Shaping Multidimensional Signal Spaces—Part II: Shell-Addressed Constellations

Amir K. Khandani and Peter Kabal

Abstract—**By appropriately selecting the boundary of a multidimensional signal constellation used for data transmission, the average energy of the constellation can be reduced. Reduction in the average energy (shaping gain) is obtained at the price of increasing the constellation-expansion ratio (CER_s) and the peak-to-average-power ratio (PAR). In this paper, we describe some practical means to select the boundary so as to achieve a point with low addressing complexity near the knee of the corresponding tradeoff curves (shaping gain versus CER_s or PAR). One class of addressing schemes is based on using a lookup table. We introduce a method to facilitate the realization of the addressing lookup table. This method is based on the decomposition of the addressing into a hierarchy of addressing steps, each of a low complexity. This avoids the exponential growth of the complexity. Based on this addressing decomposition, by using a memory of a practical size, we can move along a tradeoff curve which has negligible suboptimality. Another class of addressing schemes is based on using a Voronoi constellation in a space of half the original dimensionality. We also introduce hybrid multilevel addressing schemes which combine the two classes. These schemes provide single points with moderate addressing complexity near the knee of the optimum tradeoff curves.**

Index Terms—**Shell addressing, shell index, half integer grid, truncated cube, addressing lookup table, addressing decomposition, shell-addressed Voronoi constellation.**

I. Introduction

By appropriately selecting the boundary of a multidimensional signal constellation used for data transmission, the average energy of the constellation can be reduced. Reduction in the average energy is measured by the shaping gain γ_s. The constellation-expansion ratio (CER_s) is defined as the ratio of the employed number of points per two dimensions to the minimum necessary number of points per two dimensions [1]. The peak-to-average-power ratio (PAR) is defined as the ratio of the peak of energy per two dimensions to the average energy per two dimensions [1]. For a fixed transmission rate, the shaping gain must be traded off against CER_s and PAR. The structure of boundary optimizing these tradeoffs is introduced in [2]. In general, the initial portion of the optimum tradeoff curves has a steep slope. This means that an appreciable part of the maximum shaping gain, corresponding to a spherical boundary, can be achieved with a relatively small CER_s, PAR. In this paper, we describe some practical means to select the boundary so as to achieve a point with low addressing complexity near the knee of the optimum tradeoff curves.

In general terms, our shaping methods work as follows. The 2-D subconstellations are partitioned into a set of concentric shells, each with an equal number of signal points. We treat all the points within a shell as having equal energy. These shells are indexed in the radial direction by $0, 1, \cdots$. Assuming continuous approximation in 2-D, the average energy of a shell is proportional to its index plus a fixed offset. The shell indexes can be thought as a set of integer points in a 1-D space. Then, a block of n shell indexes corresponds to a point in an n-D space formed by taking the Cartesian product of the 1-D spaces. Shaping is achieved by selecting a subset of these n-D points. The optimum subset (corresponding to the shell sequences of the least average energy) is composed of the points with the least sum of the coordinates.

Concerning the addressing (assignment of data bits to constellation points), we discuss using a lookup table to move along the optimum tradeoff curves, and introduce a method to facilitate the realization of this lookup table. This method is based on the decomposition of addressing into a hierarchy of the addressing steps, each of a low complexity. This decomposition avoids the exponential growth of the complexity. Based on this decomposition, by using a memory of a practical size, we can move along a tradeoff curve which has negligible suboptimality.

Another class of addressing schemes is based on using a Voronoi constellation in a space of half the original dimensionality. Based on this approach, we introduce a method to achieve a single point on the optimum tradeoff curves. The point achieved has significant shaping gain with low addressing complexity and low CER_s, PAR. Finally, in a multilevel addressing scheme, we combine the Voronoi constellations of the previous method with a lookup table to move along a tradeoff curve which is nearly optimum. To further reduce the complexity of this method, we replace the lookup table by a Voronoi constellation, and thereby achieve a single point near the knee of the optimum tradeoff curves.

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A. K. Khandani is with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.

P. Kabal is with the Department of Electrical Engineering McGill University, Montreal, P.Q., Canada, H3A 2A7, and INRS—Telecommunications (Université du Québec), Verdun, P.Q., Canada H3E 1H6.

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We use both continuous approximation and discrete analysis to calculate the performance of our schemes. In calculating the shaping performance, we usually give expressions just for γ_s and CER_s. The corresponding PAR can be calculated using [2, eq. (6)].

Discrete cube / symmetrical discrete cube $C_n(K)$ / $SC_n(K)$: $C_n(K)$ is equal to the Cartesian produce $\{C_1(K)\}^n$ where $C_1(K) = \{J + 0.5,\ J = 0,\cdots, K-1\}$. The cardinality of $C_n(K)$ is equal to $|C_n(K)| = |C_1(K)|^n = K^n$. The point $\vec{J} + (0.5)^n \in C_n(K)$ is indexed by $\vec{J} = (J_0,\cdots, J_{n-1})$.

Similarly, $SC_n(K) = \{SC_1(K)\}^n$ where $SC_1(K) = \{\pm(J + 0.5),\ J = 0,\cdots, K-1\}$.

First moment shell of $C_n(K)$: $F_n(K, L)$ is the first moment shell of $C_n(K)$, i.e.,

$$F_n(K,L) \equiv \left\{ \vec{J} + (0.5)^n \in C_n(K): \sum_{p=0}^{n-1} J_p = L \right\}. \quad (1)$$

Truncated discrete cube $TC_n(K, L, T)$: $TC_n(K, L, T)$ is the set of the first moment shells $F_n(K, l)$, $0 \le l \le L-1$, and a selected subset of $F_n(K, L)$ such that $|TC_n(K, L, T)| = T$. We use the notation $TC_n(K, L)$ when all the points of $F_n(K, L)$ are included in TC_n. It is easy to show that

$$|F_n(K,L)| = \sum_{l=0}^{K-1} |F_{n-1}(K, L-l)|$$

$$|TC_n(K,L)| = \sum_{l=0}^{L} |F_n(K,l)|. \quad (2)$$

Note that the $|F_n(K, l)|$'s are the coefficients of the generating function $[g_1(z, K)]^n$ where $g_1(z, K) = 1 + z + \cdots + z^{K-1}$. Using this fact, most of the subsequent results can be expressed in terms of generating functions. However, we prefer the F_n, TC_n notation because it provides a stronger geometrical interpretation.

II. Shell-Addressed Constellations A_N

A. Basic Structure

The A_N constellations are based on a shaping region as close as possible to the optimum shaping region $\mathscr{A}_N(\psi)$ introduced in [2]. We use the notation $\mathscr{S}_N(R)$ to denote an N-D (N-dimensional) sphere of radius R. In the $\mathscr{A}_N(\psi)$ region, an $\mathscr{S}_2(R)$ is the boundary of the 2-D subspaces and an $\mathscr{S}_N(\sqrt{n\psi}R)$, $n = N/2$ is the boundary of the whole space. In [2], by a change of variable denoted as the shell mapping, the energy shells of $\mathscr{S}_2$'s are mapped to points along a dimension. As a result, the N-D space (N-domain) is mapped to $n = N/2$-D space (n-domain). The $\mathscr{A}_N$ region is mapped to an n-D hypercube of edge length one truncated by a simplex of edge length $n\psi$. This is denoted as $\mathscr{TC}_n(1, n\psi)$. This mapping has a useful property that a uniform density of points within $\mathscr{A}_N$ results in a uniform density of points within $\mathscr{TC}_n$. Using this property, shaping is achieved by partitioning the n-domain into equal volume partitions, and then selecting a subset of these partitions. For $\psi = 1/2$, we have another useful property that $\mathscr{TC}_n(1, n/2)$ is equal to the Voronoi region of the lattice D_n^* in the positive coordinates. In this paper, this property is used to facilitate the addressing as well as the partitioning of the n-domain.

In the following, the idea of shell mapping is extended to the discrete case. This is achieved in two steps. In the first step, the region $\mathscr{TC}_n$ is replaced by the discrete set TC_n. In the second step, the circular region $\mathscr{S}_2(R)$ is replaced by the circular constellation $S_2(M)$ where M denotes the cardinality.

Step I: Using K concentric circles, each $\mathscr{S}_2(R)$ is partitioned into K shells of equal volume. These are indexed by $J = 0,\cdots, K-1$. The outer radius and the average energy of the Jth shell satisfy $R(J) = \sqrt{J+1}\,\Delta R$ where $\Delta R = R/\sqrt{K}$ and $E(J) = (\Delta R)^2(J + 0.5)$. The shells are mapped to the points $Y = J + 0.5$. This results in the set $C_n(K)$ in the n-domain. Each point of $C_n(K)$ corresponds to a shaping cluster of volume $\pi^n(\Delta R)^N$ in the N-domain. The average energy of the cluster indexed by $\vec{J}$ is $E(\vec{J}) = (\Delta R)^2(0.5n + \Sigma_p J_p)$. This means that the points located on $F_n(K, l)$'s represent the clusters with equal average energy. Using this fact, shaping is *optimally* achieved by selecting a subset of $C_n(K)$ with the least $\Sigma_p J_p$ (corresponding to clusters with the least average energy). This results in the shaping set $TC_n(K, L)$ in the n-domain. The overall shaping is optimum to the extent that the resolution of the partitioning of the 2-D subspaces (K) allows.

It is easy to show that assuming $R = 1$, the average energy per two dimensions of the selected subset is equal to

$$P_2 = \frac{1}{nK|TC_n(K,L)|}\left[\frac{n}{2}|TC_n(K,L)| + \sum_{l=1}^{L} l|F_n(K,l)|\right]. \quad (3)$$

This is used to calculate the tradeoff curves given in Fig. 1, $\mathrm{CER}_s = K/|TC_n(K, L)|^{1/n}$. The two discrete sets of points correspond to the discrete analysis with $M = 128$ points per two dimensions. The computational method will be explained later. The optimum curves are extracted from [2].

The lookup table is a block of $|TC_n(K, L)|$ memory locations, each with $n\lceil \log_2 K \rceil$ bits. Fig. 2 shows the tradeoff between the γ_s and the size of the memory. Referring to Fig. 1, for small values of CER_s (which are of practical interest), $K = 4$ achieves almost all the shaping gain. This is the same effect for the first time in the pioneering work of [3]. However, referring to Fig. 2, $K = 4$ results in a substantial decrease in the memory size compared to $K = 8$.

Step II: We assume that the projection of the constellation on the 2-D subspaces is a finite portion of $Z^2 + (1/2)^2$ where $Z^N + (1/2)^N$ denotes the N-D half integer grid. This is the case for most of the known TCM (trellis-coded modulation) schemes, including the schemes of [4]. The points of $Z^2 + (1/2)^2$ are grouped in the order of increasing energy into K energy shells, each with $2^p = M/K$ points. Each shell has four-way symmetry, and con-

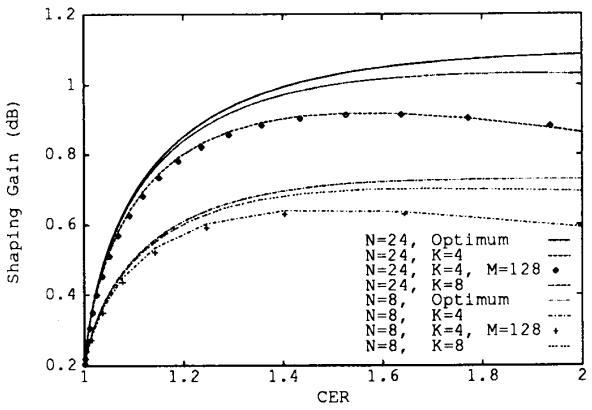

Fig. 1. Tradeoff between CER_s and γ_s in A_N constellations, $N = 8, 24$, $K = 4, 8$.

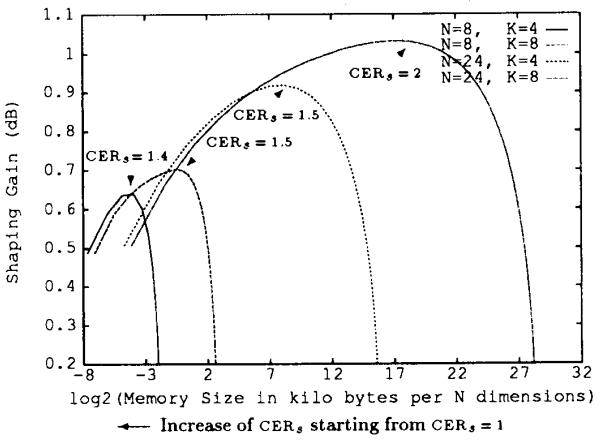

Fig. 2. Tradeoff between the memory size and γ_s in A_N constellations, $N = 8, 24$, $K = 4, 8$.

tains an equal number of points from each partition in an Ungerboeck partition chain. These are important issues for practical implementation of a multidimensional TCM scheme [4]. The Jth shell is mapped to the point $Y = J + 0.5$ along a dimension. The shaping set in the n-domain is selected as

$$TC_n(K, L, 2^t) = \left\{\vec{Y}: \vec{Y} \in F_n(K, l), l < L \text{ or } \vec{Y} \in F_n(K, L), E(\vec{Y}) \le \Theta\right\} \quad (4)$$

where L selects the first fully filled (first moment) shells and Θ selects the points corresponding to the N-D clusters, with the least average energy on the last shell. This method is not necessarily optimum because, unlike the case of Step I, the points located on the F_n's do not represent the N-D clusters with the same average energy. But, since the energy differences are small, the suboptimality is negligible.

By changing M for a fixed K, we can change the total rate of the constellation for fixed lookup table complexity, fixed CER_s, and *essentially* fixed γ_s. For continuous approximation in the 2-D subspaces (Step I), γ_s remains fixed.

The whole constellation is denoted as $A_N(M, K, 2^t)$. We have $|A_N(M, K, 2^t)| = 2^{t+pn}$ and $\text{CER}_s = K \times 2^{-t/n}$ where $2^p = M/K$ and $n = N/2$. The lookup table has t input and $n\lceil \log_2 K \rceil$ output lines.

Example: Fig. 3 shows the structure of the A_4 constellation, $M = 32$, $K = 4$. The available signal space in the n-domain is the set $C_2(4)$. Each point of the n-domain corresponds to $8 \times 8 = 64$ points in the N-domain. The shaping set in the n-domain is selected as $TC_2(4, L)$, $1 \le L \le 6$. The two dotted lines correspond to shaping sets $TC_2(4, 1)$ and $TC_2(4, 4)$, respectively. For the solid line, we have the Voronoi region of $D_2^* = \Re Z^2$ where $\Re$ denotes the rotational operator [5]. The average energies of the 2-D shells are $\{E(J),\ J = 0, 1, 2, 3\} = \{1.5, 3.5, 6.5, 8.5\}$. Assuming a continuous approximation (Step I), we obtain $(\Delta R)^2 = 8/\pi$ and $\{E(J)\} = 1.27, 3.82, 6.37, 8.91\}$. The difference between the elements of these two sets is the main cause for any suboptimality of TC_n as the shaping set. •

In the following, we introduce some methods to facilitate the realization of the addressing lookup table in the A_N constellations.

B. Structure of the Addressing Lookup Table for A_N Constellations

We assume that the shaping set in the n-domain is equal to $TC_n(K, L) \subset C_n(K)$. We use a property of $TC_n(K, L)$ to simplify the realization of the addressing lookup table. In general, the projection of $TC_n(K, L)$ on any m-D subspace is equal to the set $TC_m(K, L)$. This property of the TC_n set is the basis for a recursive addressing scheme proposed in [6]. In the present work, we partition the projections on the 2-D subspaces, which are TC_2's, into subsets such that the addressing can be achieved directly on them. This results in a logical table with many "don't care" entries. We first show that for a uniform probability density on the points of A_N (which results in a uniform probability density on the points of TC_n), the induced probability density on the TC_2's depends only on the sum of the coordinates. This means that the points of F_2's are used with equal probability. This allows us to address the points of the F_2's with a fixed length code.

The dimensions of the n-domain are labeled by $Y_0, \cdots, Y_{n-1}$. To compute the induced probability density on the points of $TC_2(K, L)$ in the Y_0, Y_1 subspace, we draw from every point $(Y_2, \cdots, Y_{n-1}) \in TC_{n-2}(K, L)$, a 2-D plane parallel to the Y_0, Y_1 subspace and find the part of it which is located inside $TC_n(K, L)$. The intersection of such a plane with $TC_n(K, L)$ is the set $TC_2(K, L - \sum_{p=2}^{n-1} J_p)$ where $J_p = Y_p - 0.5$. The points of this set are mapped to the Y_0, Y_1 subspace. By counting the number of times that a given point is used, the induced probability

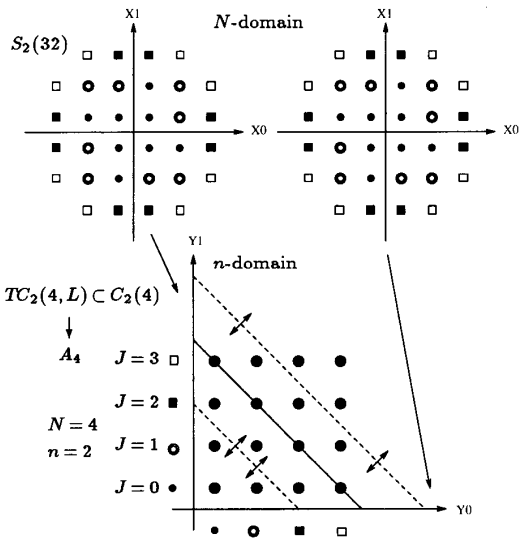

Fig. 3. Example of A_4 constellation.

density is calculated. The total number of times that the set $F_2(K, b)$ is used is equal to the total number of times that the sets $TC_2(K, a)$, $a \geq b$ are used. The number of times that the set $TC_2(K, a)$ is used is equal to $|F_{n-2}(K, L - a)|$. Using this fact, the frequency of $F_2(K, b)$ is found as

$$N_2(b) = \sum_{b \leq a \leq L} |F_{n-2}(K, L-a)| = \sum_{l=0}^{L-b} |F_{n-2}(K, l)| = |TC_{n-2}(K, L-b)|. \quad (5)$$

From (5), it is seen that for fixed K and fixed L, the frequency of the points of $F_2(K, b)$ are equal. This means that if we partition the 2-D subspaces of $TC_n(K, L)$ into $F_2(K, b)$'s $b = 0, \cdots, \min(2K-2, L)$, the set $TC_n(K, L)$ can be expressed as a subset of the $n/2$-fold Cartesian product of these partitions. In practice, to avoid complicated mathematical operations in the following addressing steps, we further partition each $F_2(K, b)$ such that the number of points in each of the final partitions is an integral power of two.

The computational approach presented here can be used to compute the shaping performance for a given number of energy shells per 2-D (K). The procedure is as follows. Using the same reasoning as used in deriving (5), the frequencies of the 2-D shells are found as

$$N_1(J) = |TC_{n-1}(K, L-J)|. \quad (6)$$

The corresponding probabilities are

$$P(J) = \frac{|TC_{n-1}(K, L-J)|}{\sum_{J=0}^{K-1} |TC_{n-1}(K, L-J)|}, \quad (7)$$

which results in the average energy per two dimensions $P_2 = \sum_{J=0}^{K-1} P(J)E(J)$. This is used to calculate the discrete set of points in Fig. 1.

Example: In the constellation $A_N(M, K, 2^t) = A_8(128, 4, 64)$, the 2-D subconstellation of 128 point (which is selected from the cross constellations [7]) is partitioned into $K = 4$ shells, each of $2^p = 32$ points. The average energy of the 2-D shells is equal to $\{E(J), J = 0, 1, 2, 3\} = \{5, 15.5, 25.5, 36\}$. A circular 2-D subconstellation would result in $\{E(J), J = 0, 1, 2, 3\} = \{5, 15.5, 25.5, 35.75\}$. Assuming continuous approximation over a 2-D circular region, we obtain $\{E(J)\} = \{5.09, 15.28, 25.47, 35.65\}$. There are $K^n = 4^4 = 256$, N-D partitons, and shaping is achieved by selecting $2^t = 64$ of them. We have $|A_N| = 2^{26}$ and $r_s = 2$ ($\text{CER}_s = 1.41$). The lookup table has six input and eight output lines (bits). The output lines are divided into four groups, each with two lines. These are used to select a shell within each 2-D subspace. Another group of 20 lines, divided into four groups of 5 lines, are used to select a point within each of the selected 2-D shells.

To specify the shaping set, namely $TC_4(4, L, 64)$, we need to find L. Using (2), we obtain

$$\{|F_4(4, l)|\} = \{1, 4, 10, 20, 31, 40, 44, 40, 31, 20, 10, 4, 1, 0, 0, \cdots\}. \quad (8)$$

It is seen that $\sum_{l=0}^{4} F_4(4, l) = 66$. This means that $L = 4$, and only 29 points of the 31 points in $F_4(4, 4)$ are included in TC_4. In the following, we discuss how to select the subset of $F_4(4, 4)$.

If all the points of $F_4(4, 4)$ were included, the frequencies of the 2-D shells would be $\{N_1(J), J = 0, 1, 2, 3\} = \{32, 20, 10, 4\}$. The two points of $F_4(4, 4)$ with the highest $E(\vec{J})$ are the points $\vec{J} = (J_0, J_1, J_2, J_3) = (1, 1, 1, 1)$ with $E(\vec{J}) = 62$ and the point $\vec{J} = (0, 1, 1, 2)$ with $E(\vec{J}) = 61.5$.[1] If we discard these two points, the induced probability density on different dimensions of $TC_4(4, 4, 64)$ will no longer be the same. For the first dimension, the frequencies are $\{31, 19, 10, 4\}$, and for the second, third, and fourth dimensions, $\{32, 18, 10, 4\}$, $\{32, 18, 10, 4\}$, and $\{32, 19, 9, 4\}$, respectively. This results in average energies of 13.258, 13.094, 13.094, and 12.938 along the first–fourth 2-D subspaces. The overall average energy is $P_2 = 13.096$, resulting in $\gamma_s = 0.614$ dB. Using continuous approximation, the maximum shaping gain ($K = \infty$) for $N = 8$ and $\text{CER}_s = 1.41$ is $\gamma_s = 0.698$ dB [2].

The finest partitioning is obtained by $K = 32$ ($P = 4$), which results in $\gamma_s = 0.727$ dB. This requires a lookup table with 18 input and 20 output lines. The size of the memory with respect to $K = 4$ (six input and eight output lines) has increased by the multiplicative factor 10240. As a result of this large increase in the complexity, the shaping gain has increased by about 0.1 dB.

To build up the lookup table, we partition $F_2(4, l)$, $l = 0, \cdots, 4$ of $TC_2(4, 4)$ into subsets, each with an integral bit rate as in Fig. 4.The first moment shells are denoted as

[1] There are 24 points with $E(\vec{J}) = 61.5$. These are the points indexed by the permutations of (0, 1, 1, 2) and (0, 0, 1, 3), 12 points from each.

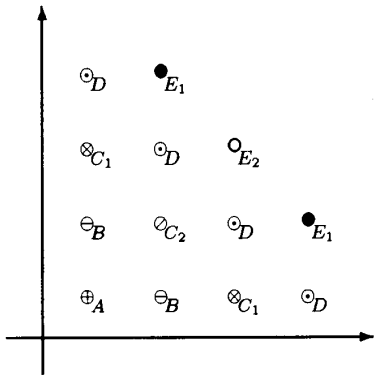

Fig. 4. The $TC_2(4,4)$ partitioned into the addressing subsets.

A, B, C, D, and E. The addressing subsets have the same subscript and also the same sign. If $F_4(4,4)$ were completely included in TC_4, we could specify all the points of TC_4 with this partitioning. For this example, we need a finer partitioning. This can be avoided by discarding the point indexed by $(0,0,2,2)$ with $E = 61$[2] instead of the point indexed by $(1,1,2,0)$ with $E = 61.5$. This results in the average energy per two dimensions $P_2 = 13.098$. The loss in the shaping gain compared to the previous case ($P_2 = 13.096$) is negligible. Table I shows the index vector of the points of TC_4. The indexes are obtained from the permutations of the given vectors. It also gives the number of points obtained from each vector $\mathcal{N}$, and the average energy of the corresponding N-D clusters E.

We reach the conclusion that the points of $F_n(K,L)$ represent the clusters with almost the same E, and selecting an arbitrary subset of them results in negligible degradation. This subset should be selected on the basis of the addressing complexity.

Table II contains a prefix code for the addressing in the n-domain. The "don't care" entries can be used to construct a logical table of reduced complexity. Obviously, for a constellation like $A_{24}(128,4,2^{22})$ which needs a lookup table with 22 input and 24 output lines, the effect of the "don't care" entries will be more pronounced.

Although the method described here can be used to reduce the complexity of the addressing lookup table, for some applications, the complexity may still remain impractical. In the following section, by recursively applying the same idea to higher dimensional constituent subspaces, we develop a method to further reduce the complexity. This method makes use of a set of prefix coding schemes to decompose the addressing into lower dimensional subspaces.

C. Prefix-Coded Addressing Decomposition

Consider the shaping set $TC_m(K,L,2^t)$ where m is an integral power of two. The $F_{m'}(K,l)$'s, $l = 0,\cdots,L$, $m' = m/2$ are partitioned into blocks such that each block has an integral bit rate and the number of blocks is minimum. Each block of $F_{m'}(K,l)$ corresponds to a nonzero element in the binary expansion of $|F_{m'}(K,l)|$. The Cartesian product of a block of $F_{m'}(K,l_1)$ with a block of $F_{m'}(K,l_2)$ results in a cluster (of integral bit rate) in $F_m(K,l_1+l_2)$. For a uniform density on the 2^t points of TC_m, it is easy to show that the m-D clusters can be labeled with a prefix code. Based on this prefix code, part of the input data stream is used to select one of the m-D clusters. The remaining bits, up to a total of t bits, are split into two parts, with the lengths equal to the bit rate of the corresponding m'-D blocks (components of the selected m-D cluster). These bits are subsequently used to select a point within those m'-D blocks.

The $m'' = m'/2 = m/4$-D constituent subspaces are similarly partitioned into blocks of integral bit rates. The m'-D blocks are expressed as the union of the m'-D clusters where, in turn, each m'-D cluster is an element in the Cartesian product of the $m'' = m'/2$-D blocks. This means that the m'-D blocks are the union of the m'-D clusters where each m'-D cluster is obtained by concatenating two $m'' = m'/2$-D blocks. This results in a recursive partitioning structure. To complete the recursion, we need a means to select a single m'-D cluster within each m'-D block. This is achieved by assigning an independent prefix coding scheme to each m'-D block and using its codewords to address the corresponding m'-D clusters.

To obtain an estimate of the complexity, for $K = 4$, $\mathrm{CER}_s = 1.19$, and $N = 16, 32, 64$, the number of codewords for the first addressing step is equal to $\{46, 150, 564\}$[3] and the total number of blocks is equal to $\{82, 280, 1048\}$,[4] respectively. For $K = 8$, $\mathrm{CER}_s = 1.414$, and $N = 16, 32, 64$, these quantities are equal to $\{113, 414, 1323\}$ and $\{206, 786, 2813\}$, respectively. For each block, we store a small prefix code (with the codewords labeling the clusters constituting that block) and two pointers for each of its clusters (labeling the lower dimensional blocks constituting that cluster). In the following, we explain an alternative (to the prefix coding) for the selection of the clusters within the blocks.

Assume that the clusters within each block are arranged in a given order. Also assume that all the elements of a higher order cluster have a larger label. Using this

TABLE I
POINTS OF THE SHAPING SET IN THE n-DOMAIN OF THE $A_8(128,4,2^6)$ CONSTELLATION

Points	$\mathcal{N}$	E	Points	$\mathcal{N}$	E
0000	1	20.0	0012	12	51.0
0001	4	30.5	0111	4	51.5
0002	4	40.5	0022	5	61.0
0011	6	41.0	0013	12	61.5
0003	4	51.0	0112	12	61.5

[2] There are six points with $E = 61$. These are the points indexed by the permutations of $(0,0,2,2)$.

[3] This is the total number of the nonzero elements in the binary expansion of $|F_m(K,l)|$'s, $l = 0,\cdots,L$, $m = N/2$.

[4] This is the total number of the nonzero elements in the binary expansion of $|F_m(K,l)|$'s, $l = 0,\cdots,L$, $m = 2,4,8,\cdots,N/2$.

TABLE II
A PREFIX CODE FOR THE ADDRESSING IN THE n-DOMAIN OF THE $A_8(128, 4, 2^6)$ CONSTELLATION. THE "×" DENOTES THE "DON'T CARE" ENTRIES.

BD	×	×	×	0	0	0	C_1A	×	1	1	1	0	1
DB	×	×	×	0	0	1	AE_1	×	0	0	1	1	0
AD	×	×	0	0	1	0	E_1A	×	0	1	1	1	0
DA	×	×	1	0	1	0	C_1C_2	×	1	0	1	1	0
BC_1	×	×	0	0	1	1	C_2C_1	×	1	1	1	1	0
C_1B	×	×	1	0	1	1	BC_2	×	0	0	1	1	1
BB	×	×	0	1	0	0	C_2B	×	0	1	1	1	1
C_1C_1	×	×	1	1	0	0	AC_2	0	0	1	1	1	1
AB	×	0	0	1	0	1	C_2A	1	0	1	1	1	1
BA	×	1	0	1	0	1	AA	0	1	1	1	1	1
AC_1	×	0	1	1	0	1	E_2A	1	1	1	1	1	1

structure, the clusters can be selected according to the range of the data value involved in that addressing step. After finding the cluster, the cardinalities of all the previous clusters are subtracted from the data value, and the residue is used to select a point within that cluster. Similar to the previous case, this is easily achieved by splitting the corresponding bit stream into two parts. This is the end of a recursion step. As all the cardinalities are integral powers of two, most of the computations are equivalent to bit operations.

In the proposed scheme, the range of each block was limited to a single $|F_m(K, l)|$, $l = 0, \cdots, L$, $m = 2, 4, 8, \cdots, N/2$. This results in the best possible tradeoff for a given number of shells per 2-D (K). By relaxing this constraint, it is possible to further reduce the complexity while the degradation is negligible. This is studied in the following section where we impose a different constraint that all the blocks have the same bit rate. This new constraint eliminates the need for the prefix coding. Obviously, relaxing both of these constraints results in a more general scheme which is not discussed here.

D. Direct Addressing Decomposition

Consider an N'-D unshaped constellation, i.e., $A_{N'} = \{S_2\}^{N'/2}$. This constellation is partitioned into K energy shells of equal volume. The 2-fold Cartesian product of the set of the partitions is shaped by using a lookup table. Assuming continuous approximation, the calculation of the tradeoff is quite similar to the one presented in [2, Section VI]. The final result is shown in Fig. 5.

The main point is that for small values of CER_s, a moderate value of K can essentially achieve the optimum tradeoff. This phenomenon is a generalization of the same effect observed for dimensionality two in the pioneering work of [3]. This property allows us to decompose the addressing into some intermediate steps achieved on the two-fold Cartesian product of a set with low cardinality.[5]

For a dimensionality $N = 2^u$, this results in $u - 1$ addressing steps. The ith step, $i \in [1, u - 1]$, is achieved on the 2^i-D subspaces and results in dimensionality 2^{i+1}. We assume that the subspaces involved in the ith addressing step are partitioned into $K_i = 2^{k_i}$ shells. The ith addressing step, $i = 1, \cdots, u - 2$, requires a lookup table with $2k_i \times 2^{2k_i}$ bits. The last step requires $2k_{u-1} \times 2^{2k_{u-1} - r_s}$ bits. An upper bound to total memory size M_s is obtained by setting $r_s = 0$. Fig. 6 shows the final tradeoff curves for $\{K_i,\ i = 1, \cdots\} = \{64, 64, 128, 256, \cdots\}$. It is seen that the suboptimality is negligible.

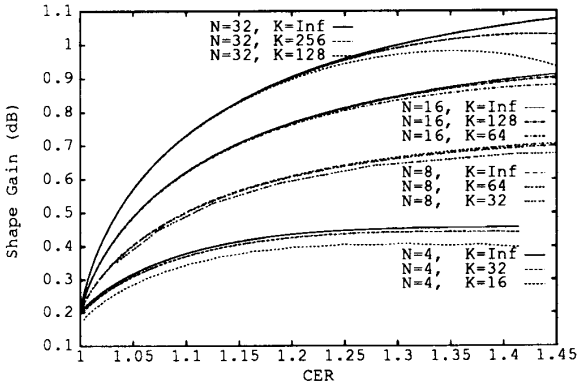

Fig. 5. Tradeoff between CER_s and γ_s using a finite number of energy shells in the $N/2$-D subspaces.

By selecting smaller values for the k_i's, one can further decrease the size of the memory at the price of some performance degradation. Specifically, it is appropriate to use a small number of partitions in the first $u - 2$ addressing steps and a large number at the last step. This is due to the facts that: 1) shaping is essentially achieved at the last step, and consequently a higher resolution at this step has a more important effect on the overall performance; and 2) even for a large k_{u-1}, due to the subtraction of r_s, the addressing of the last step does not need a large lookup table. Table III shows some examples of the performance and complexity of the proposed scheme.

The addressing decomposition methods introduced here have some similarities with the scheme of [8]. The key point to the scheme of [8] is that the weight distribution of the integer lattice (or more generally, a lattice which is equal to the Cartesian product of some lower dimensional sublattices) is equal to the convolution of the weight distributions in its subspaces. This fact is used in [8] to successively decompose the addressing into lower dimensional subspaces. The major difference is that here, by imposing the appropriate constraint(s), we have been able

[5] We already observed in [2, Fig. 5] that the same property is valid for the $\mathscr{A}_N^{N'}(1/2, \psi')$ regions. This means that the address decomposition procedure discussed here can be applied to that case too.

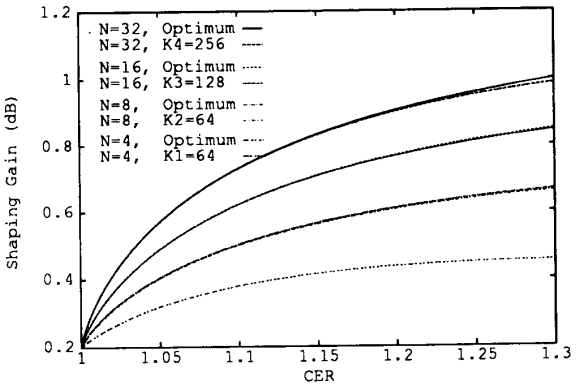

Fig. 6. Tradeoff between CER_s and γ_s using the direct address decomposition method.

TABLE III
PARAMETERS OF THE POINT ACHIEVED USING THE DIRECT ADDRESSING DECOMPOSITION METHOD. THE VALUES INSIDE PARENTHESES ARE THE OPTIMUM γ_s. COLUMN M_t IS MEMORY SIZE IN BYTES (8 b)/N-D (NO COMPUTATION).

N	CER_s	PAR	γ_s dB	M_t	$\{k_i, i = 1, \cdots\}$
16	1.19	2.69	0.73 (0.76)	$1.8k$	{3, 4, 6}
16	1.19	2.70	0.74 (0.76)	$2.0k$	{4, 4, 6}
16	1.41	3.28	0.85 (0.9)	$0.87k$	{4, 4, 6}
16	1.41	3.30	0.87 (0.9)	$2.25k$	{4, 4, 7}
32	1.19	2.76	0.85 (0.89)	$2.1k$	{4, 4, 5, 6}
32	1.19	2.77	0.87 (0.89)	$3.5k$	{4, 4, 5, 7}
32	1.41	3.38	0.98 (1.06)	$3.25k$	{4, 4, 5, 8}
32	1.41	3.40	1.00 (1.06)	$8.0k$	{4, 5, 6, 8}
32	1.41	3.41	1.02 (1.06)	$9.7k$	{4, 5, 6, 9}
64	1.19	2.82	0.94 (1.0)	$8.3k$	{4, 4, 5, 6, 8}

to avoid the computation of the convolutions. This substantially reduces the complexity. It is also possible to combine the decomposition methods proposed here with that of [8].

An alternative to the addressing by a lookup table is the use of a Voronoi constellation [9], [10]. In the following, the idea of the shell mapping is extended to this case. This is achieved by replacing the general $\mathcal{TC}_n$ boundary by the Voronoi region of a lattice.

III. SHELL-ADDRESSED VORONOI CONSTELLATIONS

Returning to our general view of shaping, the 2-D subconstellations are partitioned into a set of concentric shells of equal volume. The shells are indexed by $J = 0, \cdots, K-1$. Assuming continuous approximation in 2-D, the average energy of the Jth shell is proportional to $J + 0.5$. The shell indexes are mapped to the points of the 1-D half integer grid, $Z + (1/2)$, bounded within $[0, K]$. A block of n shell sequences is mapped to a point of the n-D half integer grid, $Z^n + (1/2)^n$, bounded within the hypercube $[0, K]^n$. In our general approach, shaping is achieved by using the TC_n set to select a subset of points of $Z^n + (1/2)^n$ with the least sum of the coordinates. This corresponds to a subset of shell sequences with the least average energy. In the shell-addressed Voronoi constellations, we use the boundary around the origin of a lattice to select this subset. This approach restricts the selection of the boundary and is potentially suboptimum. However, by using the group property of the lattices, it has a payoff in reduced addressing complexity. By using the D_n^* lattice, we can achieve a point on the optimum tradeoff curves. It is shown that this is the unique lattice achieving a nontrivial point on the optimum tradeoff curves.

Consider a lattice Λ_n^s such that the projection of its Voronoi region on any dimension is the region $[-1, 1]$. We use the notation $V_n(K\Lambda_n^s)$ to denote the subset of points of $Z^n + (1/2)^n$ bounded within the Voronoi region around the origin of $K\Lambda_n^s$. This is the Voronoi constellation based on $Z^n + (1/2)^n$ and the shaping lattice $K\Lambda_n^s$. The shell-addressed Voronoi constellations are based on using the points of such a set to shape the n-domain. The complexity of the addressing is that of a linear mapping plus the complexity of decoding of Λ_n^s. Assuming binary lattices to have an integral total rate, M and K should be integral powers of two, i.e., $M = 2^m$ and $K = 2^k$.

In the A_N constellations, the available signal space in the n-domain is restricted to positive coordinates. To obtain symmetry, the 2-D shells are further partitioned into two subshells, each with 2^{m-k-1} points. The two subshells of the Jth shell are mapped to the points $Y = \pm(J + 0.5)$. This results in the set $SC_n(2^k)$ in the n-domain. Shaping is achieved by selecting the set $V_n(2^k\Lambda_n^s) \subset SC_n(2^k)$ to shape the n-domain.

The average energy is determined by the absolute first moment of the n-domain. Consequently, in selecting Λ_n^s, one should try to minimize the absolute first moment of the lattice Voronoi region for a given volume. A lattice with a pyramidal Voronoi region results in a spherical constellation. However, we know that such a lattice exists *only* in dimensionality two [11] (this is the lattice $D_2^* = \Re Z^2$). An analytical method to calculate the absolute first moment of the lattices together with numerical results for the lattices D_n and $\Re D_n$ is given in [12].

For $\Lambda_n^s = D_n^*$, we obtain the constellation $A_N(2^m, 2^k, 2^{kn-1})$. This corresponds to the point $r_s = 1$ ($CER_s = (2)^{1/n}$) on the optimum tradeoff curves. These are the A-points in [2, Table I]. For $N = 4$ ($n = 2$), we have $D_2^* = \Re Z^2$ and the constellation is spherical. This is the case shown in Fig. 3.

The major complexity in a Voronoi constellation is that of decoding the shaping lattice. The decoding of D_n^* is efficiently achieved using the following definition [13]:

$$D_n^* = \{(2Z)^n\} \cup \{(2Z)^n + (1)^n\}. \tag{9}$$

Based on this definition, to decode a vector $\boldsymbol{x} = (x_i, i = 0, \cdots, n-1)$, we first find the two nearest integers on the two sides of each component of $\boldsymbol{x}$. The nearest integers on the two sides of the component x_i are denoted as x_i^e and x_i^o where superscripts e/o stand for even/odd. The point $\boldsymbol{x}^e = (x_i^e,\ i = 0, \cdots, n-1)$ is the

nearest point of $2Z^n$ to $\boldsymbol{x}$. Similarly, the point $\boldsymbol{x}^o = (x_i^o, i = 0,\cdots, n-1)$ is the nearest point of $2Z^n + (1)^n$ to $\boldsymbol{x}$. The nearest of the two points $\boldsymbol{x}^e, \boldsymbol{x}^o$ is the nearest point of the lattice D_n^* to $\boldsymbol{x}$. The decoding of D_n^* is much simpler than the decoding of the popular lattices (like E_8 and Λ_{24}) used in the standard Voronoi constellations.

In the following, we show that the point corresponding to the lattice D_n^* is the only nontrivial point that a shell-addressed Voronoi constellation can achieve on the optimum tradeoff curves. The other point is the trivial case of a cubic constellation.

Referring to the definition of the region $\mathscr{TC}_n$ in [2], to achieve an optimum point, the first condition is that the points $[0^{n-1}, \pm 2]$, where $[\cdot]$ denotes the set of all the points obtained by the permutations of the components within $[\cdot]$, should be the nearest point to the origin along each dimension of Λ_n^s. Also, to realize a point with the parameter $1/n \le \psi \le 1$, we should have $[(\pm 2\psi)^n] \in \Lambda_n^s$. Using the group property of the lattice, this requires that $[0^{n-1}, \pm(4 - 4\psi)] \in \Lambda_n^s$, and also $[0^{n-1}, \pm 4\psi] \in \Lambda_n^s$. This contradicts the first condition for all the range of $\psi \le 1$, except for $\psi = 1/2$ and $\psi = 1$ where $\psi = 1/2$ corresponds to the D_n^* lattice and $\psi = 1$ corresponds to the integer lattice (resulting in a cubic constellation).

For N around 12, the point achieved by the shell-addressed Voronoi constellation based on the D_n^* lattice is located near the knee of the optimum tradeoff curves. As N increases, specifically for $N > 16$, this point moves toward the initial parts of the curves. It should be mentioned that in a practical scheme, the initial part of the curves may be the most interesting part. In the following, we introduce a method to achieve a higher γ_s in spaces of higher dimensionality. This is based on a multilevel addressing procedure which combines the shell-addressed Voronoi constellation method with a lookup table to move along a curve which is nearly optimum. In the rest of the paper, we make frequent use of Fig. 7 to explain our schemes. The actual values corresponding to this figure will be written inside double braces $\{\{\cdot\}\}$.

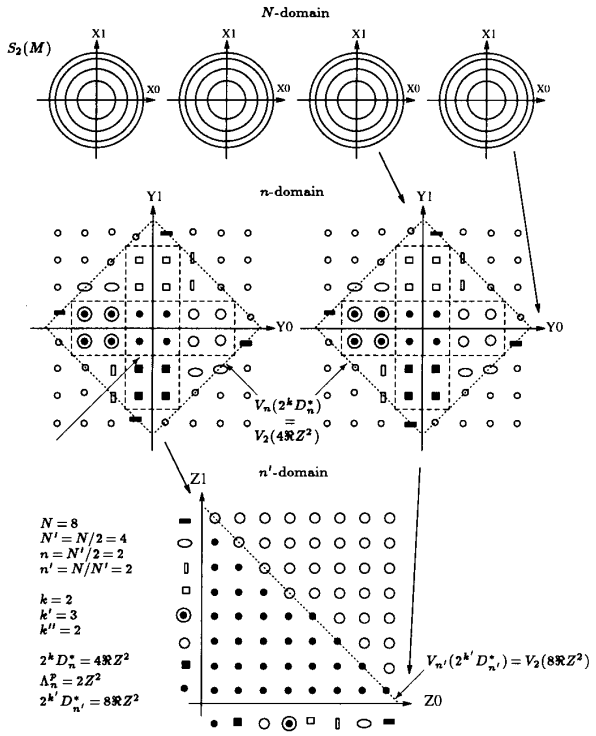

Fig. 7. Example of a multilevel shell-addressed constellation. In this example, $N = 8$, $N' = 4$, $n = 2$, $n' = 2$, $k = 2$, $k' = 3$, $k'' = 2$, $2^kD_n^* = 4\Re Z^2$, $\Lambda_n^p = 2Z^2$, and $2^{k'}D_{n'}^* = 8\Re Z^2$.

IV. Two-Level Shell-Addressed Constellations

A. Basic Structure

The two-level shell-addressed constellations $A_N^{N'}$ are based on a shaping region as close as possible to the region $\mathscr{A}_N^{N'}(1/2, \psi')$ introduced in [2]. These constellations provide a means to move along a curve which is nearly optimum. Examples of such curves are given in [2, Figs. 3 and 5]. The structure of $\mathscr{A}_N^{N'}$ $\{\{A_8^4\}\}$ constellations is as follows. A constellation $A_{N'}(2^m, 2^k, 2^{kn-1})$, $n = N'/2$ is employed along each N'-D subspace. The shaping set in the n-domain is equal to $V_n(2^kD_n^*)$ $\{\{V_2(4\Re Z^2)\}\}$. By using a partitioning lattice Λ_n^p $\{\{2Z^2\}\}$ which has $2^kD_n^*$ $\{\{4\Re Z^2\}\}$ as a sublattice, $V_n(2^kD_n^*)$ is partitioned into $2^{k'} = |\Lambda_n^p/2^kD_n^*|$ $\{\{2^3 = |2Z^2/4\Re Z^2|\}\}$ shaping clusters, each with $2^{k''} = |Z^n/\Lambda_n^p|$ $\{\{2^2 = |Z^2/2Z^2|\}\}$ points. This is based on the decomposition

$$Z^n = 2^kD_n^* + [Z^n/\Lambda_n^p] + \left[\Lambda_n^p/2^kD_n^*\right], \qquad (10)$$

obtained from the partition chain $Z^n/\Lambda_n^p/2^kD_n^*$ $\{\{Z^2/2Z^2/4\Re Z^2\}\}$. In matrix notation, we have

$$Z^n = 2^kD_n^* + \boldsymbol{a}\boldsymbol{G} + \boldsymbol{b}\boldsymbol{H} \qquad (11)$$

where $\boldsymbol{G}$ and $\boldsymbol{H}$ are generators of $[Z^n/\Lambda_n^p]$ $\{\{[Z^2/2Z^2]\}\}$ and $[\Lambda_n^p/2^kD_n^*]$ $\{\{[2Z^2/4\Re Z^2]\}\}$, respectively, $\boldsymbol{a}$ is a binary k''-tuple, and $\boldsymbol{b}$ is a binary k'-tuple. Using (11), each coset of $[Z^n/2^kD_n^*]$ $\{\{[Z^2/4\Re Z^2]\}\}$ is labeled by $(\boldsymbol{a}, \boldsymbol{b})$. Each shaping cluster in the n-domain is the set of the points with the same $\boldsymbol{b}$. This means that $\boldsymbol{b}$ determines a cluster within the n-domain and $\boldsymbol{a}$ determines a point within that cluster.

The partitioning of the N'-D subspaces results in $2^{k'n'}$, $n' = (N/N')$, $\{\{2^6\}\}$ shaping partitions in the N-domain. The third step of shaping is achieved by using a lookup table to select 2^t of these N-D partitions of the least average energy. The whole constellation is denoted as $A_N^{N'}(2^m, 2^k, \Lambda_n^p, 2^t)$. The lookup table has t input lines and $k'n'$ output lines. The output lines are divided into n' groups. Each group of the output lines is assigned to one of the N'-D subspaces, and is used as the $\boldsymbol{b}$ part of the label in (11). Another $k''n'$ data bits, divided into n' groups, are used as the $\boldsymbol{a}$ part of the label in (11). Finally, another $N(m - k - 1)/2$ data bits select one point within each 2-D subspace.

The $B_N^{N'}$ $\{\{B_8^4\}\}$ constellation is devised to achieve a single point near the tradeoff curve of $A_N^{N'}$ *without using the lookup table*. These are the marked points in [2, Fig. 5]. The shaping/partitioning of the N'-D subspaces is the same as in the $A_N^{N'}$ constellations. The shaping clusters of each of the N'-D subspaces are mapped in the order of increasing energy to the points of $Z + (1/2)$ bounded within $[-2^{k'}, 2^{k'}]$ $\{\{[-2^3, 2^3]\}\}$ along a dimension of the n'-domain. The positive and negative points are mapped to the same cluster. Each point is labeled by the label of the corresponding cluster, namely, the k'-tuple $\boldsymbol{b}$ in (11), and an extra bit which is selected according to the sign of the point. This sign bit is later used as 1 b of the k''-tuple $\boldsymbol{a}$ in (11). This results in the set $SC_{n'}(2^{k'})$ $\{\{SC_2(8)\}\}$ in the n'-domain. In Fig. 7, just the positive part of the n'-domain is shown. The shaping set in the n'-domain is selected as $V_{n'}(2^{k'}D_{n'}^*) \subset SC_{n'}(2^{k'})$ $\{\{V_2(8\Re Z^2) \subset SC_2(8)\}\}$. This allows to use a voronoi constellation for the addressing. In the $A_N^{N'}$ constellation, this part of the addressing was achieved by a lookup table. In each signaling interval, $n'(k' + 1) - 1$ data bits are used to select a point in the n'-domain. The label of each component of the selected point (which is the k'-tuple $\boldsymbol{b}$ plus a sign bit) with another $k'' - 1$ data bits are used in (11) to determine the corresponding 2-D shell indexes.

To store the labels, we require a block of memory with $M_s = k' \times 2^{k'}$ bits (compared to the $A_N^{N'}$ constellations which require $n'k' \times 2^{n'k'}$ bits.). The reduction in the complexity is based on a similar phenomenon, as in the case of the addressing decomposition methods. The key point is that the same addressing structure is used for several times. In the present context, the memory block storing the labels is the same for all the dimensions of the n'-domain.

The whole constellation is denoted as $B_N^{N'}(2^m, 2^k, \Lambda_n^p)$. The shaping redundancy is $r_s = n' + 1$ and the total rate is $mN/2 - n' - 1$. As in the case of the $A_N^{N'}$ constellations, we have the appropriate choice of $n' = 2$. In the sequel, we assume that $n' = 2$.

B. Performance Measure

To calcualte the shaping gain, first, by using (11), the average energies of the N'-D points mapped to each shaping cluster in the n-domain are calculated. Then, by adding the average energies along different dimensions of the n'-domain, the average energy of the final subset of the N-domain is found.

As an example, Table IV shows the shaping gain of the $B_{16}^8(32, 4, \Lambda_4^p)$ constellation for different partitioning lattices. For this constellation, we have $r_s = 3$ ($\text{CER}_s = 1.3$). By changing M, we can change the total rate of the constellation for fixed lookup table complexity, fixed CER_s, and *essentially* fixed γ_s. As an example, $B_{16}^8(64, 4, Z^4)$ results in $\gamma_s = 0.72$ dB and $B_{16}^8(128, 4, Z^4)$ results in $\gamma_s = 0.71$ dB.

Using a similar computational approach as used in the case of Fig. 1, the maximum shaping gain for $\text{CER}_s = 1.3$, $N = 16$, and $K = 4$ is found to be equal to $\gamma_s = 0.77$ dB.

TABLE IV
SHAPE GAIN OF THE $B_{16}^8(32, 4, \Lambda_4^p)$ CONSTELLATIONS FOR $\text{CER}_s = 1.3$; M_s DENOTES THE REQUIRED MEMORY SIZE IN BYTES (8 b) PER N DIMENSIONS

Λ_4^p	k'	M_s	γ_s dB
D_4^*	8	$0.25k$	0.62
$\Re Z^4$	9	$0.56k$	0.66
D_4	10	$1.25k$	0.70
Z^4	11	$2.75k$	0.73

V. MULTILEVEL SHELL-ADDRESSED CONSTELLATIONS

As far as the shaping region in a domain is selected as the Voronoi region of a lattice, it can be easily partitioned into shaping clusters of equal volume. This provides us with a way to achieve another level of shaping/addressing on their Cartesian product. This can be done several times to produce a multilevel (nested) form of shaping. Similarly to the $B_N^{N'}$ constellations, this can be used to achieve *single points* with high shaping redundancy near the optimum tradeoff curves.

Notation $B_N^{N_1,\cdots,N_q}(2^m, 2^k, \Lambda_{n_1}^p, \cdots, \Lambda_{n_q}^p)$ is used as the complete notation for this constellation. This constellation has a $B_{N_q}^{N_1,\cdots,N_{q-1}}(2^m, 2^k, \Lambda_{n_1}^p, \cdots, \Lambda_{n_{q-1}}^p)$ along each of its N_q-D subspaces, and the lattices $\Lambda_{n_q}^p$ and $D_{n_q}^*$, $n_q = N/N_q$ are used to partition/shape the Cartesian product of the $B_{N_q}^{N_1,\cdots,N_{q-1}}$'s. Addressing in $B_N^{N_1,\cdots,N_q}$ requires a set of q memory blocks with k_i', $i = 1, \cdots, q$ input and output lines where $2^{k_{i+1}'} = |\Lambda_{n_i}^p / 2^{k_i'} D_{n_i}^*|$, $k_1' = k$, $n_1 = N_1/2$. The total rate is equal to $mN/2 - r_s$ where $r_s = 1 + N\sum_{i=1}^{q}(1/N_i)$.

VI. COMPARISON WITH OTHER TECHNIQUES

In the following, we compare our addressing schemes with the pioneering works of [10], [3], and [14].

In the Voronoi constellations, the Voronoi region of a lattice is used as the shaping region [9], [10]. The complexity of the addressing is that of a linear mapping plus that of decoding the shaping lattice. In [3], the 2-D subspaces are partitioned into the circular shells of equal volume. Then, a multilevel shaping code is used to specify the sequence of the 2-D subregions. In [14], the Voronoi region of an infinite dimensional lattice obtained from a convolutional code is used as the shaping region. The addressing complexity is that of a linear mapping plus the decoding of the code trellis diagram.

The major problem in the Voronoi constellations based on the binary lattices is that they have a cubic 2-D subconstellation (instead of spherical). For a given CER_s, this decreases the achievable γ_s and also increases the PAR. The Voronoi constellations also suffer from the problem of ties, which occurs when some points are located on the boundary of the shaping region. The ties complicate the addressing procedure, and potentially may result in a constellation which is not symmetric.

The shell-addressed Voronoi constellations introduced here have a spherical 2-D subconstellation. Their addressing is achieved by a Voronoi constellation of *half* the original dimensionality. This reduces the addressing complexity. Also, an important class of our schemes achieving

TABLE V
COMPARISON BETWEEN THE VORONOI CONSTELLATIONS (VC) AND THE CALDERBANK, OZAROW METHOD (C / O) WITH THE OPTIMUM CONSTELLATIONS; THE VALUES IN PARENTHESES ARE THE OPTIMUM VALUES OF CER_s, PAR FOR THE GIVEN γ_s

N	γ_s	VC [10] CER_s	PAR	γ_s	C/O [3] CER_s	PAR
4	0.37	1.41 (1.09)	4.62 (2.27)	—	—	—
8	0.65	2.00 (1.26)	6.98 (2.81)	—	—	—
12	0.75	3.00 (1.26)	8.24 (2.86)	0.63	1.55 (1.13)	3.42 (2.51)
16	0.81	1.54 (1.24)	5.58 (2.88)	0.69	1.45 (1.14)	3.02 (2.55)
24	1.03	4.00 (1.50)	15.2 (3.67)	0.80	1.50 (1.16)	3.46 (2.67)
32	0.85	1.35 (1.16)	4.94 (2.70)	0.86	1.46 (1.17)	3.40 (2.72)

a point on the optimum tradeoff curve is based on the lattice D_n^* which has a simple decoding algorithm. In a shell-addressed Voronoi constellation, the ties, although still existing, do not result in an addressing problem or asymmetry.

The schemes of [3] also use a spherical 2-D subconstellation and do not have the problem of ties. To have a fair comparison of [3] with this work or with [10] and [14], it remains: 1) to find an appropriate shaping code which has an integral (or rational) bit rate per signaling interval (to avoid the problem of the nonintegral bit rate), and 2) to find an efficient addressing scheme to map the data bits to the codewords. As mentioned in [3], the addressing problem is not a major issue. However, the problem of the nonintegral bit rate needs to be further discussed.

As already mentioned, we are essentially able to closely approximate any point up to the knee of the optimum tradeoff curves. In Table V, we have compared some of the values obtained in [10] and [3] with the optimum values calculated in [2].

It should be mentioned that by applying the *peak constraint* technique [10], [14], it is possible to modify the Voronoi constellations in such a way that the 2-D points outside a circle of selected radius are not allowed. This constraint can be applied to the minimum distance decoder [15] of the lattice. Such a modification to some extent remedies the deficiencies caused by a cubic 2-D subconstellation. For example, our simulation results show that for the E_8 lattice, one can achieve almost all the shaping gain given in Table V, but with $CER_s = 1.7$ and PAR = 4 instead of $CER_s = 2$ and PAR = 6.98. It should be mentioned that most probably for the higher dimensional lattices (like Λ_{24}), the improvement due to this technique will be more pronounced.

As a more detailed comparison, a four-state trellis diagram of [14] (in conjunction with the peak constraint technique) achieves $\gamma_s = 0.97$ dB, $CER_s = 1.5$, PAR = 3.75. For $N = 32$, a two-level shaping code of [3] achieves $\gamma_s = 0.86$ dB, $CER_s = 1.46$, PAR = 3.40. For $N = 32$, our direct address decomposition method needs $M_s = 2.1$ kbytes/N dimensions to achieve $\gamma_s = 0.85$, $CER_s = 1.19$, and PAR = 2.76. As an alternative, we need $M_s = 3.25$ kbytes/N dimensions to achieve $\gamma_s = 0.98$, dB, $CER_s = 1.41$, and PAR = 3.38. The achieved points are very near the L-point/K-point in [2, Table I]. On the other hand, to realize the L-points/K-points, the appropriate number of shells per 2-D subspaces is equal to 4/8 (as an example, refer to Fig. 1). For these numbers of partitions, a direct addressing scheme requires a lookup table with $M_s = 1.05 \times 10^6/6.5 \times 10^9$ kbytes/N dimensions to closely approximate the L-point/K-point.

In addition to better performance, our address decomposition methods have two other advantages over [3] and [14].

- The examples given in [3] and [14] achieve fixed tradeoff points with relatively high CER_s. The achieved points are relatively far from the knee of the optimum tradeoff curves. In a coding scheme carrying a large bit rate per dimension, a high value of CER_s may be hard to implement. However, our methods are not confined to a fixed tradeoff point. Specifically, for $CER_s = 1.19$ (L-points in [2, Table I]), we can achieve a higher γ_s than [3] or almost all the γ_s of [14] with a substantial decrease in CER_s and PAR. It should be mentioned that it is possible to find other examples for the application of the ideas introduced in [3] and [14], achieving different, possibly better, tradeoff points.
- It seems that our methods, which have no computation, are easier to implement.

VII. SUMMARY AND CONCLUSIONS

We have introduced several practical addressing schemes. These are based on partitioning the constellation into the shaping clusters of equal volume and selecting a subset of the clusters with low average energy. In one class of schemes, addressing is achieved by a lookup table. By decomposing the addressing, we have substantially decreased the complexity of the lookup table. In another class of schemes, addressing is based on the use of Voronoi constellations. Hybrid multilevel addressing schemes which combine both classes are also considered. As an example of performance, in 32-D space, by increasing the number of the 2-D points by 41% and using 3.25 kbytes of memory (no computation), we realize a shaping gain of 0.98 dB.

In general, the shaping gain supplements the coding gain, and thereby offers an additional gain that could otherwise only be obtained through a large increase in coding complexity. It seems that after the initial coding gain associated with the Ungerboeck schemes [16] or with

the Wei schemes [4], this is the easiest way to obtain higher gains.

A. Recent Related Results

Recently, Laroia *et al.* in [17] developed an addressing decomposition scheme using ideas from a type of structured vector quantizer. In their work, an example for $N = 64$ is given which needs 15 multiply–adds/2-D, together with a total memory of 1.5 kbytes/N-D, to achieve a tradeoff point with $\text{CER}_s = 1.5$ near the optimum curve (the optimum γ_s for $N = 64$, $\text{CER}_s = 1.5$ is equal to 1.2 dB [2]). The complexity and performance of this method can be compared with our results given earlier.

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References

[1] G. D. Forney and L. F. Wei, "Multidimensional constellations—Part I: Introduction, figures of merit, and generalized cross constellations," *IEEE J. Select. Areas Commun.*, vol. 7, pp. 877–892, Aug. 1989.

[2] A. K. Khandani and P. Kabal, "Shaping multidimensional signal spaces—Part I: Optimum shaping, shell mapping," *IEEE Trans. Inform. Theory*, this issue, pp. 1799–1808.

[3] A. R. Calderbank and L. H. Ozarow, "Nonequiprobable signaling on the Gaussian channel," *IEEE Trans. Inform. Theory*, vol. 36, pp. 726–740, July 1990.

[4] L. F. Wei, "Trellis coded modulation with multidimensional constellations," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 483–501, July 1987.

[5] G. D. Forney, "Coset codes—Part I: Introduction and geometrical classification," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1123–1151, Sept. 1988.

[6] A. K. Khandani and P. Kabal, "An efficient addressing scheme for the nearly optimum shaping of multidimensional signal spaces," submitted to *IEEE Trans. Inform. Theory*, Aug. 1992.

[7] G. D. Forney, Jr., R. G. Gallager, G. R. Lang, F. M. Longstaff, and S. U. Quereshi, "Efficient modulation for bandlimited channels," *IEEE J. Select. Areas Commun.*, vol. 2, pp. 632–647, Sept. 1984.

[8] G. R. Lang and F. M. Longstaff, "A leech lattice modem," *IEEE J. Select. Areas Commun.*, vol. 7, pp. 968–973, Aug. 1989.

[9] J. H. Conway and N. J. A. Sloane, "A fast encoding method for lattice codes and quantizers," *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 820–824, Jan. 1983.

[10] G. D. Forney, Jr., "Multidimensional constellations—Part II: Voronoi constellations," *IEEE J. Select. Areas Commun.*, vol. 7, pp. 941–958, Aug. 1989.

[11] J. H. Conway and N. J. A. Sloane, "A lower bound on the average error of vector quantizers," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 820–824, Nov. 1983.

[12] A. K. Khandani, "Shaping multi-dimensional signal spaces," Ph.D. dissertation, McGill Univ., Mar. 1992.

[13] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1988.

[14] G. D. Forney, "Trellis shaping," *IEEE Trans. Inform. Theory*, vol. 38, pp. 281–300, Mar. 1992.

[15] ____, "Coset codes—Part II: Binary lattices and related codes," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1152–1187, Sept. 1988.

[16] G. Ungerboeck, "Channel coding with multilevel/phase signals," *IEEE Trans. Inform. Theory*, vol. IT-28, pp. 55–67, Jan. 1982.

[17] R. Laroia, N. Farvardin, and S. Tretter, "On SVQ shaping of multidimensional constellations—High-rate large-dimensional constellations," in *Proc. 26th Annu. Conf. Inform. Sci. Syst.*, Princeton, NJ, Mar. 1992, pp. 527–531.