all $i \in E$, we have $x_i^* \in \{m^{(0)}, ..., m^{(t)}\}$. Furthermore, $m^{(0)} = \min_{S \subset E} \frac{f(S)}{|S|}$.

From the algorithm, we have

$$m^{(j)} = \frac{f^{(j)}(S^{(j)})}{|S^{(j)}|} = \min_{S \subset E^{(j)}} \frac{f^{(j)}(S)}{|S|} < \frac{f^{(j)}(S^{(j+1)} \cup S^{(j)})}{|S^{(j+1)} \cup S^{(j)}|} = \frac{f^{(j)}(S^{(j+1)} \cup S^{(j)})}{|S^{(j+1)}| + |S^{(j)}|}. \tag{43}$$

Therefore,

$$m^{(j)} < \tfrac{f^{(j)}(S^{(j+1)} \cup S^{(j)})}{|S^{(j+1)}| + |S^{(j)}|} \quad \Longrightarrow \tag{44}$$

$$m^{(j)} < \tfrac{f^{(j)}(S^{(j+1)} \cup S^{(j)}) - m^{(j)}|S^{(j)}|}{|S^{(j+1)}|} \quad \Longrightarrow \tag{45}$$

$$m^{(j)} < \tfrac{f^{(j)}(S^{(j+1)} \cup S^{(j)}) - f^{(j)}(S^{(j)})}{|S^{(j+1)}|} \quad = m^{(j+1)}, \tag{46}$$

where (46) relies on LHS of (43). Consequently, $m^{(0)} < m^{(1)} < \ldots < m^{(t)}$ and the proof is complete. ∎

The remaining issue in Algorithm III is how to compute $\min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}$. These types of problems are known as geometric minimizations. In order to find the minimizer, the smallest value of β is desirable such that there is a set S with $f(S) = \beta|S|$. For the special case of single antenna Gaussian multiaccess channels, computing such β is very simple. For the general case, β can be computed by Dinkelbach's discrete Newton method as follows [18].

The algorithm is initialized by setting β equal to $f(E)/|E|$, which is an upper bound for optimum β. Then, a minimizer Y of $f(S) - \beta|S|$ is calculated, as will be explained later. Since $f(E) - \beta|E| = 0$, then $f(Y) - \beta|Y| \leq 0$. If $f(Y) - \beta|Y| = 0$, the current β is optimum. If $f(Y) - \beta|Y| < 0$, then we update $\beta = f(Y)/|Y|$, which provides an improved upper bound. By repeating this operation, the optimal value of β will eventually be calculated [18]. It is shown that the number of β visited by the algorithm is at most $|E|$ [18].

Using this approach, the minimization problem $\min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}$ is changed to $\min_{S \subset E, S \neq \emptyset} f(S) - \beta|S|$. By direct verification of (4), it is easy to see that $f(S) - \beta|S|$ is a submodular function. There have been a lot of research on submodular minimization problems [18]–[20]. In [19], [20], the first combinatorial polynomial-time algorithms for solving submodular minimization problems are developed. These algorithms design a strongly polynomial combinatorial algorithm for testing membership in polymatroid polyhedra.

B. Decomposition of the Time-Sharing Problem

In the following, we take advantage of the special properties of $\mathbf{x}^*$ and polymatroids to break down the time-sharing problem to some lower dimensional subproblems. In the previous subsection, a chain of nested polymatroids $\mathcal{B}(f^{(\alpha)}, E^{(\alpha)})$, $\alpha = 0, \dots, t$, is introduced, where $\mathcal{B}(f^{(\alpha-1)}, E^{(\alpha-1)}) \subset \mathcal{B}(f^{(\alpha)}, E^{(\alpha)})$ for $\alpha = 1, \dots, t$. Since $S^{(j)} \subset E^{(j)}$ for $j = 0, \dots, t$ and regarding the definition of polymatroid, $\mathcal{B}(f^{(j)}, S^{(j)})$, $j = 1, \dots, t$, is a polymatroid, which is defined on the dimensions $S^{(j)}$. According to the proof of Theorem 2, the vector $m^{(j)}\mathbf{1}_{|S^{(j)}|} \in \mathcal{B}(f^{(j)}, S^{(j)})$ is on the hyperplane $\mathbf{x}(S^{(j)}) = f(S^{(j)})$. Let $\{\pi^{(j)}_{\gamma_j}, \gamma_j = 1, \dots, |S^{(j)}|!\}$ be the set of all permutations of the set $S^{(j)}$, and $\mathbf{u}^{(j)}(\pi^{(j)}_{\gamma_j})$ be the corner point corresponding to the permutation $\pi^{(j)}_{\gamma_j}$ in the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$. Then, there exist the coefficients $0 \leq \lambda^{(j)}_{\gamma_j} \leq 1$, $\gamma_j = 1, \dots, |S^{(j)}|!$, such that

$$m^{(j)}\mathbf{1}_{|S^{(j)}|} = \sum_{\gamma_j=1}^{|S^{(j)}|!} \lambda^{(j)}_{\gamma_j} \mathbf{u}^{(j)}\big(\pi^{(j)}_{\gamma_j}\big), \tag{47}$$

where

$$\sum_{\gamma_j=1}^{|S^{(j)}|!} \lambda^{(j)}_{\gamma_j} = 1. \tag{48}$$

Note that $E = \bigcup_{j=0}^{t} S^{(j)}$. Consider a permutation $\pi^{(j)}_{\gamma_j}$ as one of the total $|S^{(j)}|!$ permutations of $S^{(j)}$, for $j = 0, \cdots, t$, then the permutation π formed by concatenating these permutations, i.e. $\pi = \left(\pi^{(t)}_{\gamma_t}, \cdots, \pi^{(0)}_{\gamma_0}\right)$, is a permutation on the set E.

Theorem 3 *Consider the permutation* $\pi = \left(\pi^{(t)}_{\gamma_t}, \cdots, \pi^{(0)}_{\gamma_0}\right)$ *of the set* E.

(I) *The corner point corresponding to the permutation* π *in the polymatroid* $\mathcal{B}(f, E)$ *is*

$$v_i(\pi) = u^{(j)}_i(\pi^{(j)}_{\gamma_j}), \quad \text{for} \quad i \in S^{(j)}, \tag{49}$$

where $\mathbf{u}^{(j)}(\pi^{(j)}_{\gamma_j})$ *is the corner point of the polymatroid* $\mathcal{B}(f^{(j)}, S^{(j)})$ *corresponding to the permutation* $\pi^{(j)}_{\gamma_j}$*, and* $u^{(j)}_i(\pi^{(j)}_{\gamma_j})$ *denotes the value of* $\mathbf{u}^{(j)}(\pi^{(j)}_{\gamma_j})$ *over the dimension* i*,* $i \in S^{(j)}$*.*

(II) *The vector* $\mathbf{x}^*$ *is in the convex hull of the set of corner points corresponding to the following set of permutations*

$$\left\{ \left(\pi^{(t)}_{\gamma_t}, \cdots, \pi^{(0)}_{\gamma_0}\right), 1 \leq \gamma_t \leq |S^{(t)}|!, \dots, 1 \leq \gamma_0 \leq |S^{(0)}|! \right\}, \tag{50}$$

where the coefficient of the corner point corresponding to the permutation $\pi = \left(\pi^{(t)}_{\gamma_t}, \cdots, \pi^{(0)}_{\gamma_0}\right)$ *is equal to* $\lambda^{(t)}_{\gamma_t} \dots \lambda^{(0)}_{\gamma_0}$*, i.e.*

$$\mathbf{x}^* = \sum_{\gamma_t=1}^{|S^{(t)}|!} \dots \sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda^{(t)}_{\gamma_t} \dots \lambda^{(0)}_{\gamma_0} \mathbf{v}\big(\big(\pi^{(t)}_{\gamma_t}, \cdots, \pi^{(0)}_{\gamma_0}\big) \big). \tag{51}$$

Proof: **Part (I)** From recursive equation (34), we can show that

$$\text{For } S \in E - \bigcup_{i=0}^{j-1} S^{(i)}, \quad f^{(j)}(S) = f\left(S \cup \left\{\bigcup_{i=0}^{j-1} S^{(i)}\right\}\right) - f\left(\left\{\bigcup_{i=0}^{j-1} S^{(i)}\right\}\right). \tag{52}$$

Consider the permutation $\pi = \left(\pi^{(t)}_{\gamma_t}, \cdots, \pi^{(0)}_{\gamma_0}\right)$. Set $\xi = \sum_{i=1}^{j} |S^{(i)}|$. By using (5) and (52), for $\xi < \kappa \leq \xi + |S^{(j+1)}|$, $v_{\pi(\kappa)}(\pi)$ is equal to

$$v_{\pi(\kappa)}(\pi) = f\left(\{\pi(1), \ldots, \pi(\kappa)\}\right) - f\left(\{\pi(1), \ldots, \pi(\kappa - 1)\}\right) \tag{53}$$

$$= f\left(\left\{\bigcup_{i=0}^{j-1} S^{(i)}, \pi(\xi+1)\ldots, \pi(\kappa)\right\}\right) - f\left(\left\{\bigcup_{i=0}^{j-1} S^{(i)}, \pi(\xi+1), \ldots, \pi(\kappa-1)\right\}\right) \tag{54}$$

$$= f^{(j)}\left(\{\pi(\xi+1)\ldots, \pi(\kappa)\}\right) - f^{(j)}\left(\{\pi(\xi+1)\ldots, \pi(\kappa-1)\}\right). \tag{55}$$

According to definition of polymatroid and its corner points, the RHS of (55) is the value of $u^{(j)}_{\pi(\kappa)}(\pi^{(j)})$ in the corresponding corner point of the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$.

Part (II) Since $\sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda^{(0)}_{\gamma_0} = 1$ and by using (47) and part (I) of the theorem, it is easy to verify that the i^{th}, $i \in S^{(0)}$, entry of

$$\sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda^{(0)}_{\gamma_0} \mathbf{v}\left(\pi^{(t)}_{\gamma_t}, \ldots, \pi^{(0)}_{\gamma_0}\right) \tag{56}$$

is equal to $m^{(0)}$. Similarly, the entry i, $i \in S^{(1)}$, of

$$\sum_{\gamma_1=1}^{|S^{(1)}|!} \lambda^{(1)}_{\gamma_1} \sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda^{(0)}_{\gamma_0} \mathbf{v}\left(\pi^{(t)}_{\gamma_t}, \ldots, \pi^{(0)}_{\gamma_0}\right), \tag{57}$$

is equal to $m^{(1)}$, while the entry i, $i \in S^{(0)}$, remains $m^{(0)}$. By continuing this procedure, part (II) of the algorithm is proven. ■

Regarding the above statements, the problem of finding time-sharing coefficients is decomposed to some lower dimensional subproblems. In each sub-problem, the objective is to find the coefficients of the time-sharing among the corner points of the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$, $j = 0, \ldots, t$, to attain $m^{(j)}\mathbf{1}_{|S^{(j)}|}$. In this part, we present an algorithm which finds the coefficients of the time-sharing over the corner points of a general polymatroid $\mathcal{B}(f, E)$ to attain a vector $\mathbf{x}$ located on the face of the polymatroid.

Algorithm IV

1) Initialize $\alpha = 1$, $\mathbf{u}_1 = \mathbf{v}(\pi^*)$ (the fairest corner point obtained by algorithm I).

2) Solve the linear optimization problem

$$\begin{aligned} &\max \tau \\ s.t. \quad &\textstyle\sum_{i=1}^{\alpha} \mu_i \mathbf{u}_i - \mathbf{x} \geq \tau \\ &0 \leq \mu_i \leq 1 \end{aligned} \tag{58}$$

Let μ_i^{α}, $i = 1, \dots, \alpha$ be the optimizing coefficients.

3) If $\mathbf{x} = \sum_{i=1}^{\alpha} \mu_i^{\alpha} \mathbf{u}_i$, Stop.

4) $\alpha \longleftarrow \alpha + 1$. Set $\mathbf{e} = \mathbf{x} - \sum_{i=1}^{\alpha} \mu_i^{\alpha} \mathbf{u}_i$ and determine the permutation π for which $\mathbf{e}_{\pi(1)} \geq \mathbf{e}_{\pi(2)} \geq \dots \geq \mathbf{e}_{\pi(|E|)}$. Set $\mathbf{u}_{\alpha} = \mathbf{v}(\pi)$ and move to step 2.

The idea behind the algorithm is as follows. In each step, the time-sharing among some corner points is performed. If the resulting vector is equal to $\mathbf{x}$, the answer is obtained; otherwise a permutation π is determined such that $\mathbf{e}_{\pi(1)} \geq \mathbf{e}_{\pi(2)} \geq \dots \geq \mathbf{e}_{\pi(|E|)}$, where the error vector $\mathbf{e}$ represents the difference between the vector $\mathbf{x}$ and resulting vector from time-sharing. We can compensate the error vector $\mathbf{e}$ by including an appropriate corner point in the set of corner points participating in time-sharing. Clearly, the best one to be included is the one which has the highest possible rate for user $\pi(1)$ and lowest possible rate for user $\pi(|E|)$. Apparently, this corner point is $\mathbf{v}(\pi)$, computed by algorithm IV.

Note that Algorithm IV can be applied over the sub-polymatroids $\mathcal{B}(f^{(j)}, S^{(j)})$, $j = 0, \dots, t$, to attain $m^{(j)} \mathbf{1}_{|S^{(j)}|}$ or directly applied over the original polymatroid to attain $\mathbf{x}^*$. If a and $|S^j|$ are relatively small numbers, the decomposition method has less complexity, otherwise applying Algorithm IV over the original problem is less complex.

C. Decomposition of Rate-Splitting Approach

As mentioned, an alternative approach to achieve any rate-vector on the sum-capacity facet of MAC is *rate splitting* [8], [9]. This method is based on splitting all input sources except one into two parts, and treating each spilt input as two virtual inputs (or two virtual users). Thus, there are at most $2a - 1$ virtual users. It is proven that by splitting the sources appropriately and successively decoding virtual users in a suitable order, any point on the sum-capacity facet can be attained.

Similar to the time-sharing part, we prove that to attain the rate vector $\mathbf{x}^*$, the rate-splitting procedure can be decomposed into some lower dimensional subproblems. Consider a MAC, where the capacity region is represented by polymatroid $\mathcal{B}(f, E)$ and the vector $\mathbf{x}^*$, derived in Algorithm III, is on its face. Assume that the users in the set $S^{(j)}$ are decoded before the set of users in $\{S^{(j-1)}, S^{(j-2)}, \ldots, S^{(0)}\}$ and after the users in the set $\{S^{(t)}, \ldots, S^{(j+2)}, S^{(j+1)}\}$ .Therefore, by similar discussion used in (36) to (42), we conclude that the rate of the users in the set $S^{(j)}$ is characterized by the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$, where the rate-vector $m^{(j)}\mathbf{1}_{|S^{(j)}|}$ is on its face. Regarding the results presented in [8], [9], we can attain the rate-vector $m^{(j)}\mathbf{1}_{|S^{(j)}|}$ by properly splitting the sources of all inputs, except for one, in the set $S^{(j)}$ to form $2|S^{(j)}| - 1$ virtual users and by choosing the proper order of the decoding of the virtual users. Consequently, using algorithm V (below), we achieve the rate-vector $\mathbf{x}^*$ in the original polymatroid.

Algorithm V

1) Apply rate-splitting approach to attain the rate-vector $m^{(j)}\mathbf{1}_{|S^{(j)}|}$ on the face of the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$, for $j = 0, \ldots, t$. Therefore, for each j, $0 \leq j \leq t$, at most $2|S^{(j)}| - 1$ virtual users are specified with a specific order of decoding.
2) Starting from $j = t$, decode the virtual users in the set $S^{(j)}$ in the order found in Step 1. Set $j \leftarrow j - 1$. Follow the procedure until $j < 0$.

V. Conclusion

We considered the problem of fairness for a class of systems for which a subset of the capacity region forms a polymatroid structure. The main purpose is to find a point on the sum-capacity facet which satisfies a notion of fairness among active users. This problem is addressed in cases where the complexity of achieving interior points is not feasible, and where the complexity of achieving interior points is feasible. For the first case, the corner point for which the minimum rate of the active users is maximized (max-min corner point) is desired for signaling. A simple greedy algorithm is introduced to find the optimum max-min corner point. For the second case, the polymatroid properties are exploited to locate a rate-vector on the sum-capacity facet which is optimally fair in the sense that the minimum rate among all users is maximized (max-min rate). In the case that the rate of some users can not increase further (attain the max-min value), the algorithm recursively maximizes the minimum rate among the rest of the users. It is shown that the problems of deriving the time-sharing coefficients and rate-spitting scheme can be solved by decomposing the problem to some lower-dimensional subproblems. In addition, a fast algorithm to compute the time-sharing coefficients to attain a general point on the sum-capacity facet is proposed.

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