Solutions for E&CE-318 Midterm Examination, Winter 2002

1.1) All the coefficients are placed in integer multiples of $\omega_0 = \pi$ then $T = \frac{2\pi}{\omega_0} = 2$.

1.2) The average value belongs to the component located in $\omega = 0$, that's 2.

1.3) If the signal passes that filter, only components located in $\omega_0 = 0, 2\pi, -2\pi$ will pass it and the other two components will be rejected by means of the filter. In fact, if we show the input signal as $x(t) = \sum_{k=-\infty}^{+\infty} X(k)\exp(jk\omega_0 t)$ then the output will be as $g(t) = \sum_{k=-\infty}^{+\infty} X(k)H(k\omega_0)\exp(jk\omega_0 t)$. By means of this relation we can see that the output will be as

$$g(t) = 2\times\frac{1}{1+0}\exp(j0t) + 3\times\frac{1}{1+2}\exp(j2\pi t) + 3\times\frac{1}{1+2}\exp(-j2\pi t)$$
$$+1\times 0\exp(j3\pi t) + 1\times 0\exp(j3\pi t) = 2 + \exp(j2\pi t) + \exp(-j2\pi t) = 2 + 2\cos(2\pi t)$$

2.1) In order to show that these signals can be an orthogonal basis, we have to show two things. First, they must be orthogonal to each other. Second, their energy must be 1. It can be easily shown that these signals are orthogonal to each other regardless of the values of A and B. On the other hand

$$\int_0^1 \phi_0{}^2(t)dt = 1 \Rightarrow \int_0^1 A^2 dt = 1 \Rightarrow A^2 = 1 \Rightarrow A = 1$$

At the above equation we assumed that A is positive.

$$\int_0^1 \phi_1{}^2(t)dt = 1 \Rightarrow \int_0^{1/2} B^2 dt = 1 \Rightarrow 0.5B^2 = 1 \Rightarrow B = \sqrt{2}$$

Similar results can be derived for the last signal.

2.2) In order to find the expansion coefficients over this basis we must find the inner product of x(t) and each coordinate.

$$a_0 = \int_0^1 x(t)\phi_0(t)dt = \int_0^1 x(t)dt = 3/4$$

$$a_1 = \int_0^1 x(t)\phi_1(t)dt = \int_0^{1/4} -\sqrt{2}x(t)dt + \int_{1/4}^{1/2}\sqrt{2}x(t)dt = -\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{8}$$

$$a_1 = \int_0^1 x(t)\phi_2(t)dt = \int_{1/2}^{3/4}\sqrt{2}x(t)dt + \int_{3/4}^{1} -\sqrt{2}x(t)dt = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{8} = \frac{\sqrt{2}}{8}$$

Now we find the energy of the error signal. One way is calculating the error signal and then finding its energy. The estimated signal is

$$\hat{x}(t) = 3/4\phi_0(t) + \frac{\sqrt{2}}{8}\phi_1(t) + \frac{\sqrt{2}}{8}\phi_2(t) = \begin{cases} 1/2 & 0 < t < 1/4 \\ 1 & 1/4 < t < 3/4 \\ 1/2 & 3/4 < t < 1 \\ 0 & elsewhere \end{cases}$$

Then the error signal is

$$e(t)=x(t)-\hat{x}(t)=\begin{cases}4t-1/2 & 0<t<1/4\\ 0 & 1/4<t<3/4\\ 4-4t-1/2 & 3/4<t<1\\ 0 & elsewhere\end{cases}$$

The energy of the above signal is 1/24.

We can compute the error signal by means of finding the signal energy and comparing it by the energy of its estimation. The energy of x(t) is 2/3. The energy of the estimated signal is $a_0^{\,2}+a_1^{\,2}+a_2^{\,2}=\frac{9}{16}+\frac{1}{32}+\frac{1}{32}=\frac{5}{8}$. Then the energy of the error is $\frac{2}{3}-\frac{5}{8}=\frac{1}{24}$.

2.3) Because the projection of x(t) on the first coordinate is greater than the others, it is more important than the others. However there is no difference between the second and the last signal from the point of view of error energy.

2.4) We know that the error signal is orthogonal to all coordinates and if we add it to the estimated signal, x(t) will be produced. So we only need to add a normalized version of e(t) to our basis to represent x(t) exactly. Then,

$\phi_3(t)=\sqrt{24}e(t)$

3.1) It is easy to show that the output is:

$$y(t)=aK^2(\alpha+\cos(\omega_m t)+A\cos(\omega_c t))^2-b(\alpha+\cos(\omega_m t)-A\cos(\omega_c t))^2$$
$$=(aK^2-b)\left((\alpha+\cos(\omega_m t))^2+(A\cos(\omega_c t))^2\right)+2(aK^2+b)(\alpha+\cos(\omega_m t))A\cos(\omega_c t)$$

We can see that the second term is the desired modulated signal so we set $aK^2-b=0$. On the other hand, for having an AM signal with modulation index one we have:

$$2(aK^2+b)(\alpha+\cos(\omega_m t))A\cos(\omega_c t)=4(\alpha+\cos(\omega_m t))Ab\cos(\omega_c t)\Rightarrow$$
$$m=\frac{1}{\alpha}=1\Rightarrow\alpha=1$$

The total power is:

$$P=(4Ab)^2(1/2+1/4)=2\Rightarrow(Ab)^2=\frac{1}{6}\Rightarrow Ab=\frac{1}{\sqrt{6}}$$

So these three equations must be satisfied.

$$aK^2-b=0$$
$$\alpha=1$$
$$Ab=\frac{1}{\sqrt{6}}$$

4.1) It is easy to see that g(t) is a signal with period 2. Then $\omega_0=\frac{2\pi}{T}=\frac{2\pi}{2}=\pi$.

$$G_k = \frac{1}{T}\int_0^T g(t)\exp(-jk\omega_0 t)dt = \frac{1}{2}\int_0^2 g(t)\exp(-jk\pi t)dt$$

$$= \frac{1}{2}\int_{0^-}^{0^+} g(t)\exp(-jk\pi t)dt + \frac{1}{2}\int_{1^-}^{1^+} g(t)\exp(-jk\pi t)dt$$

$$= \frac{1}{2} - \frac{1}{2}\exp(-jk\pi) = \begin{cases} 1 & k = 2k_1 + 1 \\ 0 & k = 2k_1 \end{cases}$$

4.2) When the signal passes this filter only coefficients placed in $-\pi, \pi$ can pass the filter. Then the output f(t) will be as follows:

$$f(t) = 1 \times \frac{1}{1 + \frac{\pi}{\pi}}\exp(j\pi t) + 1 \times \frac{1}{1 + \frac{\pi}{\pi}}\exp(-j\pi t) = 0.5\exp(j\pi t) + 0.5\exp(-j\pi t) = \cos(\pi t)$$