Problem 4.1.1

Consider the cosine wave

$$g(t) = A\cos(2\pi f_0 t)$$

Plot the spectrum of the discrete-time signal $g_\delta(t)$ derived by sampling $g(t)$ at the times $t_n = n/f_s$, where $n = 0, \pm1, \pm2, \cdots$ and

(i) $f_s = f_0$ (ii) $f_s = 2f_0$ (iii) $f_s = 3f_0$

Solution

$$g(t) = A\cos(2\pi f_0 t)$$

$$G(f) = \frac{A}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

Hence,

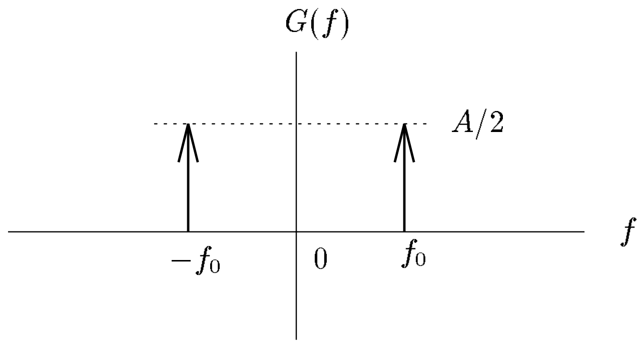
$$\begin{aligned} G_\delta(f) &= f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \\ &= \frac{Af_s}{2} \sum_{m=-\infty}^{\infty} [\delta(f - f_0 - mf_s) + \delta(f + f_0 - mf_s)] \end{aligned}$$

(i) $f_s = f_0$

$$G_\delta(f) = \frac{Af_0}{2} \sum_{m=-\infty}^{\infty} [\delta(f - f_0 - mf_0) + \delta(f + f_0 - mf_0)]$$

(ii) $f_s = 2f_0$

$$G_\delta(f) = Af_0 \sum_{m=-\infty}^{\infty} [\delta(f - f_0 - 2mf_0) + \delta(f + f_0 - 2mf_0)]$$



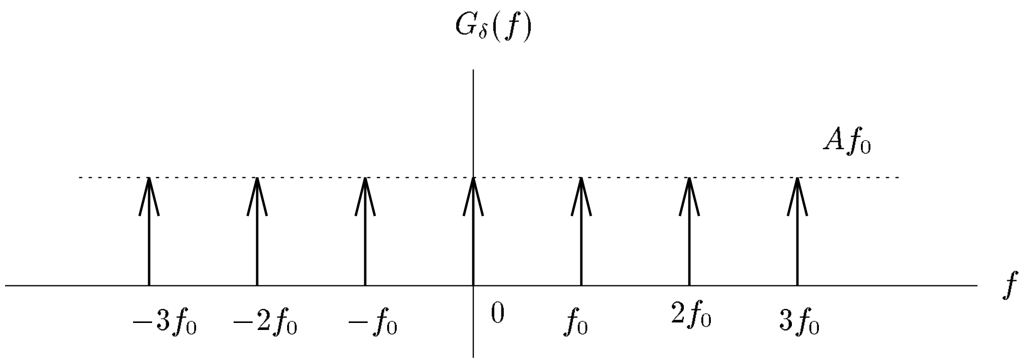


Figure 1: $f_s = f_0$

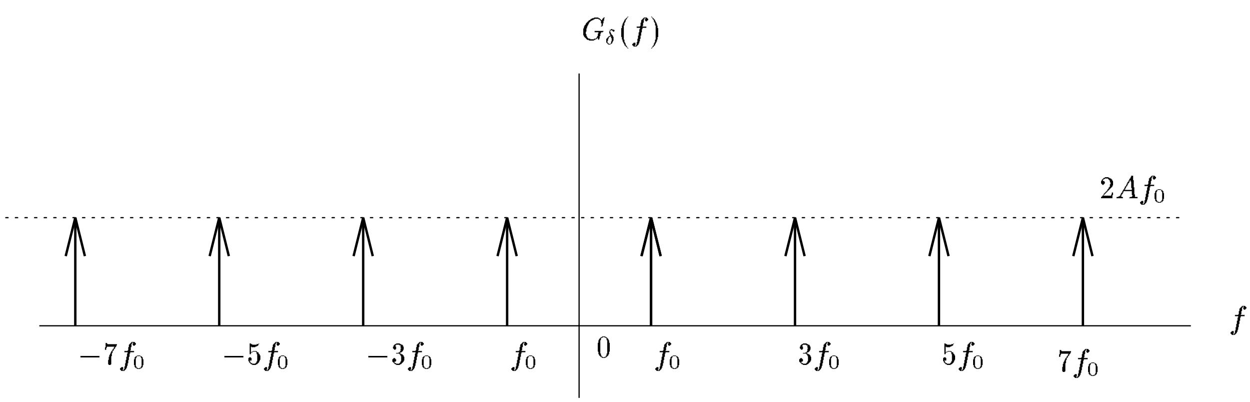


Figure 2: $f_s = 2f_0$

(iii) $f_s = 3f_0$

$$G_\delta(f) = \frac{3Af_0}{2} \sum_{m=-\infty}^{\infty} [\delta(f - f_0 - 3mf_0) + \delta(f + f_0 - 3mf_0)]$$

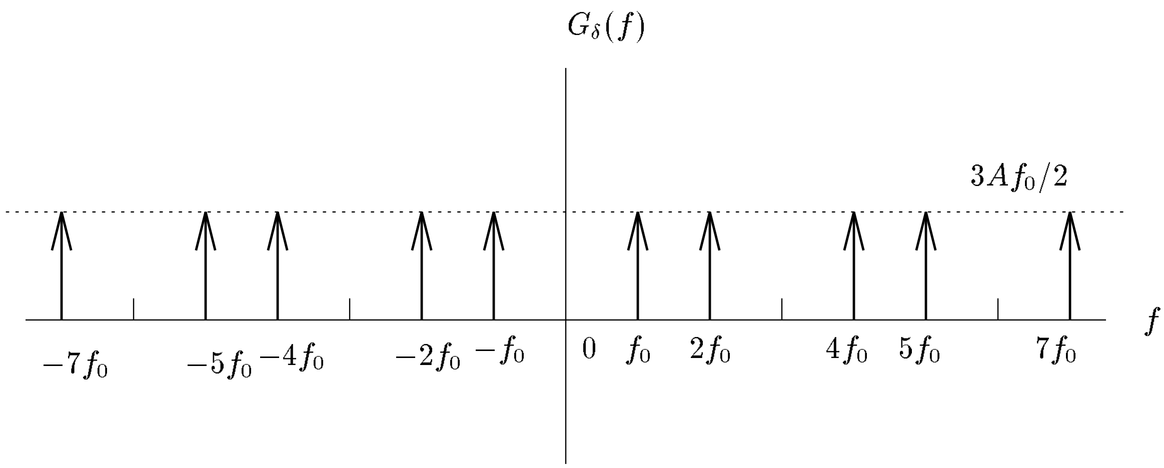


Figure 3: $f_s = 3f_0$

Problem 4.1.2

The signal

$$g(t) = 10\cos(20\pi t)\cos(200\pi t)$$

is sampled at the rate 250 samples per second.

(a) Determine the spectrum of the resulting sampled signal.

(b) Specify the cutoff frequency of the ideal reconstruction filter so as to recover $g(t)$.

(c) What is the Nyquist rate for $g(t)$.

Solution

The signal $g(t)$ is

$$\begin{aligned} g(t) &= 10\cos(20\pi t)\cos(200\pi t) \\ &= 5[\cos(220\pi t) + \cos(180\pi t)] \end{aligned}$$

Hence,

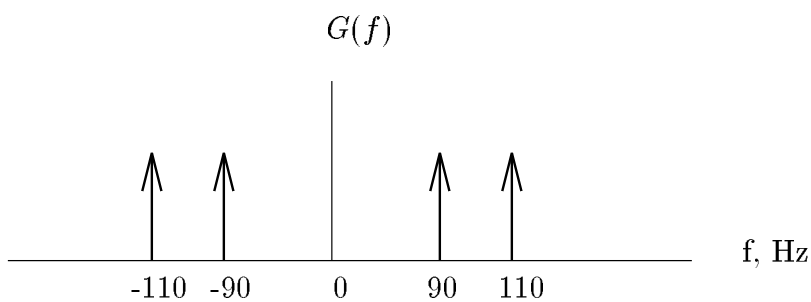
$$G(f) = 2.5[\delta(f-110) + \delta(f+110) + \delta(f-90) + \delta(f+90)]$$

Correspondingly the spectrum of the sampled version of $g(t)$ with a sampling period $T_s = 1/250$s is given by

$$\begin{aligned} G_\delta(f) &= f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \\ &= 625 \sum_{m=-\infty}^{\infty} \delta(f-110-250m)+\delta(f+110-250m)+\delta(f-90-250m)+\delta(f+90-250m)] \end{aligned}$$

(b) the spectrum $G(f)$ and $G_\delta(f)$ are illustrated in Fig. (4). From this figure we deduce that in order to reconstruct the original signal $g(t)$ from $g_\delta(t)$, we need to use a low-pass filter with a cutoff frequency greater than 110Hz but less than 140Hz.

(c) The highest frequency component of $g(t)$ is 110Hz. Hence, the Nyquist rate of $g(t)$ is 220Hz.



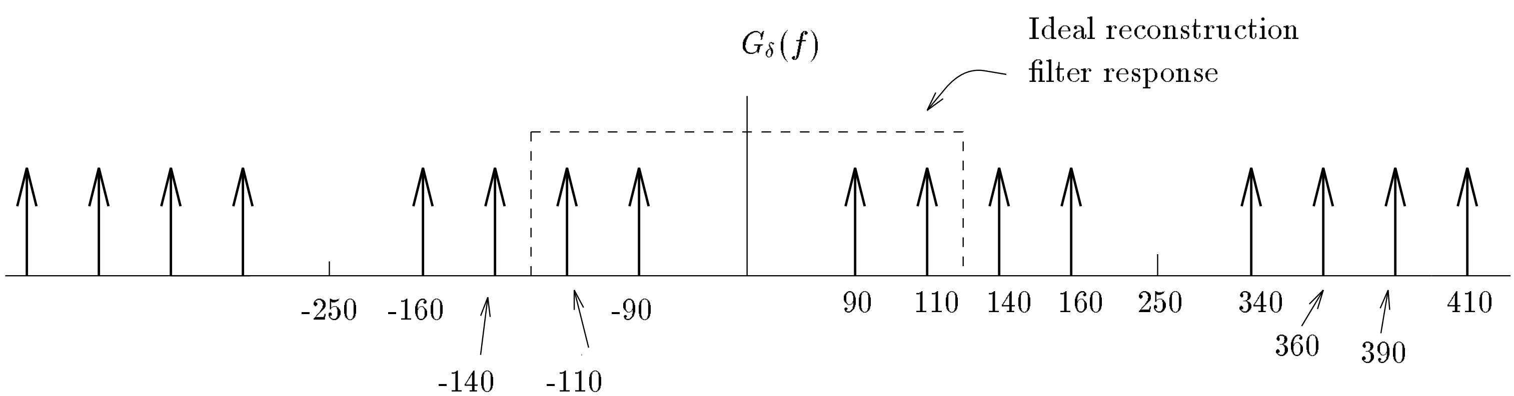


Figure 4: Problem: 4.1.2

Problem 4.1.3

A signal $g(t)$ consists of two frequency components $f_1 = 3.9$ kHz and $f_2 = 4.1$ kHz in such a relationship that they just cancel each other out when the signal $g(t)$ is sampled at the instants $t = 0, T, 2T, \cdots$, where $T = 125\mu s$. The signal $g(t)$ is defined by

$$g(t) = \cos\left(2\pi f_1 t + \frac{\pi}{2}\right) + A\cos(2\pi f_2 t + \phi)$$

find the values of amplitude A and phase ϕ of the second frequency component.

Solution

The signal at the sampling instants is

$$\begin{aligned} g(nT) &= \cos(2\pi f_1 nT + \frac{\pi}{2}) + A\cos(2\pi f_2 nT + \phi) \\ &= 0, \quad n = 0, 1, 2, \cdots \end{aligned}$$

At $n = 0$,

$$\cos(\frac{\pi}{2}) + A\cos(\phi) = 0$$

Hence,

$$A\cos(\phi) = 0 \qquad (1)$$

At $n = 1$, with $f_1 = 3.9$ kHz, $f_2 = 4.1$ kHz, and $T = 125\mu s$, we have

$$\cos(0.975\pi + \frac{\pi}{2}) + A\cos(1.025\pi + \phi) = 0 \qquad (2)$$

From (1), we deduce that A must be nonzero. Hence, $\phi = \pm\pi/2$. Accordingly, (2) simplifies as

$$-\sin(0.975\pi) \pm A\sin(1.025\pi) = 0 \qquad (3)$$

But $\sin(1.025\pi) = -\sin(0.025\pi)$ and $\sin(0.975\pi) = +\sin(0.025\pi)$. To satisfy (3), we must therefore have

$$A = 1$$

and the ambiguous sign (of ϕ) must be negative. That is

$$\phi = -\pi/2$$

Problem 4.1.4

Let E denote the energy of a strictly band-limited signal $g(t)$. Show that E may be expressed in terms of the sample values of $g(t)$, taken at the Nyquist rate, as follows:

$$E = \frac{1}{2W} \sum_{n=-\infty}^{\infty} \left| g\left(\frac{n}{2W}\right) \right|^2$$

where W is the highest frequency component of $g(t)$.

Solution

If $g(t)$ is band-limited to $-W \le f \le W$, we may express it as:

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}(2Wt - n)$$

The energy of $g(t)$ is therefore

$$\begin{aligned} E &= \int_{-\infty}^{\infty} g(t) g^*(t) dt \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \mathrm{sinc}(2Wt - n) \sum_{k=-\infty}^{\infty} g^*\left(\frac{k}{2W}\right) \mathrm{sinc}(2Wt - k) \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g\left(\frac{n}{2W}\right) g^*\left(\frac{k}{2W}\right) \int_{-\infty}^{\infty} \mathrm{sinc}(2Wt - n)\mathrm{sinc}(2Wt - k) \qquad (1) \end{aligned}$$

But

$$\int_{-\infty}^{\infty} \mathrm{sinc}(2Wt - n)\mathrm{sinc}(2Wt - k) = \begin{cases} \frac{1}{2W}, & k = n \\ 0, & k \neq n \end{cases}$$

Hence, we may satisfy (1) as

$$E = \frac{1}{2W} \sum_{n=-\infty}^{\infty} \left| g\left(\frac{n}{2W}\right) \right|^2$$

Problem 4.1.5

Consider a continuous-time signal $g(t)$ of finite energy, with a continuous spectrum $G(f)$. Assume that $G(f)$ is sampled uniformly at the discrete frequencies $f = kF_s$, thereby obtaining the sequence of frequency samples $G(kF_s)$, where k is an integer in the entire range $-\infty < k < \infty$, and F_s is the frequency sampling interval. Show that if $g(t)$ is duration-limited, so that it is zero outside the interval $-T < t < T$, then the signal is completely defined by specifying $G(f)$ at frequencies spaced $1/2T$ hertz apart.

Solution

Since $g(t) = 0$ outside the interval $-T < t < T$, we may express the Fourier transform of $g(t)$ as

$$G(f) = \int_{-T}^{T} g(t)\exp(-j2\pi ft)dt \qquad (1)$$

Expanding $g(t)$ as a Fourier series, with period $2T$, we have

$$g(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(\frac{j\pi kt}{T}\right), \qquad -T < t < T$$

where

$$c_k = \frac{1}{2T}\int_{-T}^{T} g(t)\exp\left(-\frac{j\pi kt}{T}\right)dt \qquad (2)$$

Comparing (1) and (2), we deduce that

$$c_k = \frac{1}{2T}G\left(\frac{k}{2T}\right) \quad = \quad F_s G(kF_s)$$

where $F_s = \frac{1}{2T}$
Therefore,

$$g(t) = F_s \sum_{k=-\infty}^{\infty} G(kF_s)\exp(j2\pi kF_s t), \qquad -T < t < T \qquad (3)$$

Substituting (3) in (1), we get

$$\begin{aligned} G(f) &= F_s \int_{-T}^{T} \sum_{k=-\infty}^{\infty} G(kF_s)\exp(j2\pi kF_s t)\exp(-j2\pi ft)dt \\ &= F_s \sum_{k=-\infty}^{\infty} G(kF_s)\int_{-T}^{T} \exp[-j2\pi t(f - kF_s)]dt \\ &= \sum_{k=-\infty}^{\infty} G\left(\frac{k}{2T}\right)\mathrm{sinc}(2Tf - k) \qquad (4) \end{aligned}$$

We may thus state that

1. The signal $g(t)$, and therefore its spectrum $G(f)$ is uniquely determined in terms of samples of $G(f)$ taken at the rate $F_s = 1/2T$.

2. Given the frequency samples $\{G(k/2T)\}$, $k = 0, \pm 1, \pm 2, \cdots$, the original spectrum $G(f)$ can be reconstructed without distortion.

Problem 4.2.1

The spectrum of a band-pass signal occupies a band of width 0.5 kHz, centered around ±10KHz. Find the Nyquist rate for quadrature sampling the in-phase and quadrature components of the signal.

Solution

$$g(t) = g_I(t)\cos(2\pi \times 10^3 t) - g_Q(t)\sin(\pi \times 10^3 t)$$

where $g_I(t)$ and $g_Q(t)$ are low-pass signals with a bandwidth:

$$W = \frac{1}{2} \times 0.5 = 0.25 \quad \text{kHz}$$

The Nyquist rate for $g_I(t)$ and $g_Q(t)$ is therefore

$$2W = 0.5 \quad \text{kHz}$$

Problem 4.4.1

The signals

$$g_1(t) = 10\cos(100\pi t)$$

and

$$g_2(t) = 10\cos(50\pi t)$$

are both sampled at times $t_n = n/f_s$, where $n = 0, \pm 1, \pm 2, \cdots$, and $f_s = 75$ samples per second. Show that the two sequences of samples thus obtained are identical.

Solution

We note that

1. The Nyquist rate of $g_1(t)$ is 100 Hz; hence, with a sampling rate of 75 Hz, the signal $g_1(t)$ is under-sampled by 25 Hz below the Nyquist rate.

2. The Nyquist rate of $g_2(t)$ is 50 Hz; hence, with a sampling rate of 75 Hz, the signal $g_2(t)$ is over-sampled by 25 Hz above the Nyquist rate.

3. Although $g_1(t)$ and $g_2(t)$ represent sinusoidal waves of different frequencies, by under-sampling $g_1(t)$ and over-sampling $g_2(t)$ appropriately, their sampled versions are identical.

Problem 4.4.2

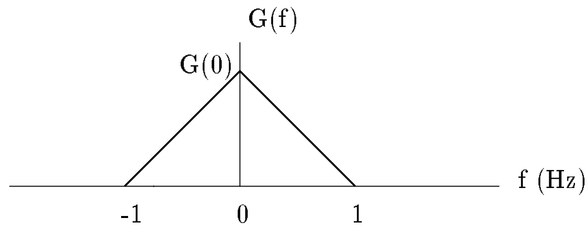


Figure 5: Problem 4.4.2

Figure 5 shows the spectrum of a low-pass signal $g(t)$. The signal is sampled at the rate of 1.5 Hz, and then applied to a low-pass reconstruction filter with cutoff frequency 1 Hz. Plot the spectrum of the resulting signal.

Solution

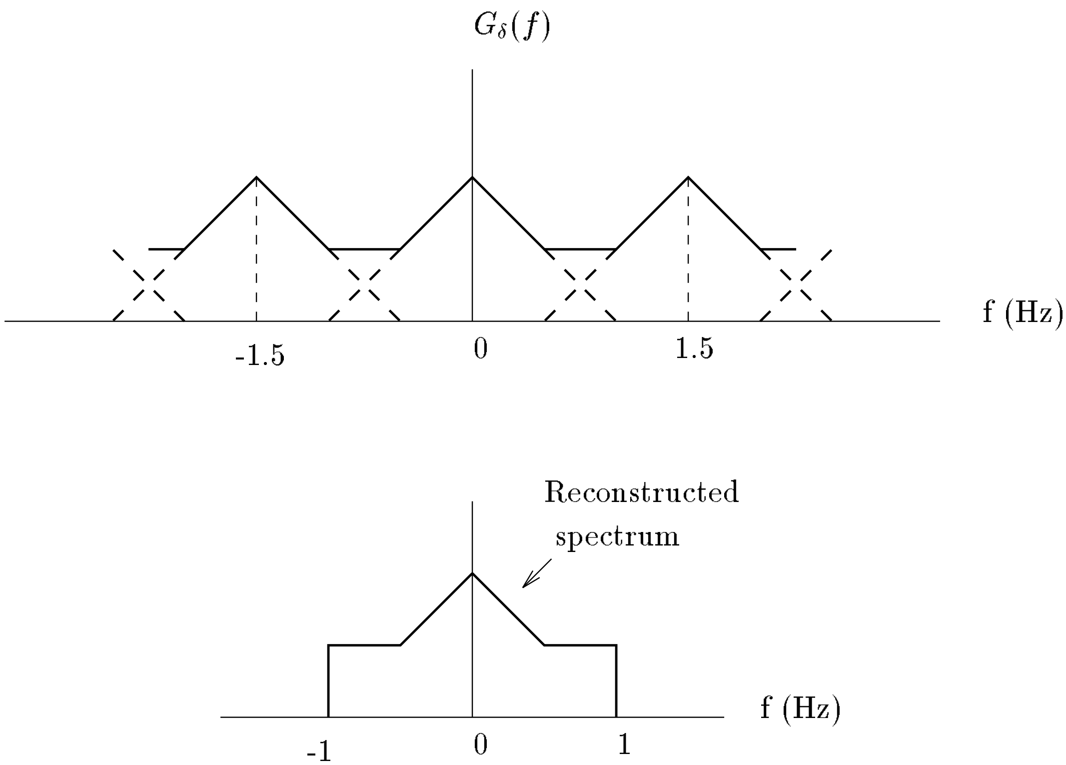


Figure 6: Problem 4.4.2

Problem 4.5.2

This problem is aimed at investigating the fact that the practical electronic circuits will not produce a sampling function that consists of exactly rectangular pulses. Let $h(t)$ denote some arbitrary pulse shape so that the sampling function $c(t)$ may be expressed as

$$c(t) = \sum_{n=-\infty}^{\infty} h(t - nT_s)$$

where T_s is the sampling period. The sampled version of an incoming analog signal $g(t)$ is defined by

$$s(t) = c(t)g(t)$$

(a) Show that the Fourier transform of $s(t)$ is given by

$$S(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s)H(nf_s)$$

where $G(f) = F[g(t)], H(f) = F[h(t)]$, and $f_s = 1/T_s$.
(b) What is the effect of using the arbitrary pulse shape $h(t)$?

Solution

(a) The Fourier transform of the sampling function is

$$\begin{aligned} C(f) &= \sum_{m=-\infty}^{\infty} H(f)\exp(-j2\pi m f T_s) \\ &= H(f) \sum_{m=-\infty}^{\infty} \exp(-j2\pi m f T_s) \\ &= \frac{H(f)}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_s}\right) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H\left(\frac{k}{T_s}\right) \delta\left(f - \frac{k}{T_s}\right) \end{aligned}$$

The Fourier transform of the sampled signal is

$$\begin{aligned} S(f) &= C(f) * G(f) \\ &= \frac{1}{T_s}\left\{ \sum_{k=-\infty}^{\infty} H\left(\frac{k}{T_s}\right) \delta\left(f - \frac{k}{T_s}\right) \right\} * G(f) \\ &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} H\left(\frac{k}{T_s}\right) G\left(f - \frac{k}{T_s}\right) \end{aligned}$$

(b) Using an arbitrary pulse shape $h(t)$ means that the sampled spectrum is no longer periodic. Instead, each replica of $G(f)$, centered at $f = k/T_s$, is multiplied by a frequency dependent constant $H(k/T_s)$. However, when the signal is reconstructed by a low-pass filter, all replicas are removed, leaving, $T_s^{-1}H(0)G(f)$. Thus, except for a scaling factor, an arbitrary sampling function will not affect the reconstructed signal.

Problem 4.7.3

Twenty-four voice signals are sampled uniformly and the time-division multiplexed. The sampling operation uses flat-top samples with 1 microsecond duration. The multiplexing operation includes the provision for synchronization by adding an extra pulse of sufficient amplitude and also 1 microsecond duration. The highest frequency component of each voice signal is 3.4 kHz.

(a) Assuming a sampling rate of 8 kHz, calculate the spacing between successive pulses of the multiplexed signal.

(b) Repeat your calculation assuming the use of Nyquist rate sampling.

Solution

(a) The sampling period is $T_s = 1/8000 = 125\mu s$. There are 24 channels and 1 sync pulse. Hence the time allotted to each channel is

$$T_c = \frac{T_s}{25} = 5\mu s$$

The pulse duration is 1 μs, and so the time between pulses is $4\mu s$.

(b) Assuming the use of sampling at the Nyquist rate (6.8 kHz), the sampling period is

$$T_s = \frac{1}{6.8 \times 10^{-3}} = 0.147 \times 10^{-3} s = 147\mu s$$

Correspondingly,

$$T_c = \frac{147}{25} = 6.68\mu s$$

$$\text{Time between pulses } = 5.68\mu s$$