plex centered at the origin. The proof can be carried out inductively with respect to the vector dimension. A constant length s for the edges arises together with a constant length nc_n for the vertices of the simplex.

b) Due to a memoryless Gaussian source, the n-dimensional pdf $f(\underline{x})$ is rotational invariant. By Lemma 1, the centroids of the quantization cells have the same length r_1 and the centroid of Z_N has the same direction as the center of S_N. Let S_N be spanned by $\underline{v}_1(n),\cdots,\underline{v}_n(n)$. Due to (18), S_N lies in the plane $x_n = c_n$ and the center of S_N lies on the x_n axis:

$$\sum_{i=1}^{n} \underline{v}_i(n) = -v_{n+1}(n) = (0, nc_n)^T.$$

Hence ((A1), (A2)),

$$r_1 = \hat{x}_n = (n+1)\int_{[0,\infty)} x_n \int_{\Pi_{n-1}(x_n/c_n S_N)} f(\underline{x})\, dx_1,\cdots, dx_{n-1}\, dx_n. \tag{A4}$$

c) Since S_N is the convex hull of $\underline{v}_1(n),\cdots,\underline{v}_n(n)$, its projection on the $x_1,\cdots,x_{n-1}$ plane is the convex hull of $\underline{v}_1(n-1),\cdots,\underline{v}_n(n-1)$ ((18)), which is the simplex $P_{n-1}(s)$. Hence, $\Pi_{n-1}(x_n/c_n S_N)$ is the simplex $P_{n-1}(s')$ with edge length ((18))

$$s' = x_n/c_n s = x_n(2n(n+1))^{-1/2}. \tag{A5}$$

d) By (A4), (A5), and the statistical independence of x_i,

$$r_1 = (n+1)\int_{[0,\infty)} x_n f_1(x_n) W_{n-1}\left(x_n(2n(n+1))^{-1/2}\right) dx_n \tag{A6}$$

holds with $W_{n-1}(s)$ denoting the probability of $P_{n-1}(s)$.

e) The functions $W_k(s)$ can be evaluated iteratively. For $k > 1$, we have ((18))

$$W_k(s) = \int_{[-kc_k, c_k]} f_1(x_k) \int_{I(P_k(s), x_k)} \cdot f_1(x_1) \cdots f_1(x_{k-1})\, dx_1,\cdots, dx_{k-1}\, dx_k, \tag{A7}$$

with $I(P_k(s), x_k)$ as the intersection of $P_k(s)$ at x_k ((A2)).

With (18), it can be shown that $I(P_k(s), x_k)$ is the simplex $P_{k-1}(s')$. Its edge length s' depends linearly on x_k with $s'(-kc_k) = 0$ and $s'(c_k) = s$. Hence,

$$W_k(s) = \int_{[-kc_k, c_k]} f_1(x_k) W_{k-1}(s(x_k + kc_k) /(c_k + kc_k))\, dx_k, \qquad k > 1. \tag{A8}$$

By (A8) and

$$W_1(s) = \int_{[-s/2, s/2]} f_1(x)\, dx,$$

the functions $W_k(s)$ and, by (A6), r_1, are given by one-dimensional integrals which can be computed numerically.

References

[1] H. S. M. Coxeter, *Regular Polytopes*. New York: Macmillan, 1963.

[2] A. Gersho and B. Ramamurthi, "Classified vector quantization of images," *IEEE Trans. Commun.*, vol. COM-34, pp. 1105–1115, 1986.

[3] P. Meissner, "Composite source models for speech—state estimation and rate-distortion functions," *ITG-Fachber.*, vol. 107, pp. 73–78, 1989.

[4] CCITT Draft revision of Recommendation H.261, Video Codec for audio-visual services at px64 kbit/s, *Signal Processing, Image Com. 2*, pp. 221–239, 1990.

[5] P. Vogel, "Source coding by classification," IEEE Trans. Commun., Second review.

[6] P. J. Sakrison, "A geometric treatment of the source encoding of a Gaussian random variable," *IEEE Trans. Inform. Theory*, vol. IT-14, pp. 481–486, 1968.

[7] N. J. A. Sloane, "Tables of sphere packings and spherical codes," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 327–338, 1981.

[8] R. M. Gray, *Source Coding Theory*. Boston: Kluver academic, 1990.

[9] M. V. Eyuboglu and G. D. Forney, Jr., "Lattice and trellis quantization with lattice- and trellis-bounded code books—High rate theory for memoryless sources," *IEEE Trans. Inform. Theory*, vol. 39, pp. 46–59, Jan. 1993.

[10] R. Laroia and N. Farvardin, "A structured fixed-rate vector quantizer derived from a variable-length scalar quantizer: Part I—Memoryless sources, *IEEE Trans. Inform. Theory*, vol. 39, pp. 851–867, May 1993.

[11] T. R. Fischer, "A pyramid vector quantizer," *IEEE Trans. Inform. Theory*, vol. IT-32, pp. 568–583, 1986.

[12] J. H. Conway and N. J. A. Sloane, "Fast quantizing and decoding algorithms for lattice quantizers and codes," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 227–232, 1982.

[13] W. C. Mitchell, "The number of lattice points in a k-dimensional hypersphere," *Math. Comput.*, vol. 20, pp. 300–310, 1966.

[14] A. Gersho, "Asymptotically optimal block quantization," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 373–380, 1979.

[15] T. J. Goblick and J. L. Holsinger, "Analog source digitization: A comparison between theory and practice," *IEEE Trans. Inform. Theory*, vol. IT-13, pp. 323–326, 1967.

[16] A. Gersho, "Quantization," *IEEE Commun. Soc. Mag.*, pp. 16–29, 1977.

[17] Y. Linde, A. Buzo, and R. M. Gray, "An algorithm for vector quantizer design," *IEEE Trans. Commun.*, vol. COM-28, pp. 84–95, 1980.

[18] A. Gersho, "On the structure of vector quantizers," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 157–166, 1982.

[19] H. Gish and J. N. Pierce, "Asymptotically efficient quantizing," *IEEE Trans. Inform. Theory*, vol. IT-14, pp. 676–683, 1968.

[20] P. Noll and R. Zellinski, "Bounds on quantizer performance in the low bit-rate region," *IEEE Trans. Commun.*, vol. COM-26, pp. 300–304, 1978.

[21] R. M. Gray and E. D. Karnin, "Multiple local optima in vector quantizers," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 256–261, 1982.

Block-based Eigensystem of the $1 \pm D$ and $1 - D^2$ Partial Response Channels

A. K. Khandani and P. Kabal

***Abstract*—We find analytical expressions for the block-based input and output eigenvectors and eigenvalues of the systems with responses $1 \pm D$ and $1 - D^2$. The input eigenvectors form an orthonormal basis which is the optimum modulator for a channel with that transfer function. The output eigenvectors form an orthonormal basis with the same spectral nulls as the corresponding system. This basis can be used to produce line codes with spectral nulls. The eigenvectors are sinusoids.**

Manuscript received November 14, 1991; revised September 20, 1993. This work was supported by the Natural Sciences and Engineering Research Council of Canada. This paper was presented in part at the Twenty-ninth Annual Allerton Conference, October 1991.

A. K. Khandani is with the Department of electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

P. Kabal is with the Department of Electrical Engineering, McGill University, 3480 University, Montreal, Quebec, Canada H3A 2A7.

IEEE Log Number 9404920.

0018-9448/94$04.00

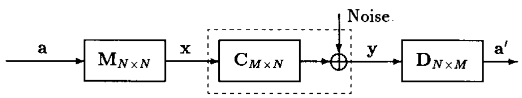


Fig. 1. Block diagram of the transmission system.

This reduces the computational complexity by allowing for fast transform algorithms to perform the modulation for a block of data.

Index Terms—**Block-based eigensystem, partial response channels.**

I. Introduction

We consider the problem of finding the input and the output eigenvectors and the eigenvalues of the block-based $1 \pm D$ and $1 - D^2$ systems. For an $M \times N$-dimensional matrix $\boldsymbol{C}$, the input eigenvectors $\boldsymbol{m}_k$, $k \in [0, N-1]$, are the eigenvectors of $\boldsymbol{C}^t\boldsymbol{C}$ with the eigenvalues ϕ_k^2. The output eigenvectors $\hat{\boldsymbol{m}}_k$, $k \in [0, M-1]$, are the eigenvectors of $\boldsymbol{CC}^t$. Assuming $M > N$, $\boldsymbol{CC}^t$ has N nonzero eigenvalues equal to the same ϕ_k^2's and $M_0 = M - N$ eigenvalues equal to zero. We have

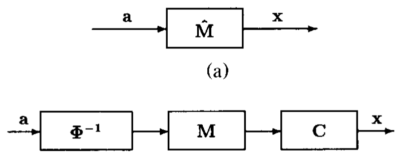
$$\boldsymbol{C}\boldsymbol{m}_k = \phi_k \hat{\boldsymbol{m}}_k,$$
$$\boldsymbol{C}^t \hat{\boldsymbol{m}}_k = \phi_k \boldsymbol{m}_k. \tag{1}$$

Since $\boldsymbol{C}^t\boldsymbol{C}$ and $\boldsymbol{CC}^t$ are both symmetric, the input and the output eigenvectors form an orthonormal basis denoted by $\boldsymbol{M}$ and $\hat{\boldsymbol{M}}$, respectively. In the following, we discuss two applications for the eigenvectors of the matrix $\boldsymbol{C}$.

The first application involves signaling over a channel with the transfer matrix $\boldsymbol{C}$. The block diagram of the transmission system under consideration is shown in Fig. 1. We use a discrete-time model and block-based processing. The block length is equal to N. Each block invokes $M = N + M_0$ channel uses, where M_0 is the memory length of the channel. M_0 zeros are transmitted between successive blocks; as a result, each block starts with zero initial conditions. Modulator matrix $\boldsymbol{M}$ is the basis for the given constellation at the channel input. The additive noise is white Gaussian with zero mean and unit variance. The demodulator matrix $\boldsymbol{D}$ is selected such that $\boldsymbol{DCM} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the $N \times N$ identity matrix. This results in a unity gain N-dimensional channel with additive Gaussian noise whose autocorrelation matrix is $\boldsymbol{DD}^t$. We assume that the decisions along different dimensions of the channel are made independently. In this case, the effective noise along the kth dimension, $k = 0, \cdots, N-1$, is a Gaussian random process with power σ_k^2, where σ_k^2 is the kth diagonal element of the matrix $\boldsymbol{DD}^t$.

It can be shown that the input eigenvectors of $\boldsymbol{C}$ are the optimum modulating basis at the channel input [1]. This basis minimizes the product of the noise powers along different dimensions. In this case, for a given total rate and given minimum distance-to-noise ratio at the demodulator output, the volume of the signal space and, consequently, the required energy at the channel input, are minimized.

The second application involves line coding. The output eigenvectors of a system $\boldsymbol{C}$ form an orthonormal basis with the same spectral nulls as the system. This basis can be used to produce line codes with spectral nulls. Fig. 2(a) shows the block diagram of such a line coder. Considering (1), multiplication (modulation) by $\hat{\boldsymbol{M}}$ can be achieved using the system shown in Fig. 2(b), where $\boldsymbol{\Phi}^{-1}$ is a diagonal matrix with the diagonal elements $1/\phi_k$. As we will see later, for the systems under consideration, modulation by $\boldsymbol{M}$ can be achieved using an even discrete sine transform algorithm. This reduces the computational complexity of a realization based on Fig. 2(b).



(b)

Fig. 2. (a) Block diagram of the line coder. (b) An equivalent form for (a).

The $1 - D$, $1 + D$, and $1 - D^2$ systems have special importance in partial response signaling [2]. The $1 - D$ system has a spectral null at zero frequency, $1 + D$ has a spectral null at the Nyquist frequency, and $1 - D^2$ has spectral nulls at both zero and the Nyquist frequencies. Let the energy per channel use be normalized to 1. The $1 \pm D$ systems have an $(N+1) \times N$-dimensional transfer matrix with ith column, $i = 0, \cdots, N-1$, equal to $[(0)^i, \sqrt{2}/2, \pm\sqrt{2}/2, (0)^{N-1-i}]^t$. The $1 - D^2$ system has an $(N+2) \times N$-dimensional transfer matrix with ith column, $i = 0, \cdots, N-1$, equal to $[(0)^i, \sqrt{2}/2, 0, -\sqrt{2}/2, (0)^{N-1-i}]^t$.

II. Eigensystem of the $1 \pm$ D and $1 - \text{D}^2$ Systems

The input eigenvectors of the $1 - D$ system are equal to

$$m_k(n) = \sqrt{\frac{2}{N+1}} \sin \frac{\pi(k+1)(n+1)}{N+1}, \qquad k, n = 0, \cdots, N-1. \tag{2}$$

The corresponding eigenvalues are given by

$$\phi_k^2 = 1 - \cos \frac{\pi(k+1)}{N+1}. \tag{3}$$

This can be verified by considering (2) as a periodic function with period $N + 1$. This function is zero at $n = i(N+1) - 1$, $\forall i$. This means that the signal itself provides zero initial conditions for the N-dimensional blocks. Consequently, the response of the system in each block is equal to its steady state. Note that in steady state, a sinusoid is the input eigenfunction of any linear system.

To give a formal proof, we consider $\boldsymbol{C}^t\boldsymbol{C}$ as the transfer matrix of a linear time-invariant system with the transfer function $H(D) = 0.5(1 - D)(1 - D^{-1})$. This is the transform of $c(n) * c(-n)$, where $c(n) = \{1/\sqrt{2}, -1/\sqrt{2}\}$ is the impulse response of the $1 - D$ system (power is normalized to 1) and $*$ denotes the convolution. To have consistency with the block-based processing, we apply a causal input and truncate the output of positive time. In this case, if $m(n)$ is an input eigenvector and $M(D)$ is its transform, we have

$$H(D)[M(D) - m(0)] + m(0)(1 - 0.5D) = \phi^2 M(D). \tag{4}$$

Calculating (4) at time zero results in

$$\phi^2 = 1 - 0.5\frac{m(1)}{m(0)}. \tag{5}$$

Substituting $H(D) = 0.5(1 - D)(1 - D^{-1})$ and (5) in (4) results in

$$M(D) = \frac{m(0)}{1 - \dfrac{m(1)}{m(0)}D + D^2}. \tag{6}$$

Eqs. (6) and (5) can be satisfied by the eigenvectors and eigenvalues given in (2) and (3). Using (2) in (1), the output eigenvec-

After this paper was accepted for publication, we became aware of the work of Honig *et al.* [4], which presents a result similar to (2).

tors of the $1 - D$ system are found as

$$\hat{m}_k(n) = \sqrt{\frac{2}{N+1}} \cos \frac{\pi(k+1)(2n+1)}{2(N+1)}, \quad n = 0, \cdots, N, \; k = 0, \cdots, N-1. \quad (7)$$

The input and output eigenvectors of the $1 + D$ system are obtained by multiplying (2) and (7) with $(-1)^n$. The eigenvalues of the $1 + D$ system are the same as the $1 - D$ system given in (3).

An N-dimensional $1 - D^2$ channel, N even, can be considered as two time-multiplexed $N/2$-dimensional $1 - D$ channels. Consequently, the eigenvalues are in pair equal to

$$\phi_k^2 = 1 - \cos \frac{\pi(k+1)}{(N/2)+1}, \qquad k = 0, \cdots, (N/2) - 1. \quad (8)$$

The two eigenvectors corresponding to a pair of eigenvalues are of the general form $\alpha_1 m_k(2n) + \alpha_2 m_k(2n+1)$, where $\alpha_1^2 + \alpha_2^2 = 1$ and $m_k(n)$ is the eigenvector of the $1 - D$ channel given in (2).

The product of the nonzero eigenvalues of C is equal to

$$\prod_{k=0}^{N-1} \phi_k^2 = |C^t C|, \quad (9)$$

where $|C^t C|$ is the determinant of $C^t C$. This product is an important parameter of the systems based on C. For example, in the transmission system shown in Fig. (1), the volume of the Voronoi region around each constellation point at the channel input is proportional to $(\prod_k \phi_k)^{-1}$ and the required energy is proportional to $(\prod_k \phi_k)^{-2/N}$.

For the $1 \pm D$ channels, assuming $|C^t C| = 2^{-N} \times A_N$ and expanding the determinant, we obtain $A_N = A_{N-1} + 1$. Solving this recursive equation with the initial value $A_1 = 2$ results in $A_N = N + 1$. Consequently, for the $1 \pm D$ channels, we have

$$\prod_{k=0}^{N-1} \phi_k^2 = 2^{-N} \times (N+1). \quad (10)$$

For the $1 - D^2$ channel, we have

$$\prod_{k=0}^{N-1} \phi_k^2 = 2^{-N} \times [(N/2) + 1]^2. \quad (11)$$

For all three channels, modulation with the input eigenvectors can be performed by using the even discrete sine transform. For modulation with the output eigenvectors, we can use the block diagram shown in Fig. (2). Using (7), modulation with the output eigenvectors can also be achieved using an $N + 1$-point even discrete cosine transform. In this case, samples of the modulating vector are shifted by one sample and filled with zero. Ref. [3] shows how both the discrete sine transform and the discrete cosine transform can be efficiently calculated.

III. Summary

The input and output eigenvectors and the eigenvalues of the $1 \pm D$ and $1 - D^2$ systems are calculated. The product of the nonzero eigenvalues are found in closed form. In all cases, the multiplication by the input or output eigenvectors can be achieved by using fast transform algorithms.

References

[1] S. Kasturia, J. T. Aslanis, and J. M. Cioffi, "Vector coding for partial response channels," *IEEE Trans. Inform. Theory*, vol. 36, pp. 741–762, July 1990.

[2] P. Kabal and S. Pasupathy, "Partial-response signaling," *IEEE Trans. Commun.*, vol. 23, pp. 921–934, Sept. 1975.

[3] A. K. Jain, "A sinusoidal family of unitary transforms," *IEEE Trans. Pattern Anal. Machine Intell.*, vol. PAMI-1, no. 4, pp. 356–365, Oct. 1979.

[4] M. L. Honig, K. Steiglitz, and S. A. Norman, "Optimization of signal sets for partial-response channels—Part I," *IEEE Trans. Inform. Theory*, vol. 37, pp. 1327–1341, Sept. 1991.

Number of Nearest Neighbors in a Euclidean Code

Kenneth Zeger and Allen Gersho

Abstract—A Euclidean code is a finite set of points in n-dimensional Euclidean space $\mathscr{R}^n$. The total number of nearest neighbors of a given codepoint in the code is called its touching number. We show that the maximum number of codepoints F_n that can share the same nearest-neighbor codepoint is equal to the maximum kissing number τ_n in n dimensions, that is, the maximum number of unit spheres that can touch a given unit sphere without overlapping. We then apply a known upper bound on τ_n to obtain $F_n \le 2^{n(0.401 + o(1))}$, which improves upon the best known upper bound of $F_n \le 2^{n(1 + o(1))}$. We also show that the average touching number T of all the points in a Euclidean code is upper bounded by τ_n.

I. Introduction

A *Euclidean code* is a finite set Y of $M > 1$ points in n-dimensional Euclidean space $\mathscr{R}^n$. A vector quantizer codebook and a code (or signal constellation) for the Gaussian channel are both examples of Euclidean codes. In both cases, the nearest-neighbor partition (also known as the *Voronoi* partition) of the space induced by the code is of particular importance in evaluating the performance of the code. For vector quantizers, a source vector is encoded by identifying in which region of the partition it lies. For Gaussian channels, a selected codepoint is corrupted by an additive Gaussian noise vector and the maximum *a posteriori* decoder identifies in which region of the Voronoi partition the resulting vector lies.

A special case of a Euclidean code is a *uniform code* (e.g., a lattice code), defined by the property that every codepoint has the same nearest-neighbor distance, $d_{\min}$. Each point of a uniform code can be viewed as the center of a sphere of radius $r_0 = d_{\min}/2$ so that each sphere is contained in the closure of a nearest-neighbor region.

The nearest-neighbor region (or Voronoi cell) of a given point α in a Euclidean code is the set of points in $\mathscr{R}^n$ closer to α than to any other codepoint. This region is a convex set bounded by faces of dimension $n - 1$ that are subsets of hyperplanes. Each such hyperplane is defined as the locus of points equidistant from α and some other codepoint β.

Manuscript received September 23, 1993; revised February 1, 1994. This research was supported in part by the National Science Foundation under Grants NCR-92-96231 and NCR-89-14741. This paper was presented in part at the IEEE International Symposium on Information Theory, Trondheim, Norway, June 1994.

K. Zeger is with the Coordinated Science Laboratory, Department of Electrical Engineering, University of Illinois, 1308 West Main Street, Urbana, IL 61801.

A. Gersho is with the Department of Electrical and Computer Engineering University of California, Santa Barbara, CA 93106.

IEEE Log Number 9405094.