Shaping the boundary of a multi-dimensional signal constellation with a non-flat power spectrum

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Indexing terms: Multi-dimensional signal constellation, Non-flat power spectrum, Peak-to-average power ratio

Abstract: The author considers the selection of the boundary (shaping region) of a multi-dimensional signal constellation with the objective of minimising the constellation energy for a given rate and subject to some constraints on the resulting power spectrum. This shaping region is selected such that the efficient addressing technique of 'shell mapping' is applicable. Numerical results are presented for the asymptotic performance in an infinite dimensional space, and also for the finite dimensional case. These results show that, by using the proposed method in a space of moderate dimensionality (say 16), one can obtain: a substantial saving in energy (about 1dB), and a decrease in the PAR (peak-to-average-power-ratio) by a factor of about 2–3.

1 Introduction

The boundary of a constellation is conventionally selected to minimise the average energy for a fixed rate [1]. In the present work, we impose some additional constraints on the power spectrum. To satisfy these constraints, the matrix of constellation basis is selected to be non-diagonal and non-equal values of energy are allocated to different dimensions of this matrix. This is denoted as an asymmetrical shaping problem.

The conventional approaches to obtain asymmetry in a constellation are based on using a rectangular shaping region which employs non-equal numbers of points along the 1-D (one-dimensional) subspaces. In the present work, however, the number of points along the space dimensions are the same and the asymmetry is obtained by selecting an appropriate shaping region in a higher dimensional space.

The problem of designing an asymmetrical rectangular constellation for signalling over a set of dimensions with non-equal noise powers is discussed in [2] and [3]. The design objective in [2] and [3] is to minimise the probability of symbol error, while in the present work, we concentrate on the shape of the resulting power spectrum.

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2 System block diagram

Fig. 1 shows the block diagram of the system under consideration. Each signalling interval (block) is composed of N_m successive time periods. In each block, a data vector **i** is transmitted. The shaping block maps the vector **i** to a point **a** in the baseband constellation **A**. This is a finite set of the N-D points $N \le N_m$, selected from a geometrically uniform array of points (like a lattice or lattice translate), and bounded within the shaping region $\mathcal{R}_a$. We use continuous approximation and normalise the volume associated with constellation point (Voronoi region) to unity.

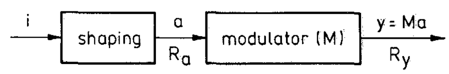

Fig. 1 *System block diagram*

We assume that the points of **A** are used with equal probability. Using continuous approximation and normalising the volume of the Voronoi region around each constellation point to unity, the rate of **A** is found as:

$$H(\mathbf{A}) = \log[V(\mathcal{R}_a)] \qquad (1)$$

where $V(\mathcal{R}_a)$ is the volume of $\mathcal{R}_a$. The average energy (second moment) along the ith dimension of $\mathcal{R}_a$ is denoted as λ_i. The diagonal matrix $\mathbf{\Lambda}_a$ is defined as: $\mathbf{\Lambda}_a = \text{diag}\,[\lambda_0, ..., \lambda_{N-1}]$.

The columns of the $N_m \times N$ matrix **M** are the dimensions of the constellation $\mathbf{y} \in \mathbf{Y}$. We have $\mathbf{M}^t\mathbf{M} = \mathbf{I}$ where **I** is the $N \times N$ identity matrix. This results in an isometry between the space containing **A** and the space containing **Y**. We have $V(\mathcal{R}_y) = V(\mathcal{R}_a)$, and consequently, $H(\mathbf{Y}) = H(\mathbf{A})$, while the distance property and consequently the performance in noise of the two constellations are identical. For $N < N_m$, we can have up to $N_m - N$ nulls in the power spectrum of **Y**.

There exists a tradeoff between the rate $H(\mathbf{Y})$, and the total average energy $\Sigma_i\lambda_i$. The objective is to optimise this tradeoff subject to some constraints on the power spectrum of **Y**. We assume that the matrix **M** is fixed and we use only $\mathbf{\Lambda}_a$ to optimise the constellation. For a spectrum with spectral nulls, **M** is selected as an orthonormal basis with the same set of nulls. (It can be shown that if the system **A** has spectral null at certain frequencies, its output eigenvectors form an orthonormal basis with the same set of nulls.)

Using the results of [4], the power spectrum of **y** is equal to:

$$S_y(\omega) = \sum_{k=0}^{N_m-1} \sum_{|i-j|=k} R_y(i,j)\cos(\omega k) \qquad (2)$$

where $R_y(i, j)$s are the elements of the autocorrelation matrix $\mathbf{R}_y = \mathbf{E}\,[\mathbf{y}\mathbf{y}^t]$. If $\mathbf{m}_i$ denotes the ith column of $\mathbf{M}$ and $S_i(\omega)$ denotes the power spectrum associated with the autocorrelation matrix $\mathbf{m}_i\mathbf{m}_i^t$, we obtain:

$$S_y(\omega) = \sum_{i=0}^{N-1} \lambda_i S_i(\omega) \tag{3}$$

3 Figures of merit

We consider two reference rectangular regions $\mathcal{C}_1$, $\mathcal{C}_2$, both with the same volume as $\mathcal{R}_a$. Region $\mathcal{C}_1$ is of dimensionality N and has the average energy $\gamma_s\lambda_i$ along its ith dimension. The factor γ_s, called the shaping gain of $\mathcal{R}_a$, reflects the reduction in the average energy of the shaping region $\mathcal{R}_a$ with respect to the reference region $\mathcal{C}_1$. The proportionality of the energies guarantees that the corresponding power spectra are identical within a scale factor.

Region $\mathcal{C}_2$ is of dimensionality N_m and has equal values of energy along all its dimensions. The equal allocation of energy results in a white power spectrum. The increase in the average energy of the region $\mathcal{C}_1$ with respect to region $\mathcal{C}_2$ is measured by P_l (performance loss). The overall changing in the average energy of $\mathcal{R}_a$ with respect to $\mathcal{C}_2$ is obtained by multiplying the shaping gain, γ_s, with the performance loss, P_l.

As a generalisation to the definition given in [1], we define the asymmetrical constellation-expansion-ratio CER_u, as the ratio of the maximum number of point per dimension $C_{\max}$ to the minimum necessary number of points per dimension, i.e.,

$$\text{CER}_u = \frac{C_{\max}}{[V(\mathcal{R}_a)]^{1/N}} \tag{4}$$

The value of CER_u for the reference region $\mathcal{C}_1$ is denoted as CER_r. Using eqn. 4, we obtain,

$$\text{CER}_r = \frac{\lambda_{\max}^{1/2}}{\left(\prod_{i=0}^{N-1} \lambda_i^{1/2}\right)^{1/N}} \tag{5}$$

where $\lambda_{\max}$ is the maximum value of λ_i for $i = 0, \ldots, N-1$.

4 Spectral shaping using a non-rectangular constellation

The computational method discussed in this Section is a generalisation of the method proposed in [1].

4.1 Asymptotic behaviour in an infinite dimensional space

Assume that the space dimensionality is extended to infinity, while the modulation is achieved on sub-blocks of dimensionality N. The 1-D subconstellations are identical and are bounded within the range $[-A, A]$. The probability distribution which is zero outside region $[-A, A]$ and maximises the rate (entropy) for a given average energy (second moment) is a truncated Gaussian distribution. For the ith dimension $i = 0, \ldots, N-1$ we have,

$$P_i(X) = \begin{cases} C(\alpha_i \exp\left(\frac{-\alpha_i X^2}{A^2}\right) & \text{for } X \in [-A, A] \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where,

$$C(\alpha_i) = \left[2A \int_0^1 \exp\left(-\alpha_i x^2\right) dx\right]^{-1} \tag{7}$$

The average energy and the rate along the ith dimension are computed as:

$$E_i = A^2 \frac{\int_0^1 x^2 \exp\left(-\alpha_i x^2\right) dx}{\int_0^1 \exp\left(-\alpha_i x^2\right) dx} \tag{8}$$

$$H_i = \frac{\alpha_i \lambda_i}{A^2} - \log C(\alpha_i) \tag{9}$$

We use A as a parameter to adjust the tradeoff point. The α_is, $i = 0, \ldots, N-1$, are selected such that $E_i = \lambda_i$ where E_i is given in eqn. 8 and the λ_is are computed using the method explained in the Appendix.

The shaping performances is computed by replacing,

$$V(\mathcal{R}_a) = \exp\left(\sum_{i=0}^{N-1} H_i\right), \quad \text{and} \quad C_{\max} = 2A \tag{10}$$

in the relationships given in Section 3. The result of these computations is shown in Fig. 2.

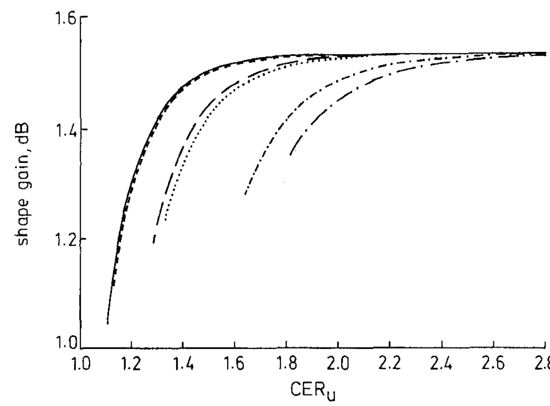

Fig. 2 *Examples of tradeoff curves in infinite dimensional space, sine basis, F_p-constraint with $f_c = 0.2$, $\text{CER}_u \geq \text{CER}_r$*
——— $N = 16$, $F_p = 0.3$, $\text{CER}_r = 1.11$, $P_l = -0.12$dB
- - - - $N = 8$, $F_p = 0.3$, $\text{CER}_r = 1.13$, $P_l = -0.13$dB
— — — $N = 16$, $F_p = 0.2$, $\text{CER}_r = 1.28$, $P_l = -0.55$dB
.... $N = 8$, $F_p = 0.2$, $\text{CER}_r = 1.33$, $P_l = -0.63$dB
– · – · – $N = 16$, $F_p = 0.1$, $\text{CER}_r = 1.64$, $P_l = -1.71$dB
——·—— $N = 8$, $F_p = 0.1$, $\text{CER}_r = 1.80$, $P_l = -1.97$dB

4.2 Asymmetrical shaping in a space of finite dimensionality

Consider a conventional shaping problem in a space of dimensionality N. The energy of each 1-D point is considered as a cost associated with that point. The final constellation is selected as a subset of the N-D points of the least (additive) cost and a total cardinality T. Addressing is a one-to-one mapping between the set of the integer numbers $[0, T-1]$ and the set of the constellation points.

In shaping, we are usually concerned with a set of points of a huge cardinality. This fact complicates the addressing and also the performance analysis of a shaped constellation. A major attribute of the shaping problem is due to the additivity property of the cost in a Cartesian product space. This property allows us to reduce the complexity of the problem by using the notion of shell (defined as the collection of points of the same cost).

In the present case, to take the effect of the spectral constraints into account, the cost is considered as: 'energy plus some Lagrange multipliers times the spectral constraints'. We assume that the spectral constraints are of the general form: $\mathbf{L}[S(\omega)] \leq e$, where $\mathbf{L}$ is a linear operator, i.e.,

$$\mathbf{L}\left[\sum_{i-0}^{N-1} \lambda_i S_i(\omega)\right] = \sum_{i=0}^{N-1} \lambda_i \mathbf{L}\left[S_i(\omega)\right] \qquad (11)$$

A spectral constraint of the form given in eqn. 11 can be decomposed as the sum of some components associated with the 1-D subspaces. This decomposition property is the basis for some of the most powerful techniques known in shaping [5–11]. Note that the F_p-constraint has this decomposability property.

Assume that the shells in a given subspace are indexed in the order of increasing cost. Some of the computational and addressing techniques discussed in [5, 11] are based on the property that the cost of a given shell is an affine function of its index (cost of the ith shell is equal to $c_0 + i\Delta$). In the following, we provide a link between the present shaping problem and those techniques.

The cost of the pth point along the ith dimension is considered as, $E_p(1 + \Sigma_l \xi_l \mathbf{L}_l[S_i])$ where E_p is the energy of the point and ξ_ls are a set of the Lagrange multipliers. The 1-D points along each dimension are aggregated into K macro-shells with a fixed increment in cost, Δ. Assuming that the costs of the 1-D points are in the range $c \in [c_{\min}, c_{\max}]$, we set $\Delta = (c_{\max} - c_{\min})/K$. The costs of the points in the ith macro-shell satisfy, $c_{\min} \leq c \leq c_{\min} + \Delta$ for $i = 0$ and $c_{\min} + i\Delta < c \leq c_{\min} + (i + 1)\Delta$ for $i = 1, ..., K - 1$. Taking the Cartesian product of the set of the macro-shells, the higher-dimensional macro-shells are defined as the collection of the elements in the Cartesian product set with a fixed sum of the indices. This results in a recursive merging rule for the macro-shells. There are $N(K - 1) + 1$ macro-shells in an N-D space. The final constellation is selected as the set of the N-D macro-shells with the indices $0 \leq M \leq M_{\max}$, where $M_{\max}$ is selected such that the total cardinality is equal to T. The Lagrange multipliers are selected to satisfy the spectral constraints. In the case of the F_p constraint, we are dealing with only one Lagrange multiplier which can be easily computed numerically using a bisectional method.

4.2.1 Shaping performance:

Consider a discrete set of points $\psi \in \Psi$. A cost $c(\psi)$ is assigned to each point $\psi \in \Psi$. The weight distribution of Ψ is defined as:

$$W_\Psi(q) = \sum_{\psi \in \Psi} q^{c(\psi)} = \sum_v C_\Psi(v) q^v \qquad (12)$$

where $C_\Psi(v)$ is the number of points of Ψ of a cost v.

Assume that the cardinality of the jth macro-shell in the ith 1-D subspace is equal to $C_i(j)$. By replacing q by a proper transform operator, one can relate eqn. 12 to the discrete Fourier transform (DFT) of the sequence $\mathbf{C}_i = [C_i(0), ..., C_i(K - 1)]$. Using this fact, in conjunction with the multiplicativity property of the weight distributions (due to the additivity property of the cost), we conclude that the cardinality of the N-D macro-shell indexed by m is DFT_m^{-1} $[\Pi_{i=0}^{N-1}$ $\mathrm{DFT}(\mathbf{C}_i)]$ where DFT is the discrete Fourier transform of length $N(K - 1) + 1$, the multiplication of DFTs is achieved on an element-by-element basis and DFT_m^{-1} is the mth element of the corresponding inverse DFT. Using this relationship, one can show that the probability induced on a given macro-shell M, along a given dimension d, is equal to:

$$P_d(M) = \frac{1}{T} \sum_{m=0}^{M_{\max}-M} \mathrm{DFT}_m^{-1}\left[\prod_{i=0, i \neq d}^{N-1} \mathrm{DFT}(\mathbf{C}_i)\right] \qquad (13)$$

where T is the cardinality of the constellation. The induced probabilities are used to compute the average energy along different dimensions. Knowing the average energies, computation of the shaping performance and the power spectrum is straightforward. Fig. 3 shows some examples of the corresponding tradeoff curves subject to the F_p-constraint. Fig. 4 shows the corresponding power spectra.

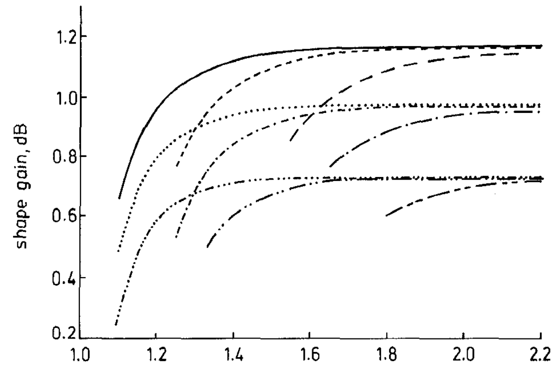

Fig. 3 *Examples of tradeoff curves in spaces of finite dimensionality, sine basis, F_p-constraint with $f_c = 0.2$, $K = 256$, $CER_u \geq CER_r$*

——— $F_p = 0.3$, $N = 32$, $CER_r = 1.10$, $P_1 = 0.10$dB
– – – – $F_p = 0.2$, $N = 32$, $CER_r = 1.25$, $P_1 = 0.51$dB
— — — $F_p = 0.1$, $N = 32$, $CER_r = 1.55$, $P_1 = 1.56$dB
·········· $F_p = 0.3$, $N = 16$, $CER_r = 1.11$, $P_1 = -0.12$dB
– · – · – $F_p = 0.2$, $N = 16$, $CER_r = 1.28$, $P_1 = -0.55$dB
—·—·— $F_p = 0.1$, $N = 16$, $CER_r = 1.64$, $P_1 = -1.71$dB
-·-·-·- $F_p = 0.3$, $N = 8$, $CER_r = 1.13$, $P_1 = -0.13$dB
——·· $F_p = 0.2$, $N = 8$, $CER_r = 1.33$, $P_1 = -0.63$dB
— - - — $F_p = 0.1$, $N = 8$, $CER_r = 1.80$, $P_1 = -1.97$dB

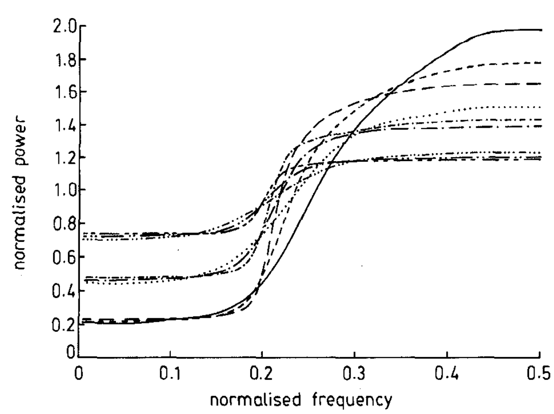

Fig. 4 *Examples of power spectra achieved in spaces of finite dimensionality, sine basis, F_p-constraint with $f_c = 0.2$*

——— $F_p = 0.1$, $N = 8$, $CER_r = 1.80$, $P_1 = -1.97$dB
– – – – $F_p = 0.1$, $N = 16$, $CER_r = 1.64$, $P_1 = -1.71$dB
— — — $F_p = 0.1$, $N = 32$, $CER_r = 1.55$, $P_1 = -1.56$dB
·········· $F_p = 0.2$, $N = 8$, $CER_r = 1.33$, $P_1 = -0.63$dB
– · – · – $F_p = 0.2$, $N = 16$, $CER_r = 1.28$, $P_1 = -0.55$dB
—·—·— $F_p = 0.2$, $N = 32$, $CER_r = 1.25$, $P_1 = -0.51$dB
-·-·-·- $F_p = 0.3$, $N = 8$, $CER_r = 1.13$, $P_1 = -0.13$dB
——·· $F_p = 0.3$, $N = 16$, $CER_r = 1.11$, $P_1 = -0.12$dB
— - - — $F_p = 0.3$, $N = 32$, $CER_r = 1.10$, $P_1 = -0.10$dB

5 Behaviour of the peak of energy

5.1 Behaviour of the PAR in a rectangular constellation

Consider a rectangular constellation where the energy allocated to the ith dimension is equal to λ_i, $i = 0, ...,$

$N - 1$. Assume that the elements of the modulation matrix $\mathbf{M}$ are equal to $m(t, i)$ where $t = 0, ..., N_m - 1$ is the time index and $i = 0, ..., N - 1$ is the dimension index. Using continuous approximation, it is easy to show that the PAR at time t is equal to:

$$\mathrm{PAR} = \frac{\left[\sum_{i=0}^{N-1} |m(t,i)|\sqrt{3\lambda_i}\right]^2}{\sum_{i=0}^{N-1} \lambda_i/N} \quad (14)$$

Here, we have taken the absolute value of the $m(i, t)$ to count for the symmetry of the constellation. Table 1 shows some examples for the minimum and maximum values of PAR, together with some other relevant parameters, for the case of a rectangular constellation.

5.2 Behaviour of PAR in a shaped constellation

To compute the PAR for a shaped constellation, we generalise the structure of a macro-shell to the case of having a two-tuple index (called a cluster) where the two indices reflect the effect of the cost and amplitude, respectively. The two-tuple indices of the clusters are arranged in a two-by-two array.

To perform the computations, the 1-D subspaces are partitioned into K_1K_2 clusters and the absolute amplitude and the cost of the points within the 1-D clusters are approximated with the corresponding mid-values. The higher-dimensional clusters are defined as the set of the elements in the Cartesian product of the 1-D subspaces with a fixed two-tuple for the sum of the corresponding 1-D indices. Assume that the costs of the 1-D points are in the range $c \in [c_{\min}, c_{\max}]$ and the absolute amplitudes of the 1-D points at time t are in the range $a \in [a_{\min}, a_{\max}]$. We set $\Delta_c = (c_{\max} - c_{\min})/K_1$ and $\Delta_a = (a_{\max} - a_{\min})/K_2$. The cost and the absolute amplitude of the points in the (i, j)th 1-D cluster satisfy:

$$\begin{aligned} &c_{\min} \le c \le c_{\min} + \Delta_c, \quad i = 0 \quad \text{and} \\ &c_{\min} + i\Delta_c < c \le c_{\min} + (i+1)\Delta_c, \\ &\qquad\qquad i = 1, \ldots, K_1 - 1 \\ &a_{\min} \le a \le a_{\min} + \Delta_a, \quad i = 0 \quad \text{and} \\ &a_{\min} + i\Delta_a < a \le a_{\min} + (i+1)\Delta_a, \\ &\qquad\qquad i = 1, \ldots, K_2 - 1 \end{aligned} \quad (15)$$

The approximate values for the sum of the absolute amplitudes and sum of the costs for the points in the N-D cluster indexed by (i, j) are equal to $Nc_{\min} + i\Delta_c + \Delta_c/2$ and $Na_{\min} + j\Delta_a + \Delta_a/2$, respectively.

Assume that the cardinality of the (i, j)th cluster in the kth 1-D subspace is equal to $C_1^k(i, j)$. The cardinality of the N-D cluster indexed by (i, j) is equal to:

$$|C_N(i,j)| = \mathrm{DFT2}^{-1}_{(i,j)}\left[\prod_{k=0}^{N-1} \mathrm{DFT2}(\mathbf{C}_1^k)\right] \quad (16)$$

where DFT2 is the two-dimensional DFT of length $[N(K_1 - 1) + 1] \times [N(K_2 - 1) + 1]$, $\mathbf{C}_1^k$ is an $[N(K_1 - 1) + 1] \times [N(K_2 - 1) + 1]$ matrix composed of the elements $C_1^k(i, j)$.

The final constellation is selected as the set of the N-D clusters with the row index (indicating cost) in the range $0 \le M \le M_{\max}$ where $M_{\max}$ is selected such that the total cardinality is equal to T. It is easy to show that the PAR at the time instant t is equal to,

$$\mathrm{PAR}_t = \frac{N\left(a_{\min} + J_{\max}\Delta_a + \Delta_a/2\right)^2}{\sum_{k=0}^{N-1} \lambda_k/N} \quad (17)$$

where $J_{\max}$ is the maximum value of the column index such that at least one of the N-D clusters indexed by (i, j), $0 \le i \le M_{\max}$, is non-empty.

Table 1 shows some examples of the value of the PAR, together with some other relevant parameters. Referring to Table 1, it is observed that the proposed shaping scheme, in addition to the saving of energy, results in a decrease in PAR by a factor of about 2–3.

6 Conclusion

We have studied the selection of an efficient constellation with a non-flat power spectrum. This is based on selecting a shaping region which tries to minimise the average energy of the constellation subject to: fixed total rate, fixed number of points for the 1-D subconstellations, and some constraints on the power spectrum. To do this, the cost of a point is defined as its energy plus some Lagrange multiplier(s) times the spectral constraint(s). Then, a subset of points of the least cost is selected. For a wide class of the spectral constraints, the cost in a Cartesian product space has additivity property. This allows us to apply most of the known shaping techniques in this new context.

7 Acknowledgment

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Table 1: Minimum and maximum values of PAR, together with some other relevant parameters, for the case of a rectangular and a shaped constellation, sine basis, F_p-constraint with f_c = 0.2

		Rectangular			Shaped				
N	F_p	CER_r	PAR_{min}	PAR_{max}	CER_u	PAR_{min}	PAR_{max}	P_l	γ_s
8	0.1	1.80	15	17	2.00	7	8	−1.97	0.70
8	0.2	1.33	17	20	1.60	8	9	−0.63	0.70
8	0.3	1.13	18	21	1.35	8	9	−0.13	0.70
16	0.1	1.64	33	36	2.00	16	17	−1.71	0.95
16	0.2	1.28	37	40	1.65	14	15	−0.55	0.95
16	0.3	1.11	40	41	1.45	16	17	−0.12	0.95

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9 Appendix: Computation of the optimum allocation of energy

Normalising the energy per time dimension to unity (i.e. $\Sigma_{i=0}^{N-1} \lambda_i = N_m$), and using eqn. 3, the F_p-constraint is formulated as:

$$\sum_{i=0}^{N-1} \lambda_i B_i(\omega_c) = \pi N_m F_p \qquad (18)$$

where,

$$B_i(\omega_c) = \int_0^{\omega_c} S_i(\omega) d\omega \qquad (19)$$

This results in the following convex optimisation problem to maximise the rate of the constellation,

$$\begin{cases} \text{maximise} & \sum_{i=0}^{N-1} \log(\lambda_i) \\ \text{subject to:} & \sum_{i=0}^{N-1} \lambda_i B_i(\omega_c) = \pi N_m F_p \\ & \sum_{i=0}^{N-1} \lambda_i = N_m \quad \lambda_i \geq 0 \end{cases} \qquad (20)$$

Using the Lagrange method, we obtain,

$$\lambda_i = \frac{1}{\xi_1 B_i(\omega_c) + \xi_2} \qquad (21)$$

where ξ_1 and ξ_2 are determined by solving,

$$\sum_{i=0}^{N-1} \frac{B_i(\omega_c)}{\xi_1 B_i(\omega_c) + \xi_2} = \pi N_m F_p$$

and

$$\sum_{i=0}^{N-1} \frac{1}{\xi_1 B_i(\omega_c) + \xi_2} = N_m \qquad (22)$$