E&CE318 Problems

Chapter2:

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Chapter3:

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Chapter4:

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Chapter5:

5.1.2 — 5.1.3 — 5.1.4 — 5.1.6 — 5.1.10 — 5.2.4 — 5.2.6 — 5.2.10 — 5.2.11 — 5.4.2 — 5.5.1 — 5.6.1 — 5.7.1 — 5.7.2 — 5.7.4

Problem 2.2.1 Classify the following signals as energy or power signals and find the normalized energy or power for each. (All signals are defined over $-\infty < t < \infty$.)

(a) 4

(b) $\cos t + \cos 2t$

(c) $e^{-2|t|}$

(d) $e^{j2\pi t}$

Solution: Recall that the total energy of a signal $f(t)$ is given by

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

A signal is an "energy signal" if its total energy content is finite. If $E = \infty$, then if

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt < \infty,$$

the signal is called a power signal.

Thus in our case:

(a)

$$\int_{-\infty}^{\infty} |4|^2 dt = \infty$$

is NOT an Energy signal, while,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |4|^2 dt = 16W$$

shows that it is a power signal.

(b)

$$\int_{-\infty}^{\infty} |\cos t + \cos 2t|^2 \; dt = \infty$$

(Consider the integral as an area under the curve of the integrand). This shows that it is not an energy signal, while,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\cos t + \cos 2t|^2 dt =$$

$$\lim_{T\to\infty} \frac{1}{T} \left[\int_{-T/2}^{T/2} \cos^2 t + \cos^2 2t + 2\cos t \cos 2t dt \right]$$

Using the identities: (i) $\cos^2 t = \frac{1}{2}(1+\cos 2t)$, and, (ii) $2\cos 2t \cos t = \cos(2t+t) + \cos(2t-t) = \cos 3t + \cos t$, the value of the integral is computed as,

$$P = \frac{1}{2} + \frac{1}{2} = 1 \; W$$

(c) Observe that,

$$\int_{-\infty}^{\infty} (e^{-2|t|})^2 dt = \int_{-\infty}^{\infty} e^{-4|t|} dt = 2\int_{0}^{\infty} e^{-4t}\ dt = \frac{2}{-4}(0-1) = 0.5\ J$$

This is therefore an energy signal.

(d)

$$\int_{-\infty}^{\infty} |e^{j2\pi t}|^2 dt = \int_{-\infty}^{\infty} |1|^2 dt = \infty \quad \Rightarrow \quad \text{Not an energy signal.}$$

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |1|^2 dt = 1\ W \quad \Rightarrow \quad \text{Power signal.}$$

Problem 2.2.2 Determine if each of the following signals is periodic. If a signal is periodic, determine its periodic.

(a) $\cos(\sqrt{3}t)$
(b) $e^{(j2\pi t - \pi/4)}$
(c) $\sin^2(2t)$
(d) $\sin(2t^2)$

Solution: Recall that a signal $f(t)$ is said to be periodic if there exists a number T such that for all t, $f(t) = f(t+T)$. The number T is the period of the signal.

(a) $\cos(\sqrt{3}t)$ is periodic because a number T can be found such that $\cos(\sqrt{3}t) = \cos[\sqrt{3}(t+T)]$. This number is easily found to be $T = 2\pi/\sqrt{3}$ (note that we are referring to the smallest possible number here.).

(b)

$$e^{[j2\pi(t+T)-\pi/4]} = e^{j2\pi t}e^{-\pi/4}e^{j2\pi T} = e^{j2\pi t}e^{-\pi/4} \quad (\text{if } T = 1)$$

(since $e^{j2\pi} = 1$). This means that the function is periodic with period $T = 1$.

(c)

$$\sin^2(2t) = \frac{1}{2}[1 - \cos(4t)] \tag{1}$$

$$\sin^2 2(t+T) = \frac{1}{2}[1 - \cos 4(t+T)] \tag{2}$$

If the function is periodic then (1) and (2) above must be equal $\forall t$. This happens if, $\cos[4(t+T)] = \cos(4t)$, or if, $4T = 2\pi$, or $T = \pi/2$.

(d) Given $f(t) = \sin(2t^2)$, recall that a sine function is periodic with period $2K\pi$. If $f(t)$ is periodic, then

$$\sin\left(2t^2 + 2K\pi\right) = \sin\left[2(t+T)^2\right]$$

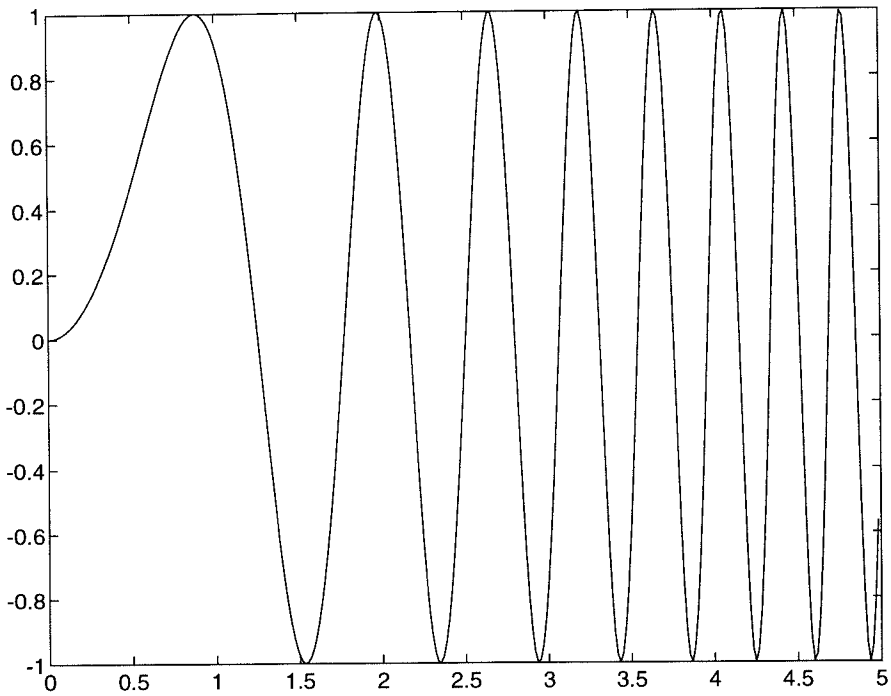

Figure 1: The function $f(t) = \sin(2t^2)$.

where T is the period. This means that, $2t^2+2K\pi = 2(t+T)^2$. This simplifies to: $2T^2+4Tt = 2K\pi$. This means that $f(t)$ is not periodic, since we cannot get a constant value for the period T from the above equation.

Problem 2.3.1 Let $f(t)$ be the input to a given system and let $g(t)$ be the corresponding output. The input-output relationships of several systems are given below. Classify the systems whether they are linear, time invariant, causal.

(a) $g(t) = \frac{d}{dt}[f(t)]$

(b) $g(t) = f(t) \ + \ f(-t)$

(c) $g(t) = f(t/2)$

(d) $g(t) = \exp\left[2f(t)\right]$

Solution:

(a) Consider two different inputs to the system, $f_1(t)$ and $f_2(t)$. The output corresponding to the input $\alpha f_1(t) + \beta f_2(t)$ is :

$$\begin{aligned} \frac{d}{dt}[\alpha f_1(t) + \beta f_2(t)] &= \frac{d}{dt}[\alpha f_1(t)] \ + \ \frac{d}{dt}[\beta f_2(t)] \\ &= \alpha g_1(t) + \beta g_2(t) \end{aligned}$$

where $g_1(t)$ and $g_2(t)$ are the outputs corresponding to inputs $f_1(t)$ and $f_2(t)$ $\Rightarrow$ Linearity is proved.

To prove time invariance, let $g_1(t)$ be the output corresponding to the input $f_1(t)$ and $g_2(t)$ be the output corresponding to the time shifted input $f_2(t) = f_1(t - t_0)$.

$$g_1(t) = \frac{d}{dt}[f_1(t)]$$

$$g_2(t) = \frac{d}{dt}[f_2(t)] = \frac{d}{dt}[f_1(t - t_0)] = g_1(t - t_0)$$

Since $g_2(t) = g_1(t - t_0)$ it follows that the system is time invariant.
To check for causality, recall that for a causal system, the output of the system at any time t_0, $g(t_0)$, is only dependent on the values of $t \leq t_0$. Considering this definition, we conclude that the system is causal.

(b) The output corresponding to inputs $\alpha f_1(t) + \beta f_2(t)$ is

$$\begin{aligned} &= \alpha f_1(t) + \alpha f_1(-t) + \beta f_2(t) + \beta f_2(-t) \\ &= \alpha[f_1(t) + f_1(-t)] + \beta[f_2(t) + f_2(-t)] \\ &= \alpha g_1(t) + \beta g_2(t) \end{aligned}$$

$\Rightarrow$ Linearity is proved.
We have,

$g(t) = \mathcal{T}\{f(t)\} = f(t) + f(-t)$
Output for the input$f(t - t_0) \Rightarrow \mathcal{T}\{f(t - t_0)\} = f(t - t_0) + f(-t - t_0)$
$g(t)$ shifted to $t_0 \Rightarrow f(t - t_0) + f(-(t - t_0)) = f(t - t_0) + f(-t + t_0)$

$$g(t - t_0) = f(t - t_0) + f(-t + t_0) \neq f(t - t_0) + f(-t - t_0)$$

$\Rightarrow$ Not time invariant.

This is not a causal system because, if we consider (for instance) a step input function $f(t) = \alpha u(t)$, for $t \leq 0$, we have, $f(t) = 0$ but $g(t) = \alpha[u(t) + u(-t)] = \alpha$, which depends on $f(t)$ for $t > 0 \Rightarrow$ the system is non-causal.

(c) The system is linear since when $f(t) = \alpha f_1(t) + \beta f_2(t)$, the corresponding output is, $g(t) = \alpha f_1(t/2) + \beta f_2(t/2) = \alpha g_1(t) + \beta g_2(t)$.
The system is not time invariant because the output corresponding to $f(t - t_0)$ is $f[(t/2) - t_0]$ which is not equal to $f[(t - t_0)/2]$.
The system is causal.

(d)

$$\exp[2f_1(t) + 2f_2(t)] \neq \exp[2f_1(t)] + \exp[2f_2(t)]$$

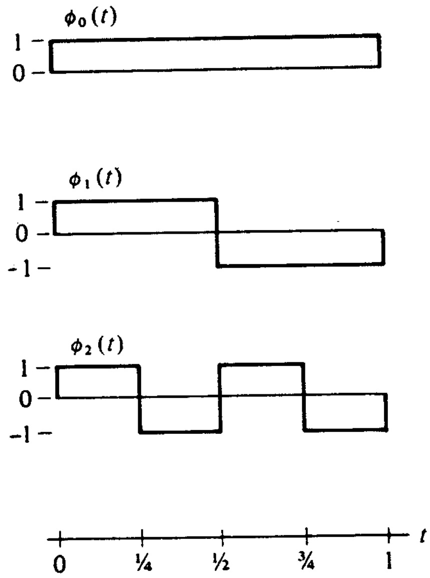

Figure 2: Set of functions related to the problem 2.5.3.

⇒ System is not linear

It is easy to show that the system is time invariant and causal.

Problem 2.5.3: A given set of functions, $\Phi_n(t)$, is shown in Fig. 2.

(a) Show that these functions form an orthogonal set over the interval $(0, 1)$. Is the set an orthonormal set?

(b) Represent the given signal $f(t) = 2t$ over the interval $(0, 1)$ using this set of orthogonal functions.

(c) Sketch $f(t)$ and the representation of $f(t)$ on the same graph and compare.

(d) Compute the energy in each term of the series and the error energy remaining after each term is included.

Solution: (a) Recall that two Complex valued functions $\Phi_n(t)$ and $\Phi_m(t)$ are said to be Orthogonal over the interval (t_1, t_2) if :

$$\int_{t_1}^{t_2} \Phi_n(t)\Phi_m^*(t)dt = \begin{cases} 0 & \text{if } n \neq m \\ K_m & \text{if n = m} \end{cases}$$

Using the given functions it can therefore be shown that:

$$\int_0^1 \Phi_0(t)\Phi_1(t)dt = 0$$

$$\int_0^1 \Phi_0(t)\Phi_2(t)dt = 0$$

and

$$\int_0^1 \Phi_1(t)\Phi_2(t)dt = 0$$

likewise.

$$\int_0^1 \Phi_0^2(t)dt = \int_0^1 \Phi_1^2(t)dt = \int_0^1 \Phi_2^2(t)dt = 1$$

Thus the set is orthogonal and orthonormal over the given interval.

(b) $f(t) \approx \sum_{n=0}^{2} f_n \Phi_n(t)$ over the interval $(0, 1)$, where $f_n = \int_0^1 f(t)\Phi_n^*(t)dt$.
Evaluating, we get: $f(t) \approx \Phi_0(t) - \frac{1}{2}\Phi_1(t) - \frac{1}{4}\Phi_2(t)$ over (0,1).

(c) Shown in Fig. (3).

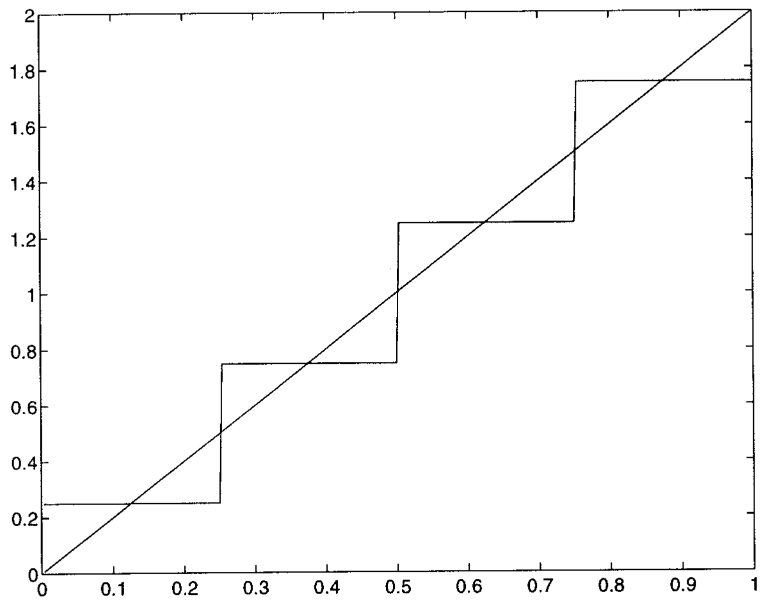

Figure 3: Expansion of the function $f(t) = 2t$ over the basis of problem 2.5.3.

(d) Energies of the terms $f_n\phi_n$ are equal to:
$E_0 = (1)^2(1) = 1$
$E_1 = (1/2)^2(1) = 1/4$
$E_2 = (1/4)^2(1) = 1/16$
To compute the energy of the error signal after each successive term is added, recall that the energy of the error signal in each case is equal to the difference between the energy of the actual signal $f(t)$ and the sum of the energies of the individual terms, $f_n\Phi_n(t)$, used in representing $f(t)$.
The total energy of the original signal is $E_s = \int_0^1 (2t)^2 dt = 4/3$
Energy of the error signal using $\phi_0 \Rightarrow E_s - E_0 = 4/3 - 1 = 1/3$.
Energy of the error signal using $\phi_0, \phi_1 \Rightarrow E_s - E_0 - E_1 = 4/3 - 1 - 1/4 = 1/12$.

Energy of the error signal using $\phi_0, \phi_1, \phi_2 \;\Rightarrow\; E_s - E_0 - E_1 - E_2 = 4/3 - 1 - 1/4 - 1/16 = 1/48$.

Note that ϕ_0, ϕ_1, ϕ_2 form a basis for the set of functions which have a constant value over each of the subranges $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, $[3/4, 1]$.

Problem 2.7.1: The complex exponential Fourier series of the signal $f(t) = a\exp(-a|t|)$ over a certain symmetric interval $(-T/2, T/2)$ is :

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{a^2(1 - e^{-a}\cos \pi n)}{a^2 + n^2\pi^2} e^{jn\pi t}$$

(a) Determine the value of T.

(b) What is the average value of $f(t)$? (Use two ways.)

(c) The component of $f(t)$ at a certain frequency can be expressed as $A\cos 3\pi t$. Determine the numerical value of the constant A.

(d) Write the first three non-zero terms in a series expansion for the value $f(t)$ at $t = 0$.

Solution: (a) Recall the representation

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}, \qquad t_1 < t < t_2$$

It is seen that $n\omega_0 = n\pi$ and therefore,

$$\omega_0 \equiv 2\pi/T = \pi \quad \Rightarrow \quad T = 2 \text{ seconds}$$

(b) Method I

The average value of $f(t)$ is equal to F_0 obtained by putting $n = 0$ in the expression for F_n, i.e.,

$$F_0 = \frac{a^2(1 - e^{-a}\cos 0)}{a^2 + 0\pi^2} = 1 - e^{-a}$$

Method II

Average value =

$$\frac{1}{T}\int_{-T/2}^{T/2} ae^{-a|t|}dt = \frac{2a}{T}\int_0^{T/2} e^{-at}dt = a\int_0^1 e^{-at}dt = 1 - e^{-a} \quad \text{(Note that } T = 2)$$

(c) The terms F_3 and F_{-3} contribute to the coefficient of $A\cos 3\pi t$ as seen in the expression below

$$\begin{aligned} A\cos 3\pi t &= \frac{a^2(1 - e^{-a}\cos 3\pi)}{(a^2 + 3^2\pi^2)} e^{j3\pi t} + \frac{a^2(1 - e^{-a}\cos 3\pi)}{[a^2 + (-3)^2\pi^2]} e^{-j3\pi t} \\ &= \frac{2a^2(1 - e^{-a}\cos 3\pi)}{a^2 + 9\pi^2} \cos 3\pi t \end{aligned}$$

Therefore,

$$A = \frac{2a^2(1 - e^{-a}\cos 3\pi)}{a^2 + 9\pi^2}$$

(d) By the same argument as above:

$$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\pi t \quad \Rightarrow \quad f(0) = C_0 + \sum_{n=1}^{\infty} C_n$$

where $C_0 = F_0 = 1 - e^{-a}$ and

$$C_n = \frac{2a^2(1 - e^{-a}\cos n\pi)}{a^2 + n^2\pi^2}$$

The other terms are therefore given by:

$$C_1 = \frac{2a^2(1 + e^{-a})}{a^2 + \pi^2}$$

$$C_2 = \frac{2a^2(1 - e^{-a})}{a^2 + 4\pi^2}$$

The first three term series expansion of $f(t)$ at $t = 0$ is then given by:

$$1 - e^{-a} + 2a^2 \left[\frac{1 + e^{-a}}{a^2 + \pi^2} + \frac{1 - e^{-a}}{a^2 + 4\pi^2} \right]$$

Problem 2.7.3: Find the exponential Fourier series representation of the raised-cosine pulse signal defined by :

$$f(t) = \begin{cases} 1 + \cos 2\pi t & \text{if} \quad |t| < 1/2 \\ 0 & \text{elsewhere} \end{cases}$$

over the interval $(-1, 1)$.

Solution:

$$f(t) = \sum_{n=-\infty}^{+\infty} F_n e^{jn\omega_0 t}$$

where F_n is defined as

$$F_n = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} dt$$

In this case $(t_2 - t_1) = 2$ and $\omega_0 = 2\pi/2 = \pi$. The value of F_n can then be obtained as :

$$\begin{aligned} F_n &= \frac{1}{2} \int_{-1/2}^{+1/2} \left(1 + \frac{e^{j2\pi t}}{2} + \frac{e^{-j2\pi t}}{2} \right) e^{-jn\pi t} dt \\ &= \frac{1}{2} \frac{\sin(n\pi/2)}{(n\pi/2)} + \frac{1}{4} \frac{\sin[(n-2)\pi/2]}{(n-2)\pi/2} + \frac{1}{4} \frac{\sin[(n+2)\pi/2]}{(n+2)\pi/2} \end{aligned}$$

Problem 2.7.4: The expressions for the coefficients in the exponential Fourier Series for a given $f(t)$ over (t_1, t_2) is given by the equation:

$$F_n = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} dt$$

A new function $g(t)$, is formed from $f(t)$ using the operations below. The series coefficients for $g(t)$ are designated by G_n. Determine the relations for G_n in terms of F_n for each of the following casess:

(a) $g(t) = f(at)$
(b) $g(t) = f(t - t_0)$ (assuming periodic $f(t)$)
(c) $g(t) = e^{j\omega_0 t} f(t)$

Solution (a) Note that $g(t) = f(at)$ is defined over $[t_1/a, t_2/a]$. Consequently, the co-efficients G_n are computed by replacing t_1, t_2, ω_0 by $t_1/a, t_2/a, a\omega_0$, respectively, in the formula for F_n. This results in,

$$G_n = \frac{1}{(t_2 - t_1)/a} \int_{t_1/a}^{t_2/a} f(at) e^{-jn\omega_0 at} dt$$

Use the change of variable $x = at$ and $dx = adt$ to obtain:

$$G_n = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(x) e^{-jn\omega_0 x} dx$$

This shows that the fourier series coefficients are invariant with time scaling.

(b)

$$G_n = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t - t_0) e^{-jn\omega_0 t} dt$$

Use the change of variable $x = t - t_0$ and $dx = dt$ so that:

$$\begin{aligned} G_n &= \frac{1}{(t_2 - t_1)} \int_{t_1 - t_0}^{t_2 - t_0} f(x) e^{-jn\omega_0 (x + t_0)} dx \\ &= \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(x) e^{-jn\omega_0 (x + t_0)} dx \\ &= \frac{e^{-jn\omega_0 t_0}}{(t_2 - t_1)} \int_{t_1}^{t_2} f(x) e^{-jn\omega_0 x} dx = e^{-jn\omega_0 t_0} F_n \end{aligned} \quad (3)$$

(c)

$$G_n = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{j\omega_0 t} e^{-jn\omega_0 t} dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{-j(n-1)\omega_0 t} dt = F_{n-1}$$

Problem 2.9.1 (a) Determine the trigonometric Fourier series representation for the signal

$$f(t) = \begin{cases} 2, & -1/2 < t < 1/2 \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

over the interval $(-1, 1)$. (b) Compare your results with those obtained in example 2.5.1.

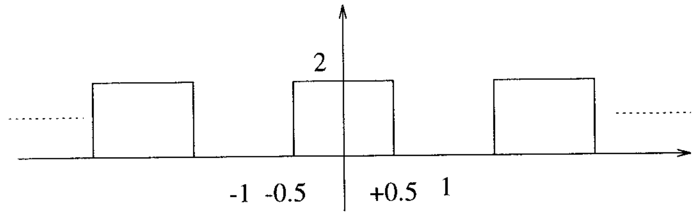

Figure 4: The function related to problem 2.9.1.

Solution: **(a)** (Note: If we already have the coefficient for a periodic square wave, those for $f(t)$ can be found by suitable adjustments.)

For the average value(a_0), we have

$$a_0 = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} f(t)dt = \frac{1}{1-(-1)}\int_{-1/2}^{1/2} 2dt = 2\left(\frac{1}{2}\right)\int_0^{1/2} 2dt = 1 \tag{5}$$

and

$$a_n = \frac{2}{t_2 - t_1}\int_{t_1}^{t_2} f(t)\cos n\omega_0 t dt, \qquad \omega_0 = \frac{2\pi}{t_2 - t_1} = \frac{2\pi}{2} = \pi \tag{6}$$

$$a_n = \frac{2}{1-(-1)}\int_{-1}^{1} f(t)\cos n\omega_0 t\ dt \tag{7}$$

$$a_n = 2\int_0^{1/2} 2\cos n\pi t dt = \frac{4}{n\pi}\sin n\pi t\Big|_0^{1/2} \tag{8}$$

$$= \frac{4}{n\pi}\sin\frac{n\pi}{2}, \quad n \neq 0 \tag{9}$$

$$a_n = 2\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right) = 2\mathrm{Sa}\left(\frac{n\pi}{2}\right) \tag{10}$$

Obviously $b_n = 0$ because of the even symmetry. Finally,

$$f(t) = 1 + 2\sum_{n=1}^{\infty}\mathrm{Sa}\left(\frac{n\pi}{2}\right)\cos n\pi t \tag{11}$$

(b): Expanding $f(t)$, we have,

$$f(t) = 1 + \frac{4}{\pi}\cos \pi t - \frac{4}{3\pi}\cos 3\pi t + \frac{4}{5\pi}\cos 5\pi t - \ldots \tag{12}$$

Comparison with the example 2.5.1 which gives the expansion of the function,

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \end{cases} \tag{13}$$

as,

$$f(t) = \frac{4}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi t + \frac{1}{7}\sin 7\pi t + \ldots\right)$$

- Has the same harmonics (i.e., multiples of ω_0),
- Has an average or DC value of 1, rather than 0,
- Expands in cosines as a result of even symmetry.

Note from a comparison with Drill Problem 2.5.1 that time scaling affects ω_0, but does not affect the Fourier series coefficients.

Problem 2.9.2 Represent the signal,

$$f(t) = \begin{cases} 2 & -1 < t < 0 \\ 1 & 0 < t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

over the interval $(-2, 2)$.

(a) Use the exponential Fourier series.

(b) Use the trigonometric Fourier series.

(c) Compare your results using equations (2.49)-(2.51).

Solution:

(a) Recall the representation

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{-jn\omega_0 t},$$

where

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt.$$

We have $T = 4$, and

$$\omega_0 = 2\pi/T = 2\pi/4 = \pi/2$$

Note that the even and the odd part of $f(t)$ are equal to: $f_e(t) = 3/2, \ -1 < t < 1$ and $f_o(t) = 1/2, \ -1 < t < 0 \ \& \ f_o(t) = -1/2, \ 0 < t < 1$. Breaking the function into even and odd parts, we obtain,

$$F_n = \frac{1}{4}\int_{-1}^{1} \frac{3}{2}e^{-j(n\pi/2)t}dt + \frac{1}{4}\int_{-1}^{0} \frac{1}{2}e^{-j(n\pi/2)t}dt - \frac{1}{4}\int_{0}^{1} \frac{1}{2}e^{-j(n\pi/2)t}dt$$

$$F_n = \frac{3}{4}\frac{\sin(n\pi/2)}{(n\pi/2)} + \frac{1}{8}\frac{\sin(n\pi/4)}{(n\pi/4)}e^{jn\pi/4} - \frac{1}{8}\frac{\sin(n\pi/4)}{(n\pi/4)}e^{-jn\pi/4}$$

$$F_n = \frac{3}{4}\frac{\sin(n\pi/2)}{(n\pi/2)} + \frac{j}{4}\frac{\sin^2(n\pi/4)}{(n\pi/4)}$$

(b)

$$a_0 = \frac{1}{4}\int_{-1}^{0} 2dt + \frac{1}{4}\int_{0}^{1} 1dt = \frac{3}{4}$$

$$a_n = \frac{2}{4}\int_{-1}^{1} \frac{3}{2}\cos(n\pi/2)tdt = \frac{3}{2}\int_{0}^{1}\cos(n\pi/2)tdt = \frac{3}{2}\frac{\sin(n\pi/2)}{(n\pi/2)}$$

$$b_n = -\int_{0}^{1}\frac{1}{2}\sin(n\pi/2)tdt = \frac{1}{2}\frac{\cos(n\pi/2)-1}{(n\pi/2)}$$

(c) It is seen that $a_0 = F_0$, $a_n = 2\mathcal{R}\{F_n\}$, $b_n = -2\mathcal{I}\{F_n\}$.

Problem 2.10.1: One method used in the design of direct current(dc) power supplies is to rectify input waveform. If the rectifier input is $f(t)$ and its output is $g(t)$, then the output of a full-wave rectifier is given by $g(t) = |f(t)|$. In this case let the input to the full-wave rectifier be: $f(t) = A\cos\omega_0 t$. (a) Sketch the input and output waveforms. (b) Determine the dc component in $f(t)$ and in $g(t)$. (c) Find the coefficients of the trigonometric Fourier series of $f(t)$ and $g(t)$ at the frequencies ω_0 and $2\omega_0$.

Solution: (a)

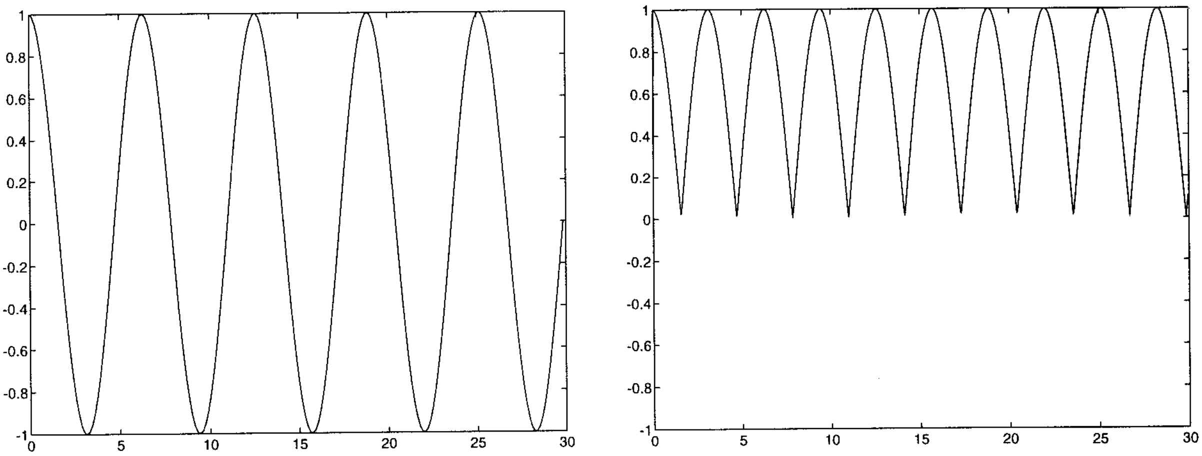

Figure 5: Functions $f(t) = A\cos\omega t$ and $g(t) = A|\cos\omega t|$.

(b) Since $a_0 = F_0 = 0$, the dc component in $f(t)$ is zero. The dc component in $g(t)$ is (Note that $g(t)$ has a period of $T/2$).

$$a_0 = \frac{4A}{T}\int_{0}^{T/4}\cos\omega_0 tdt, \quad \omega_0 = \frac{2\pi}{T} \tag{14}$$

$$a_0 = \frac{4A}{T}\int_{0}^{T/4}\cos\frac{2\pi t}{T}dt = \frac{4A}{T}\frac{1}{\frac{2\pi}{T}}\sin\frac{2\pi t}{T}\Big|_{0}^{\frac{T}{4}} \tag{15}$$

$$= \frac{2A}{\pi}\sin\frac{\pi}{2} = \frac{2A}{\pi} = 0.637A \quad (16)$$

(c) For $f(t)$,

$$a_n = \frac{2A}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\frac{2\pi t}{T}\cos\frac{2n\pi t}{T}\,dt \quad (17)$$

Using the relation $\cos A \cos B = [\cos(A-B) + \cos(A+B)]/2$

$$a_n = \frac{2A}{T}\int_0^{\frac{T}{2}} \cos{(n-1)}\frac{2\pi}{T}t + \frac{2A}{T}\int_0^{\frac{T}{2}} \cos{(n+1)}\frac{2\pi}{T}tdt \quad (18)$$

$$= \frac{A}{(n-1)\pi}[\sin(n-1)\pi] + \frac{A}{(n+1)\pi}[\sin(n+1)\pi] \quad (19)$$

$$= A\mathrm{Sa}[(n-1)\pi] + A\mathrm{Sa}[(n+1)\pi] \quad (20)$$

For $n = 1$, we have

$$a_1 = A\mathrm{Sa}(0) + A\mathrm{Sa}(2\pi) = A(1) + A(0) = A \quad (21)$$

and $a_n = 0$ for $n \geq 2$.
For $g(t)$, we have $b_n = 0$ since there is even symmetry. The a_n's are:

$$a_n = \frac{2A}{T/2}\int_{-\frac{T}{4}}^{\frac{T}{4}} g(t)\cos\frac{2n\pi t}{T/2}dt \quad (22)$$

$$= \frac{8A}{T}\int_0^{\frac{T}{4}} g(t)\cos\frac{4n\pi t}{T}dt \quad (23)$$

$$a_1 = \frac{8A}{T}\int_0^{\frac{T}{4}} \cos\frac{2\pi t}{T}\cos\frac{4\pi t}{T}dt = \frac{4A}{3\pi} \quad (24)$$

and,

$$a_2 = \frac{8A}{T}\int_0^{\frac{T}{4}} \cos\frac{2\pi t}{T}\cos\frac{4\pi t}{T/2}dt = -\frac{4A}{15\pi} \quad (25)$$

(26)

Problem 2.11.1 Show that Parseval's theorem for the trigonometric Fourier series is:

$$\frac{1}{T}\int_0^T [f(t)]^2 dt = a_o^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2). \quad (27)$$

Solution: In terms of the complex Fourier series, the Parseval theorem is as follows:

$$\frac{1}{T}\int_0^T |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2 \quad (28)$$

where $F_n = (a_n - jb_n)/2, \quad n \neq 0$. This results in $|F_n|^2 = (a_n^2 + b_n^2)/4, \quad n \neq 0$. Summing over positive n, this results in,

$$\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{29}$$

also $a_0 = F_0$. The final result is then given by:

$$\frac{1}{T}\int_0^T [f(t)]^2 dt = a_o^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{30}$$

Problem 2.13.1 The periodic signal in problem 2.7.1[1] is applied to the input of a system whose frequency transfer function is:

$$H(\omega) = \begin{cases} 1/(1 + j\omega/\pi) & |\omega| < 3\pi/2 \\ 0 & \text{elsewhere} \end{cases} \tag{31}$$

a.) Determine an expression for the output, $g(t)$, of the system.

b.) Find the average power in the system output.

Solution:

a.) From problem 2.7.1.

$$f(t) = \sum_{n=\infty}^{\infty} \frac{a^2(1 - e^{-a}\cos n\pi)e^{jn\pi t}}{a^2 + n^2\pi^2} \tag{32}$$

We have $T = 2$ s, $f_0 = 1/T = 0.5$ Hz, and $\omega_0 = 2\pi f_0 = \pi$ rad/s.
Recall that the response $g(t)$ of a system to a periodic input $f(t)$ is given by,

$$g(t) = \sum_{n=-\infty}^{+\infty} H(n\omega_0)F_n e^{jn\omega_0 t} \tag{33}$$

Here, as $\omega_0 = \pi$ and $H(\omega) = 0$ for $|\omega| < 3\pi/2$, we obtain,

$$H(n\omega_0) = \begin{cases} \dfrac{1}{1 + jn} & |n| \leq 1 \\ 0 & \text{elsewhere} \end{cases} \tag{34}$$

This means that $H(n\omega_0)$ is nonzero only for $n = -1, 0, 1$. For the signal $f(t)$, we have,

$$F_0 = \frac{a^2(1 - e^{-a})}{a^2} = (1 - e^{-a}) \tag{35}$$

$$F_1 = F_{-1} = \frac{a^2(1 + e^{-a})}{a^2 + \pi^2} \tag{36}$$

[1]This function is equal to, $f(t) = a\exp(-a|t|)$ over $t \in [-T/2, T/2]$.

Then, we obtain,

$$g(t) = (1 - e^{-a}) + \frac{a^2(1+e^{-a})}{a^2+\pi^2}\frac{1}{1+j}e^{j\pi t} + \frac{a^2(1+e^{-a})}{a^2+\pi^2}\frac{1}{1-j}e^{-j\pi t} \tag{37}$$

$$g(t) = (1 - e^{-a}) + \frac{a^2(1+e^{-a})}{a^2+\pi^2}\frac{1}{\sqrt{2}e^{j\pi/4}}e^{j\pi t} + \frac{a^2(1+e^{-a})}{a^2+\pi^2}\frac{1}{\sqrt{2}e^{-j\pi/4}}e^{-j\pi t} \tag{38}$$

or,

$$g(t) = (1 - e^{-a}) + \frac{\sqrt{2}a^2(1+e^{-a})}{a^2+\pi^2}\cos(\pi t - \pi/4) \tag{39}$$

b.) The average output power is,

$$\begin{aligned} P_g &= \sum_{n=-\infty}^{\infty} |H(n\omega_0)|^2 |F_n|^2 \\ &= (1-e^{-a})^2 + \left[\frac{a^2(1+e^{-a})}{a^2+\pi^2}\right]^2 \end{aligned}$$

Problem 2.14.2 A potential problem in push-pull amplifiers is that a voltage pedestal ϵA may be introduced into the sinusoidal waveform, as shown in Fig. P-2.14.2.

a) Develop an expression for the resulting total harmonic distortion (THD).

b) Compute the THD for $\epsilon = 0.10$.

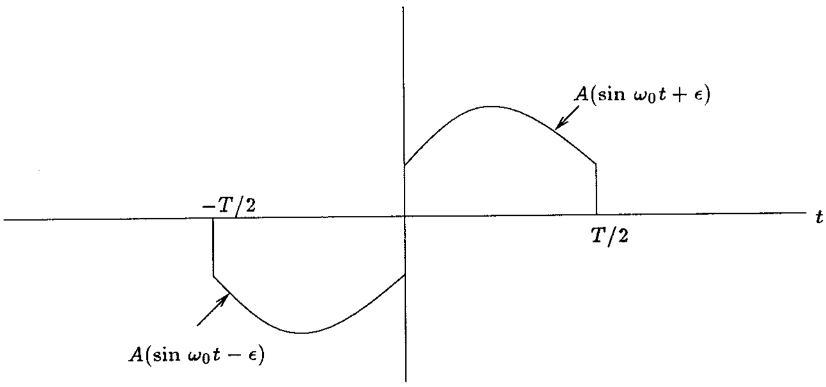

Figure 6: P-2.14.2

Solution:

$$a_0 = 0 \text{ (No dc component)} \tag{40}$$

$$a_n = 0 \text{ (Odd symmetry)} \tag{41}$$

$$b_n = \frac{4}{T}\int_0^{T/2} A(\epsilon + \sin\omega_0 t)\sin n\omega_0 t\, dt \tag{42}$$

$$= \frac{4A}{T}\int_0^{T/2} \sin\omega_0 t \sin n\omega_0 t\, dt + \frac{4A\epsilon}{T}\int_0^{T/2}\sin n\omega_0 t\, dt \tag{43}$$

$$= \begin{cases} A + \dfrac{4A\epsilon}{\pi} & n = 1 \\ \dfrac{4A\epsilon}{\pi n} & n > 1 \quad \text{and odd} \\ 0 & n > 1 \quad \text{and even.} \end{cases} \tag{44}$$

Note that, $\sin mx \sin nx = [\cos(m-n)x - \cos(m+n)x]/2$, and,

$$\int \sin mx \sin nx dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \tag{45}$$

From equation 2.8.1,

$$\text{THD} = \frac{\sum_{n=2}^{\infty}(a_n^2 + b_n^2)}{a_1^2 + b_1^2} \tag{46}$$

which in our case is equal to,

$$\text{THD} = \frac{\sum_{n=2}^{\infty} b_n^2}{b_1^2} = \frac{\sum_{n(odd)=3}^{\infty} (4A\epsilon/\pi n)^2}{[A + (4A\epsilon/\pi)]^2} = \frac{\sum_{n(odd)=3}^{\infty} (4\epsilon/\pi n)^2}{[1 + (4\epsilon/\pi)]^2} \tag{47}$$

Using a result of problem 2.11.2, for the sum of the series

$$\sum_{n(odd)=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \Rightarrow \quad \sum_{n(odd)=3}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} - 1 \tag{48}$$

Applying this result to the numerator, the THD is written as,

$$\text{THD} = \frac{(4\epsilon/\pi)^2[(\pi^2/8) - 1]}{[1 + (4\epsilon/\pi)]^2}. \tag{49}$$

b) Let $\epsilon = 0.1$, then, THD $= 0.298$.

Problem 2.14.3 A proposed method for generation of harmonic content is the triangular wave form with fixed slope shown in Fig. P-2.14.3, where $0 < \tau < T/2$.

a) Determine the amplitude of a given harmonic n.

b) Find the optimum value of τ for the generation of a given harmonic n.

c) For this optimum value of τ, compute the average power in a given harmonic, normalized to the average power in the triangular waveform.

Solution:

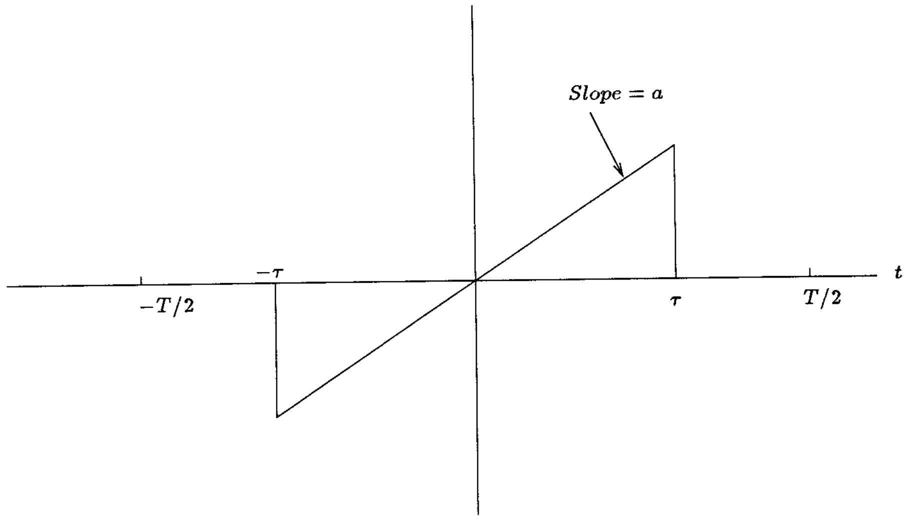

Figure 7: P-2.14.3

a)

$$a_0 = 0 \quad \text{(No dc component)} \tag{50}$$

$$a_n = 0 \quad \text{(Odd symmetry)} \tag{51}$$

$$b_n = \frac{4}{T}\int_0^{\tau} at \sin n\omega_0 t dt = \frac{2a}{\pi\omega_0 n^2}(\sin n\omega_0\tau - n\omega_0\tau \cos n\omega_0\tau). \tag{52}$$

b)

$$\frac{\partial b_n}{\partial \tau} = \frac{2a}{\pi\omega_0 n^2}(n^2\omega_0^2\tau \sin n\omega_0\tau) \tag{53}$$

To maximize the amplitude of the desired harmonic, we set: $\partial b_n/\partial\tau = 0$, from which: $n\omega_0\tau = \pi$, or, $\tau_{opt} = \pi/n\omega_0 = T/2n$, and,

$$(b_n)_{opt} = \frac{2a}{\pi\omega_0 n^2}(\sin \pi - \pi \cos \pi) = \frac{2a}{\omega_0 n^2} \tag{54}$$

c) Average power is,

$$P = \frac{2}{T}\int_0^{\tau}(at)^2 dt = \frac{2a^2\tau^3}{3T} \tag{55}$$

The desired ratio is equal to,

$$\frac{[(b_n)_{opt}]^2/2}{P} = \frac{1}{2}\left(\frac{2a}{\omega_0 n^2}\right)^2\left(\frac{3T}{2a^2\tau^3}\right) = \frac{6}{\pi^2 n}. \tag{56}$$

Problem 2.15.1 The line spectrum, in volts, of a certain periodic function $f(t)$ is shown in Fig. P-2.15.1.

a) What is the value of the period, T?

b) What is the value of $f(0)$?

c) What is the value of $f(1/2)$?

d) A second function, $g(t)$, is related to $f(t)$ by $g(t) = f(t/2)$. Sketch the line spectrum of $g(t)$.

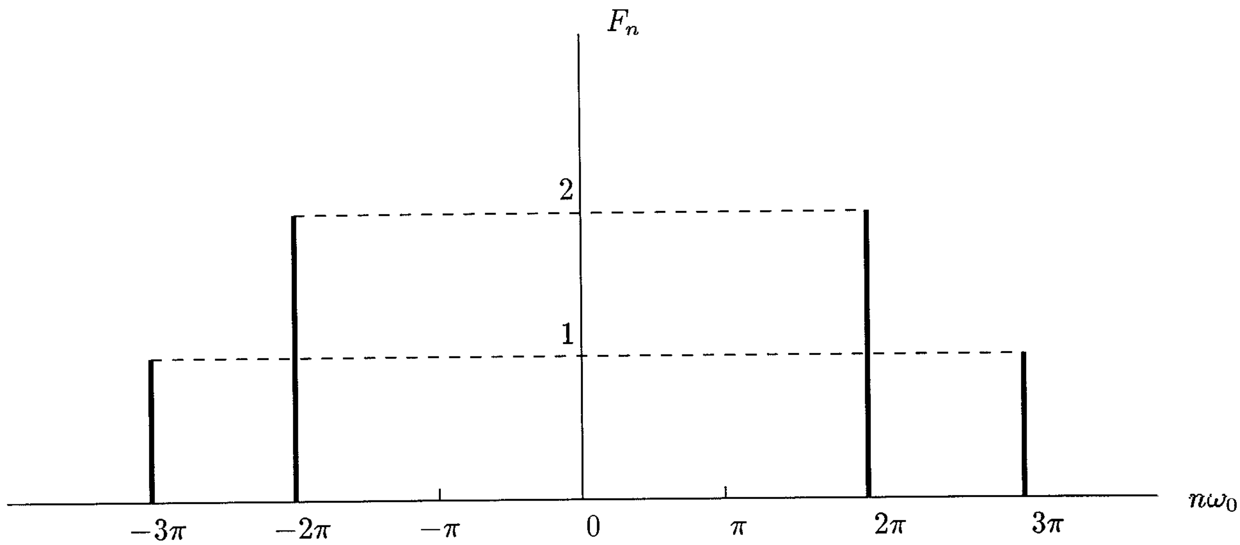

Figure 8: P-2.15.1

Solution:

a) $T = 2\pi/\pi = 2$ seconds.

b) $f(0) = e^{-j3\pi(0)} + 2e^{-j2\pi(0)} + 2e^{j2\pi(0)} + e^{j3\pi(0)} = 1 + 2 + 2 + 1 = 6.$

c) $f(1/2) = e^{-j3\pi(1/2)} + 2e^{-j2\pi(1/2)} + 2e^{j2\pi(1/2)} + e^{j3\pi(1/2)} = j - 2 - 2 - j = -4.$

d) $G_n = \frac{1}{2T}\int_{-T}^{T} f(t/2)e^{-jn[2\pi/(2T)]t}dt = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-jn\omega_0 x}dx = F_n.$

Thus the line spectrum is the same as that for the F_n, with the exception of a change in the frequency scaling (horizontal axis).

Problem 2.15.3 The output, $g(t)$, of a half-wave rectifier for input $f(t)$ can be written as

$$g(t) = \begin{cases} f(t) & \text{if } f(t) > 0, \\ 0 & \text{if } f(t) \leq 0. \end{cases} \tag{57}$$

a) Compute the trigonometric Fourier series for the output if the input is $f(t) = \sin 2\pi t$.

b) Sketch the Fourier magnitude line spectrum for both $f(t)$ and $g(t)$.

Solution: a) Using table 2.2, p. 35, from item #5, we have,

$$f(t) = \begin{cases} \sin \omega_0 t & \text{for } \; 0 \le t \le T/2, \\ 0 & \text{for } \; -T/2 \le t < 0 \end{cases} \quad \Rightarrow \quad F_n = \begin{cases} 1/\pi(1-n^2) & \text{for } \; n \text{ even}, \\ -j/4 & \text{for } \; n = \pm 1, \\ 0 & \text{otherwise.} \end{cases} \tag{58}$$

$g(t)$ is neither even nor odd, so both a_n, b_n are in general nonzero,

$$a_0 = 1/\pi = F_0 \tag{59}$$

$$a_n = 2\mathcal{R}\{F_n\} = \begin{cases} 2/\pi(1-n^2) & \text{for } \; n \text{ even}, \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{60}$$

$$b_n = -2\mathcal{I}\{F_n\} = \begin{cases} 1/2 & \text{for } \; n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{61}$$

$$g(t) = \frac{1}{\pi} + \sum_{n(even)=2}^{\infty} \frac{2}{\pi(1-n^2)} \cos 2n\pi t + \frac{1}{2} \sin 2\pi t \tag{62}$$

b) The corresponding graphs are composed of a set of lines at points corresponding to integer values of n where the height of each line is equal to the corresponding fourier series coefficient.

Problem 2.15.4 The following facts are known about a given waveform $f(t)$:

i) It has zero average value.

ii) It is real-valued and has even symmetry in t.

iii) Coefficients for harmonic terms $|n| = 1$ and $|n| \ge 5$ are zero.

iv) The following sample points are known: $f(0) = 4$, $f(T/4) = -1$, $f(T/2) = 2$.

a) Sketch the Fourier line spectrum of the exponential and trigonometric series for $f(t)$ if $T = 1$.

b) Calculate the percentage of the total average power in the waveform for $|n| > 2$.

Solution:

a) Based on the first three facts, we can write:

$$f(t) = a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + a_4 \cos 4\omega_0 t \tag{63}$$

$$= a_2 \cos 2\frac{2\pi t}{T} + a_3 \cos 3\frac{2\pi t}{T} + a_4 \cos 4\frac{2\pi t}{T}, \tag{64}$$

$$f(0) = a_2 \cos(0) + a_3 \cos(0) + a_4 \cos(0) = a_2 + a_3 + a_4 = 4, \tag{65}$$

$$f(T/4) = a_2 \cos(\pi) + a_3 \cos(3\pi/2) + a_4 \cos(2\pi) = -a_2 + a_4 = -1, \tag{66}$$

$$f(T/2) = a_2 \cos(2\pi) + a_3 \cos(3\pi) + a_4 \cos(4\pi) = a_2 - a_3 + a_4 = 2, \tag{67}$$

$$\tag{68}$$

from which: $a_2 = 2,\ a_3 = 1,\ a_4 = 1.$

b) $[(1)^2 + (1)^2]/[(2)^2 + (1)^2 + (1)^2] = 1/3 = 33.3\%.$

Problem 2.18.1 Show that a periodic train of unit impulse functions, spaced T seconds apart, may be represented by the trigonometric Fourier series:

$$\frac{1}{T} + \frac{2}{T}\sum_{n=1}^{\infty} \cos n\omega_0(t-\tau) \quad \omega_0 = 2\pi/T, \tag{69}$$

if one of the impulses is at $t = \tau$.

Solution: If $\tau = 0$, the train of impulses has even symmetry, so that the $b_n's = 0$, and,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, dt = \frac{1}{T}, \tag{70}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta(t) \cos n\omega_0 t\, dt = \frac{2}{T}. \tag{71}$$

Using a delay of τ units, we obtain the answer given.

Problem 2.18.2 Evaluate the following integrals:

a) $\int_{-\infty}^{\infty} \delta(t-2)e^{-t} \cos \pi(t-1)\, dt$

b) $\int_{-\infty}^{t} \delta(\tau-3)e^{-\tau}\, d\tau$

c) $\int_{-\infty}^{\infty} (t^2+4)\delta(4-t)\, dt$

d) $\int_{-\infty}^{\infty} (t^2+4)\delta(4-2t)\, dt$

e) $\int_{0}^{3}\int_{-\infty}^{\infty} (t^2+4)\delta(t-\tau)\, dt\, d\tau$

Solution: $\delta(t-a)$ is zero everywhere except at $t = a$. Using this fact, the integrals are simply calculated as:

a) $\int_{-\infty}^{\infty} \delta(t-2)e^{-t} \cos \pi(t-1)\, dt = e^{-2}\cos[\pi(2-1)] = -0.135$

b) $\int_{-\infty}^{t} \delta(\tau-3)e^{-\tau}\, d\tau = e^{-3}u(t-3) = 0.05u(t-3)$

c) $\int_{-\infty}^{\infty} (t^2+4)\delta(4-t)\, dt = (4)^2 + 4 = 20$

d) $\int_{-\infty}^{\infty} (t^2+4)\delta(4-2t)\, dt = (1/2)\int_{-\infty}^{\infty} (t^2+4)\delta(2-t)\, dt = [(2)^2+4]/2 = 4$

e) $\int_0^3 \int_{-\infty}^{\infty} (t^2+4)\delta(t-\tau)\, d\tau\, dt = \int_0^3 (\tau^2+4)\, d\tau = (3)^3/3 + 4(3) = 21.$

Problem 2.18.3 Evaluate the following integrals:

a) $\int_{-\infty}^{\infty} tu(t)u(2-t)\, dt$

b) $\int_{-\infty}^{\infty} t[u(t)-u(t-2)]\, dt$

c) $\int_{-\infty}^{\infty} e^{-(t-t_1)}u(t-t_1)\, dt$

d) $\int_{-\infty}^{\infty} \delta(t-t_0)e^{-(t-t_1)}u(t-t_1)\, dt$

e) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t+\tau)}u(t+\tau)\delta(t-\tau)\, d\tau\, dt$

Solution: Similar to the P-2.18.2, we can write,

a) $\int_{-\infty}^{\infty} tu(t)u(2-t)\, dt = \int_0^2 t\, dt = 2$

b) $\int_{-\infty}^{\infty} t[u(t)-u(t-2)]\, dt = \int_0^2 t\, dt = 2$

c) $\int_{-\infty}^{\infty} e^{-(t-t_1)}u(t-t_1)\, dt = \int_{t_1}^{\infty} e^{-(t-t_1)}\, dt = \int_0^{\infty} e^{-x}\, dx = 1$

d) $\int_{-\infty}^{\infty} \delta(t-t_0)e^{-(t-t_1)}u(t-t_1)\, dt = e^{-(t_0-t_1)}u(t_0-t_1) = e^{-(t_0-t_1)}$ (if $t_0 > t_1$)&0 (if $t_0 < t_1$)

e) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(t+\tau)}u(t+\tau)\delta(t-\tau)\, d\tau\, dt = \int_0^{\infty} e^{-2t}\, dt = 1/2.$

Problem 2.18.4 Show that the following properties hold for the derivative of an impulse function defined under the operation of integration.

a) $\int_{-\infty}^{\infty} f(t)\delta'(t)\, dt = -f'(0)$ where $f'(t) = \frac{df}{dt}$

b) $\int_{-\infty}^{\infty} f(t)\delta^{(n)}(t)\, dt = (-1)^n f^{(n)}(0)$

c) $t\delta'(t) = -\delta(t)$

Solution:

a) Assuming continuity of $f(t)$ and using integration by parts, we obtain,

$$\int_{-\infty}^{\infty} f(t)\delta'(t)\, dt = \int_{-\infty}^{\infty} f(t) d\delta(t) = f(t)\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta(t)\, dt = -f'(0) \tag{72}$$

b) Assuming continuity of $f(t)$, using integration by parts and recognizing that $\delta^{(n)}(t)f(t) = 0, t \neq 0$, and also using the result of the previous part, we obtain,

$$\int_{-\infty}^{\infty} f(t)\delta''(t)\, dt = \int_{-\infty}^{\infty} f(t)d\delta'(t) = f(t)\delta'(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\delta'(t)\, dt = f''(0) \tag{73}$$

etc.,

c)

$$\int_a^b t\delta'(t)\,dt = \int_a^b t d\delta(t) = t\delta(t)\Big|_a^b - \int_a^b \delta(t)\,dt = -\int_a^b \delta(t)\,dt, \tag{74}$$

therefore $t\delta'(t) = -\delta(t)$ within the operation of integration.