Optimum non-integer rate allocation using integer programming

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The authors present a new method for optimum rate allocation which does not require any assumption about either the integrality of the allocated bit rate, or on the convexity of the distortion curves. Numerical results are presented for the DCT coding of still images showing a noticeable improvement in the performance with no increase in the complexity.

Problem formulation: Rate allocation is a major concern in any coding scheme where a given total rate must be efficiently distributed among a number of different quantisers. The conventional methods to the rate allocation are based on one of the following two main approaches [1]:

(i) treating the allocated bits (and also the quantiser distortion curves) as being continuous and using a Lagrangian optimisation method to compute the bit allocation

(ii) allocating the available bits in an incremental manner by first assigning zero bit to all the quantisers and then distributing the available bits one by one between the quantisers such that the decrease in the overall distortion due to the allocation of each bit is maximised.

It can be shown that this second method results in the optimum bit allocation if the quantiser distortion curve has a certain convexity property [1, 2]. However, a serious drawback of this method is that it restricts the number of levels allocated to each quantiser to be an integer power of two (integer bit rate). In the following, we present an integer programming formulation of the rate allocation problem which does not require any of these assumptions or have their shortcomings. This method can be also applied when the quantiser partitions are labelled by a set of Huffman codes (with the objective of minimising the distortion subject to having a fixed average bit rate).

Consider a quantisation system which is used to quantise a set of K random variables, say $X^0, X^1, ..., X^{K-1}$, each with zero mean and with variance $E[(X^i)^2]$, $i = 0, 1, ..., K-1$. Assume that X^i is quantised with a scalar quantiser composed of N_i levels where N_i is a non-negative integer. Define the normalised quantiser profile $W_i(N_i)$ as the mean square error incurred in quantising X^i with N_i levels. The gain factor G_i accompanies $W_i(N_i)$ to count for the variance of the input signal. In most cases, the overall performance of the quantisation system is determined by the sum of the distortions associated with different quantisers. The level allocation problem is to find an allocation vector $\vec{N} = (N_0, N_1, ..., N_{K-1})$, which minimises

$$D(\vec{N}) = \sum_{i=0}^{K-1} G_i W_i(N_i) \tag{1}$$

subject to

$$\sum_{i=0}^{K-1} \log_2(N_i) \leq B \qquad 1 \leq N_i \leq u_i \quad i = 0, ..., K-1 \tag{2}$$

where B is the fixed quota of available bits, and u_i values are the upper limits for the admissible number of levels. In all cases, we have selected the u_is at a high enough value such that they do not impose any constraint on the optimum solution.

To formulate the rate allocation problem, we define the binary variables $\delta_i(j)$ as

$$\delta_i(j) = \begin{cases} 1 & \text{if } j \text{ levels are allocated to } i\text{th quantiser} \\ 0 & \text{otherwise} \end{cases}$$

Using this notation, we obtain the following integer programming (IP) formulation for the optimisation problem given in eqns. 1 and 2:

$$\text{minimise } D = \sum_{i=0}^{K-1} \sum_{j=1}^{u_i} \delta_i(j) G_i W_i(j) \tag{3}$$

subject to

$$\sum_{j=1}^{u_i} \delta_i(j) = 1 \quad \forall i \in \{0, ..., K-1\} \tag{4}$$

$$\sum_{i=0}^{K-1} \sum_{j=1}^{u_i} \delta_i(j) \log_2(j) \leq B \tag{5}$$

$$\delta_i(j) \in \{0, 1\} \tag{6}$$

Note that eqn. 4 ensures that each quantiser has only one value for the number of the allocated levels. In the case of using a variable rate code, eqn. 5 is modified to

$$\sum_{i=0}^{K-1} \sum_{j=1}^{u_i} \delta_i(j) L_i(j) \leq B \tag{7}$$

where $L_i(j)$ is the average length of the corresponding Huffman code for the case that j levels are allocated to the ith quantiser.

Numerical results: We have applied the proposed rate allocation method for the quantisation of the DCT coefficients of still images where the corresponding scalar quantisers are designed using the method of [3]. The corresponding integer optimisation problem is solved using software called the general algebraic modelling system (GAMS) version 2.25. Table 1 contains some examples of the results obtained. The numbers can be interpreted as follows: Case a is obtained by allocating B = 0.5, 1, 1.5, 2 bit/pixel using the formulation given in eqns. 3 – 6. Case b is obtained by allocating B = 0.5, 1, 1.5, 2 bit/pixel using the formulation given in eqns. 3 – 6 where the rates allocated are restricted to have integer values. Note that case b corresponds to the traditional bit allocation problem (which is solved here optimally using the proposed integer programming formulation). Finally, case c is obtained by using eqn. 7 as the constraint on the rate, where the value of B is set equal to the average length of a set of Huffman codes designed to label the quantisers obtained as the outcome of case b.

Table 1: Signal-to-noise ratio in dB for different rate allocation methods (for Lena image)

Rate	Case a	Case b	Case c
bit/pixel	dB	dB	dB
0.5	20.0	19.9	20.2
1.0	22.8	22.7	23.1
1.5	24.7	24.6	25.2
2.0	26.3	26.2	27.0

In general, the complexity of solving a linear integer optimisation problem is substantially higher than the complexity of solving the underlying linear program. The important point is that the linear solution of the optimisation problem involved in this work satisfies the corresponding integrality constraints in the majority of the cases (no counter-example observed). It can be shown that in the event of the rare cases that this property does not hold, an integer solution can easily be computed (which is possibly slightly sub-optimal) by simply rounding off the corresponding linear solution. This property of the proposed method is advantageous for applications where the bit allocation problem needs to be solved dynamically (with a small computational complexity).

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