Communications Systems (E&CE 318)

Course Notes and Problem Sets

By: A. K. Khandani

Email: khandani@shannon

Office: DC2740 (x5324)

Department of Electrical & Computer Engineering

University of Waterloo

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1 Preliminaries

A communication system is a means of transmitting information from one *place*, and/or *form*, and/or *time* to another. Examples for these three items are as follows:

- Place: Telephone, Radio, TV
- Form: Translation from one language to another, Digital-to-Analog (D/A) & Analog-to-Digital (A/D) conversion, Encoder & Decoder, Modulator & Demodulator
- Time: Recording

A complicated transmission system may involve a combination of these items.
The overall objective is to have a *reliable* and *efficient* transmission of information. Usually, the two objectives of reliability and efficiency are in contradiction with each other. In some cases, *security* of the transmission is also an issue of importance.
We focus on communication systems which are based on changing the level of some electrical signal. A signal which has some information in it should have some random features. In other words information is always associated with some kind of randomness. The exact mathematical definition of the information is based on the probability density function of the signal. As a rule of thumb, bandwidth, which is a measure of how rapidly a signal can change, can be also in some cases used as a measure of how much information a signal has. For example an HDTV signal has a much higher bandwidth than a normal TV because it shows much more details of the video signal.
Figures 1, 2, 3 show the block diagram of three different kinds of communication system. Simplex channel in one way (Fig. 1), half-duplex channel is two way but not simultaneously in both directions (Fig. 2) and full-duplex channel is two way (Fig. 3).
The channel can be either analog or digital. An analog channel transmits continuous signals. A digital channel transmits numbers (usually binary numbers, called binary channel). In either form, the channel has usually noise, interference, attenuation and distortion.
Noise is usually modeled as an additive factor. Reliability of the transmission is usually determined by the ratio of the signal power to the noise power. This is denoted as the

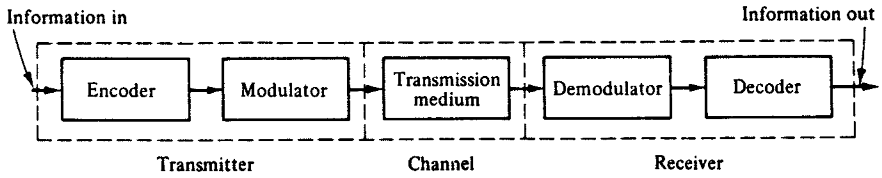

Figure 1: Block diagram of a simplex communication system.

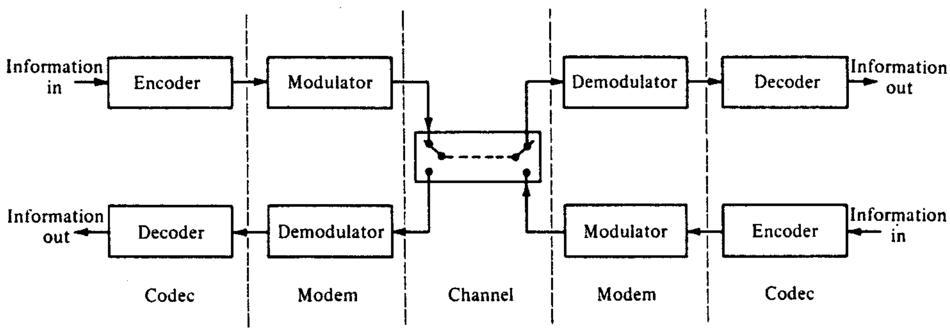

Figure 2: Block diagram of a half-duplex communication system.

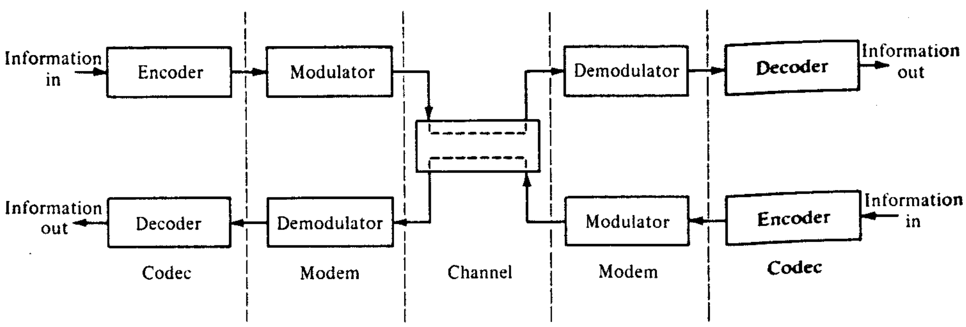

Figure 3: Block diagram of a full-duplex communication system.

Signal-to-Noise Ratio (SNR).

In the present course, we focus on analog channels. Digital channels are discussed in two subsequent courses (E&CE411, E&CE412).

Now let us look at different components of a typical transmission system (as shown in Figures 1, 2 and 3).

The *encoder* transforms the information signal (message to be transmitted) into a new form which will increase the reliability and/or the efficiency of the transmission. The decoder at the channel output is the inverse of the encoder and the combination of the encoder/decoder is called a *codec*. The role of encoder is more important in the case of a digital channel. In the following, we briefly discuss some simple examples of coding in a digital transmission system. Our discussion is based on a binary channel. A binary channel has the following properties:

- It can transmit two different numbers (symbols), namely zero and one.
- The maximum number of symbols transmitted per second is limited to a given integer R.
- Due to the lack of full reliability, a transmitted symbol (zero or one) will be received with a given probability p in error (a "zero" will be changed with probability p to a "one" and vice versa).

Examples of Coding for a binary channel: Consider the transmission of an analog signal through a binary channel. The first step in coding is to convert the signal into numbers. This is achieved by sampling and then using an analog to digital converter. According to the theory of sampling, an analog signal of bandwidth ω can be precisely represented using 2ω (uniformly spaced) samples per second. We will discuss the sampling in details later in the course.

An analog to digital converter is used to convert the samples into numbers. In the context of communication systems, this is usually called a quantizer. If the quantizer has 2^r levels, it results in r binary numbers (bits) per sample of the analog source.

Combination of the sampling and quantization results in $2\omega r$ bits per second. This can be transmitted through the binary channel assuming that $2\omega r \leq R$.

In the following, we briefly explain some more complicated aspects of coding which can be used to increase the reliability and/or the efficiency of the transmission. The efficiency is related to the number of binary symbols per second (or to the number of binary symbols per sample of the analog source). The reliability is related to the probability of error.

Coding for increasing the efficiency: Assume that each sample of the analog source is quantized to four different levels denoted as A, B, C and D. Assume that the probability of these four levels are equal to: $P_A = 0.5$, $P_B = 0.2$, $P_C = 0.2$ and $P_D = 0.1$. In the straight forward approach, we need two bits to represent these four symbols, for example using the following table:

$$\begin{array}{ll} A & 00 \\ B & 01 \\ C & 10 \\ D & 11 \end{array} \tag{1}$$

As an alternative, we can use the following table:

$$\begin{array}{ll} A & 0 \\ B & 10 \\ C & 110 \\ D & 111 \end{array} \tag{2}$$

It is seen that the letters which have a higher probability are represented with a smaller number of binary symbols (A is represented with one bit, B is represented with two bits, and C, D are each represented with three bits). Note that the binary symbols are selected such that any sequence of symbols corresponds in a unique way to a sequence of binary digits and vice versa. This property is due to the fact that the binary combinations of a smaller length do not occur at the beginning of the binary combinations of a larger length. For example 0 does not occur at the beginning of 10, 110, 111 and 10 does not occur at the beginning of 110, 111. Considering the probabilities, the average number of binary symbols per sample of the analog source is equal to:

$$1 \times P_A + 2 \times P_B + 2 \times P_C + 3 \times P_D = 1 \times 0.5 + 2 \times 0.2 + 3 \times 0.2 + 3 \times 0.1 = 1.8 \quad (3)$$

This is less than the previous case where we had two bits per sample of the analog source. This means that the efficiency has increased. A common example of such a coding scheme to increase the efficiency is in the case of the Morse alphabets.

Coding for increasing the reliability: Assume that each zero and one is transmitted for three times. In this case, if the corresponding three bits received at the channel output are composed of two or three zeros, we assume that zero was transmitted and if it is composed of two or three ones, we assume that one was transmitted. It is easy to verify that this system is more reliable compared to the normal case of a single transmission of zero and one. However, the efficiency is decreased by a factor of three.

The basic theorem of coding is as follows:

If the number of binary symbols per second at the input of a digital channel (with a nonzero probability of error) is less than a given number C (called the channel capacity), we can always find coding schemes which achieve the transmission with zero error probability. (C. E. Shannon, 1949)

A practical coding system has usually much more complicated components than those explained here. As mentioned before, these components are used to increase the efficiency and/or the reliability of the transmission.

Modulator/Demodulator: Modulator acts as an interface between the encoder and the channel (refer to Figures 1, 2 and 3). It matches the characteristics of the corresponding electrical signal to that of the channel. A common example is to adapt the frequency characteristics of the signal to that of the channel. Modulation may also serve as a means of multiplexing. This allows us to transmit several signals at the same time.

The demodulator at the channel output is the inverse of the modulator and the combination of the modulator/demodulator is called a *modem.*

In general, the encoder and the modulator have similar objectives and in some cases it is not possible to clearly separate them from each other.

In the present course, we mainly focus on discussing different aspects of the modulation/demodulation operations in the case of an analog channel.

2 Orthogonality and signal representation

A time-signal is a single-valued real, or complex function of time. Time axis is always real. We make use of complex functions because it facilitates the mathematical formulation and manipulation of the problems. Of course, in reality all signals are real. We can think of a complex signal as a collection of two independent, real-valued signals. In fact, complex signals make use of two orthogonal dimensions. Later, we will discuss the concept of orthogonality in more details. There may also be a physical interpretation behind using complex notations (e.g., movement in a 2-D plane).

2.1 Classification of signals

Definition of Energy: Consider a voltage signal $e(t)$ through a resistor R. The energy in a short interval of Δt is defined as $e^2(t)\Delta t/R$ (in Joules) and the instantaneous power is defined as $e^2(t)/R$ (in Watts). Similarly, for a current signal $i(t)$, the instantaneous power is equal to: $i^2(t)R$ Watts. It is seen that R acts as a normalization factor. We usually set this factor to $R = 1$. Using the normalization $R = 1$, the total energy of a signal $f(t)$ in the time interval $[t_1, t_2]$ is equal to,

$$E = \int_{t_1}^{t_2} |f(t)|^2 dt \quad \text{Joules} \tag{4}$$

For a complex signal $f(t) = f_r(t) + jf_i(t)$, we have $|f(t)|^2 = f_r^2(t) + f_i^2(t)$.
Average energy of $f(t)$ in the interval $[t_1, t_2]$ is equal to,

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |f(t)|^2 dt \tag{5}$$

Energy signal: $f(t)$ is called an *energy signal* if,

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt < \infty \tag{6}$$

Energy signals are usually in the form of a pulse in the sense that they have a non-zero amplitude only in a limited period of time.

Power signal: $f(t)$ is called a *power signal* if,

$$P \equiv \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|f(t)|^2 dt < \infty \tag{7}$$

where P is called the power of the signal.

Periodic signal: A period signal is a signal which repeats itself after a fixed length of time. This means that $f(t+T) = f(t)$. The smallest such positive number T is called a period. A period signal is a power signal if its average energy per period is finite. In this case, the value of the corresponding power can be found by computing the integral in (7) over only one period.

Random signals: These are signals with some random parameters (like a sinusoid with a random phase or a random amplitude). In some cases, the random parameters do not change with time and once we have known these parameters, the signal becomes deterministic (non-random). Usually, if we observe a finite duration of such a signal it determines the unknown parameters, and consequently, the future. In some other cases, the random parameters are not fixed (their values change with time). Such parameters are usually described by their probability density function.

System: A system is a rule of the form:

$$g(t) = \mathcal{T}\{f(t)\} \tag{8}$$

where t denotes the time. A system may be discrete or continuous time.

Connection of systems:

$$g(t) = \mathcal{T}_1\{\mathcal{T}_2[f(t)]\} = \mathcal{T}\{f(t)\} \tag{9}$$

where $\mathcal{T}$ expresses the equivalent system corresponding to the concatenation of $\mathcal{T}_1$ and $\mathcal{T}_2$. A system may have some random features.

A Linear System is a system for which superposition principle applies. That is, if

$$g_1(t) = \mathcal{T}\{f_1(t)\}, \tag{10}$$

and

$$g_2(t) = \mathcal{T}\{f_2(t)\}, \tag{11}$$

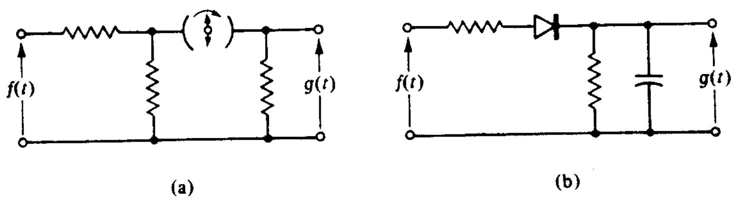

Figure 4: (a) Linear, time varying system, (b) nonlinear, time-invariant system.

then,

$$\mathcal{T}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 g_1(t) + a_2 g_2(t) \tag{12}$$

A system is time invariant if its properties do not change with time. In this case, a given time-shift in the input results in the same time-shift in the output. That is, if

$$g(t) = \mathcal{T}\{f(t)\} \tag{13}$$

then

$$g(t - t_0) = \mathcal{T}\{f(t - t_0)\} \tag{14}$$

Note that in general time-invariance and linearity are two separate properties and we can find linear systems which are time-varying or non-linear systems which are time-invarying (refer to Fig. 4).

A system is called realizable or causal if the value of its output at a given time instant t_0 depends only on the values of input prior to t_0, i.e., $t \leq t_0$.

In the theory of linear systems, using complex representation provides an easy way to work with sinusoidal signals. This is due to the property that a sinusoidal signal at the input of a linear system results in a sinusoid of the same frequency at the output where the amplitude changes by a multiplicative factor and the phase changes by an additive factor. Using complex notation allows to reflect both of these facts in a single complex operation. To study this effect consider the signal,

$$f(t) = A\cos(\omega t + \phi) \tag{15}$$

The output of a linear system **H** to this signal is of the form

$$g(t) = A|H(\omega)|\cos(\omega t + \angle H(\omega) + \phi) \tag{16}$$

We represent $\cos(\omega t + \phi)$ as $\mathcal{R}[e^{j(\omega t+\phi)}]$ (where $\mathcal{R}$ denotes the real part) and the linear system $\mathbf{H}$ as $|H(\omega)|e^{j\angle H(\omega)}$. Then, the output can be computed by a multiplication of the form:

$$g(t) = \mathcal{R}[|H(\omega)|e^{j\angle H(\omega)} \times Ae^{j(\omega t+\phi)}] = \mathcal{R}[A|H(\omega)|e^{j(\omega t+\angle H(\omega)+\phi)}] \tag{17}$$

To facilitate the presentation, we usually remove the $\mathcal{R}$ notation.

As we will see later, the Fourier analysis allows us to express a signal in terms of sinusoids. This property in conjunction with the superposition principle of linear systems gives more importance to the complex representation.

2.2 A brief review of vectors

2.2.1 Basic definitions

A vector of dimension N is the set of N numbers in a *given order*. For example, $(1, -2, 3.5)$ is a three dimensional vector.

Important operations on vectors: Consider two vectors $X = (x_1, x_2, \ldots, x_N)$ and $Y = (y_1, y_2, \ldots, y_N)$.

Addition: $X + Y = (x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N)$

Multiplication by numbers: $aX = (ax_1, ax_2, \ldots, ax_N)$

Inner product: $X.Y = x_1y_1 + x_2y_2 + \ldots + x_Ny_N$. Two vectors X and Y are called orthogonal if $X.Y = 0$. Inner product of a vector by itself is the square length of the vector.

2.2.2 Expansion of vectors

Consider a given set of vectors $\mathcal{V}$. The collection of vectors $B_1, B_2, \ldots, B_K$ is called a basis for the set $\mathcal{V}$ if each element of $\mathcal{V}$ can be expressed as a linear combination of $B_1, B_2, \ldots, B_K$. As an example, consider the set of all the two dimensional vectors. A basis for this set is of the form $B_1 = (1, 0)$ and $B_2 = (0, 1)$ and a given vector $X = (x_1, x_2)$ can be expressed as: $X = x_1B_1 + x_2B_2$.

The basis for a set of vectors is not unique. For example, for the set of the two dimensional vectors, one can also use $B_1 = (1, 1)$ and $B_2 = (1, -1)$ as the basis. Using this basis, a vector $X = (x_1, x_2)$ is expressed as:

$$(x_1, x_2) = \frac{x_1 + x_2}{2} B_1 + \frac{x_1 - x_2}{2} B_2 \tag{18}$$

The number of vectors in a basis is less than or equal to the dimensionality of the set. As an example, consider the set of the two dimensional vectors $\mathcal{V}$ where the sum of the two components of each vector is equal to zero, i.e,

$$\mathcal{V} = \{(x_1, x_2) \text{ such that } x_1 + x_2 = 0\} \tag{19}$$

A basis for this set is composed of the single vector $B = (1, -1)$ because any vector $(x_1, -x_1) \in \mathcal{V}$ can be expressed as: $x_1 B$. Obviously, it is possible to use a basis with two (or even more) vectors for representation however this is redundant.

Consider a set of vectors $\mathcal{V}$ with basis vectors: $B_1, B_2, \ldots, B_K$. An element $V \in \mathcal{V}$ can be expressed as:

$$V = \sum_{i=1}^{N} v_i B_i \tag{20}$$

This is called an expansion of V on the corresponding basis. The $v_i B_i$ is called the projection of V on B_i. The coefficient v_i in some sense measure the similarity of V with the basis vector B_i.

Usually, we work with a set of basis vectors which are orthogonal. In this case, the coefficient v_n can be computed using,

$$V = \sum_{i=1}^{N} v_i B_i \tag{21}$$

$$B_n.V = \sum_{i=1}^{N} v_i B_n.B_i \tag{22}$$

Replacing $B_n.B_i = 0$ for $i \neq n$, we obtain,

$$v_n = \frac{B_n.V}{B_n.B_n} \tag{23}$$

In expanding a vector over an orthogonal set of dimensions, we have the following properties:

1. The projection on a given basis vector is fixed and is independent of the selection of other basis vectors.

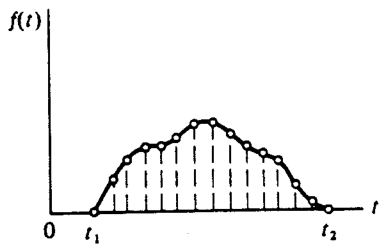

Figure 5: Specifying a time signal by its values at distinct time instances.

2. We have

$$\text{if} \quad V = \sum_{i=1}^{N} B_i v_i \qquad \text{then} \qquad V.V = \sum_{i=1}^{N} v_i^2 B_i.B_i = \sum_{i=1}^{N} v_i^2 ||B_i||^2 \tag{24}$$

where $||B_i||^2 = B_i.B_i$ is the square length of B_i. Usually, we normalize the square length of each basis vector to unity. An orthogonal basis with $B_i.B_i = 1, \forall i$ is called an orthonormal basis. For an orthonormal basis, we have $V.V = \sum_i v_i^2$.

2.3 Expressing time-signals as vectors

Any time signal can be specified by its values at distinct time instances (refer to Fig. 5). We can look at these distinct values as the components of a vector. Any finite interval of a discrete time signal results in a vector with a finite number of dimensions. For a continuous time signal, this results in an infinite dimensional vector and, consequently, the approach is not very meaningful. The important point is that in this case, as we will later, the minimum necessary number of dimensions to represent the signal is related to bandwidth of the signal and is usually finite.

Consider two continuous signals $f_1(t)$, $f_2(t)$ defined over the time interval $[t_1, t_2]$. The inner product of these signals is defined as:

$$f_1(t).f_2(t) = \int_{t_1}^{t_2} f_1(t) f_2^*(t) dt \tag{25}$$

The reason for using the complex conjugate is that in this way we have,

$$f_1(t).f_1(t) = \int_{t_1}^{t_2} |f_1(t)|^2 dt \tag{26}$$

which is the energy of the signal.

Consider a set of orthogonal signals, $\phi_n(t)$, $n=1,\ldots,N$, over the interval $[t_1,t_2]$ such that,

$$\int_{t_1}^{t_2} \phi_n(t)\phi_m^*(t) = \begin{cases} 0 & n \neq m \\ K_n & n = m \end{cases} \tag{27}$$

Consider a signal $f(t)$ and the expansion:

$$f(t) \simeq \sum_{n=1}^{N} f_n \phi_n(t) \tag{28}$$

We use the notation $\simeq$ because the set of functions $\phi_n(t)$, $n = 1,\ldots,N$, may not be a basis for $f(t)$. The error between $f(t)$ and its expansion on the orthogonal set $\phi_n(t)$, $n = 1,\ldots,N$, is computed as:

$$\epsilon_N(t) = f(t) - \sum_{n=1}^{N} f_n \phi_n(t) \tag{29}$$

The energy of the error signal is equal to:

$$\int_{t_1}^{t_2} |\epsilon_N(t)|^2 \, dt = \int_{t_1}^{t_2} \left| f(t) - \sum_{n=1}^{N} f_n \phi_n(t) \right|^2 dt \tag{30}$$

Assume that we wish to minimize this quantity by the proper choice of the coefficients f_n, $n = 1,\ldots,N$.

$$\int_{t_1}^{t_2} |\epsilon_N(t)|^2 \, dt = \int_{t_1}^{t_2} \left[f(t) - \sum_{n=1}^{N} f_n \phi_n(t) \right] \left[f(t) - \sum_{n=1}^{N} f_n \phi_n(t) \right]^* dt =$$

$$\int_{t_1}^{t_2} |f(t)|^2 \, dt + \sum_{n=1}^{N} \left[-f_n^* \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt - f_n \int_{t_1}^{t_2} f^*(t)\phi_n(t)dt + K_n |f_n|^2 \right] \tag{31}$$

In the following, we attempt to derive an alternative expression for the term inside the summation in (31). To do this, let us compute the value of,

$$\left| K_n^{1/2} f_n - \frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt \right|^2 \tag{32}$$

Note: just keep reading without trying to figure out the relationship between (31) and (32), or asking yourself why we are bringing (32) into the picture. It will become clear later. Any ways, for (32), we can write (note that K_n is a real, positive number):

$$\left|K_n^{1/2} f_n - \frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt\right|^2 =$$

$$\left[K_n^{1/2} f_n - \frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt\right] \left[K_n^{1/2} f_n^* - \frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f^*(t)\phi_n(t)dt\right] =$$

$$K_n |f_n|^2 - f_n \int_{t_1}^{t_2} f^*(t)\phi_n(t)dt - f_n^* \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt + \left|\frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f(t)\phi_n^*(t)\right|^2 dt \tag{33}$$

Combining (31) and (33), we obtain,

$$\int_{t_1}^{t_2} |\epsilon_N(t)|^2 dt = \int_{t_1}^{t_2} |f(t)|^2 dt + \sum_{n=1}^{N} \left|K_n^{1/2} f_n - \frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt\right|^2 - \left|\frac{1}{K_n^{1/2}} \int_{t_1}^{t_2} f(t)\phi_n^*(t)\right|^2 dt \tag{34}$$

Only the second term in (34) depends on the choice of f_n and is minimized by selecting:

$$f_n = \frac{1}{K_n} \int_{t_1}^{t_2} f(t)\phi_n^*(t)dt = \frac{\int_{t_1}^{t_2} f(t)\phi_n^*(t)dt}{\int_{t_1}^{t_2} |\phi_n(t)|^2 dt} \tag{35}$$

or,

$$\int_{t_1}^{t_2} f(t)\phi_n^*(t)dt = f_n K_n \tag{36}$$

It is seen that we obtain a similar relationship as the one in (23). For this value of f_n, the second term in (34) is equal to zero and by replacing (36), we obtain,

$$\int_{t_1}^{t_2} |\epsilon_N(t)|^2 dt = \int_{t_1}^{t_2} |f(t)|^2 dt - \sum_{n=1}^{N} |f_n|^2 K_n \tag{37}$$

If the set of the functions, $\phi_n(t)$, $n = 1, \ldots, N$, is a basis for $f(t)$, we obtain,

$$\int_{t_1}^{t_2} |\epsilon_N(t)|^2 dt = 0 \tag{38}$$

and, consequently,

$$\int_{t_1}^{t_2} |f(t)|^2 dt = \sum_{n=1}^{N} |f_n|^2 K_n \tag{39}$$

This is known as the *Parseval's Theorem.*

2.4 Exponential Fourier series

Similar to the general case of expanding a vector on a basis, the set of the basis vectors is not unique (different basis are related by linear relationships). An important set of orthogonal basis are of the form:

$$\phi_n(t) = e^{jn\omega_0 t} \tag{40}$$

where $n = 0, \pm 1, \pm 2, \ldots$ and ω_0 is a constant computed in the following to assure the orthogonality of the basis functions over a given time interval $[t_1, t_2]$.

The inner product of the basis functions over the time interval $[t_1, t_2]$ is equal to:

$$\int_{t_1}^{t_2} \phi_n(t)\phi_m^*(t)dt = \int_{t_1}^{t_2} e^{jn\omega_0 t} e^{-jm\omega_0 t} = \int_{t_1}^{t_2} e^{j(n-m)\omega_0 t} = \tag{41}$$

$$\frac{1}{j(n-m)\omega_0}\left[e^{j(n-m)\omega_0 t_2} - e^{j(n-m)\omega_0 t_1}\right] = \frac{1}{j(n-m)\omega_0} e^{j(n-m)\omega_0 t_1}\left[e^{j(n-m)\omega_0 (t_2-t_1)} - 1\right] \tag{42}$$

As $n-m$ is an integer, this inner product will be zero if $\omega_0(t_2-t_1)=2\pi$, or $\omega_0 = (t_2-t_1)/2\pi$ (note that $e^{j2\pi I} = 1$, for integer values of I). For this value of ω_0, we have,

$$\int_{t_1}^{t_2} e^{jn\omega_0 t} e^{-jm\omega_0 t} = \begin{cases} (t_2 - t_1) & n = m \\ 0 & n \neq m \end{cases} \tag{43}$$

The expansion of a function $f(t)$ over this basis is of the form,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}, \qquad t_1 < t < t_2 \tag{44}$$

This is called the exponential Fourier series representation.

Using the general rules of expanding signals, the coefficients F_n can be computed by multiplying the two sides of (44) by $e^{-jn\omega_0 t}$ and computing the integral over $[t_1, t_2]$. This results in,

$$F_n = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} dt \tag{45}$$

2.5 Complex functions

Using complex notation is a powerful mathematical tool which enable us to perform the computations concerning the amplitude and the phase of a sinusoid simultaneously.

Important identities:

$$
\begin{aligned}
f &= f_r + j f_i \\
f^* &= f_r - j f_i \\
f_r &= \frac{1}{2}(f + f^*) \\
f_i &= \frac{1}{2j}(f - f^*) \\
|f|^2 &= f f^* = |f_r|^2 + |f_i|^2
\end{aligned}
\tag{46}
$$

We also have,

$$
\begin{aligned}
e^{j\omega_0 t} &= \cos \omega_0 t + j \sin \omega_0 t \\
\cos \omega_0 t &= \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \\
\sin \omega_0 t &= \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)
\end{aligned}
\tag{47}
$$

2.6 Importance of the Fourier representation

The most important property of linear, time-invariant systems is that their input and output are related by a differential equation with constant coefficients. For example, a differential equation of the form

$$
a_0 g(t) + a_1 \frac{dg(t)}{dt} + \ldots = b_0 f(t) + b_1 \frac{df(t)}{dt} + \ldots \tag{48}
$$

where $f(t)$ is input, $g(t)$ is the output and the coefficients a's and b's are constant. Now consider the input signal $e^{j\omega t}$. The key point is the differentiations of this function are of the same general form, i.e.,

$$
\frac{d^k}{dt^k} e^{j\omega t} = (j\omega)^k e^{j\omega t} \tag{49}
$$

As a result, the output of the linear system $\mathbf{H}$ to the input $f(t) = e^{j\omega t}$ is of the form,

$$
g(t) = H(\omega) e^{j\omega t} \tag{50}
$$

where

$$
H(\omega) = \frac{\sum_k b_k (j\omega)^k}{\sum_m a_m (j\omega)^m} \tag{51}
$$

The multiplicative factor $H(\omega)$ is called the frequency transfer function of the linear system. If we consider

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)} \tag{52}$$

then, the input $f(t) = Ae^{j(\omega t+\phi)}$ results in the output,

$$g(t) = AH(\omega)e^{j(\omega t+\phi)} = A|H(\omega)|e^{j[\omega t+\phi+\angle H(\omega)]} \tag{53}$$

This means that a sinusoid of a given frequency at the input of a linear system results in a sinusoid of the same frequency at the output. The amplitude of this sinusoid changes by the multiplicative factor $|H(\omega)|$ and its phase changes by the additive factor $\angle H(\omega)$. Using the superposition principles, for an input signal of the form,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{54}$$

the output is equal to,

$$g(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) F_n e^{jn\omega_0 t} \tag{55}$$

3 Trigonometric Fourier series

In this section, we will find alternative expressions for the Fourier series representation of a real signal. Consider a complex signal, $f(t) = f_r(t) + jf_i(t)$. The corresponding Fourier series representation is of the form,

$$f(t) = f_r(t) + jf_i(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{56}$$

We know that in general, for any two complex quantity A and B, we have,

$$\mathcal{R}\{AB\} = \mathcal{R}\{A\}\mathcal{R}\{B\} - \mathcal{I}\{A\}\mathcal{I}\{B\} \tag{57}$$

where $\mathcal{R}$ and $\mathcal{I}$ denote the real and the imaginary parts, respectively. Using this relationship and equating the real parts of the two sides of (56), we obtain,

$$f_r(t) = \sum_{n=-\infty}^{\infty} \mathcal{R}\{F_n\}\mathcal{R}\{e^{jn\omega_0 t}\} - \sum_{n=-\infty}^{\infty} \mathcal{I}\{F_n\}\mathcal{I}\{e^{jn\omega_0 t}\} = \tag{58}$$

$$f_r(t) = \sum_{n=-\infty}^{\infty} \mathcal{R}\{F_n\}\cos n\omega_0 t - \sum_{n=-\infty}^{\infty} \mathcal{I}\{F_n\}\sin n\omega_0 t \tag{59}$$

Recall that,

$$F_n = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} \tag{60}$$

Assuming a real signal, $f(t) = f_r(t)$, we have,

$$\begin{aligned} F_n^* &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) e^{jn\omega_0 t} = F_{-n} \\ \mathcal{R}\{F_n\} &= \frac{1}{2}[F_n + F_n^*] = \frac{1}{2}[F_n + F_{-n}] \\ \mathcal{I}\{F_n\} &= \frac{1}{2j}[F_n - F_n^*] = \frac{1}{2j}[F_n - F_{-n}] \end{aligned} \tag{61}$$

To simplify the representation, we define,

$$\begin{aligned} a_0 &\equiv F_0 \\ a_n &\equiv 2\mathcal{R}\{F_n\} = F_n + F_{-n}, \ n \neq 0 \\ b_n &\equiv -2\mathcal{I}\{F_n\} = -\frac{1}{j}[F_n - F_{-n}] = j[F_n - F_{-n}] \end{aligned} \tag{62}$$

From (62), we obtain,

$$\begin{aligned} a_n &= a_{-n} \\ b_n &= -b_{-n} \end{aligned} \tag{63}$$

Substituting (62) in (59), we obtain,

$$f_r(t) = a_0 + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{a_n}{2} \cos n\omega_0 t + \sum_{n=-\infty}^{\infty} \frac{b_n}{2} \sin n\omega_0 t \tag{64}$$

Now we break the range of the two summations in (64) from $n \in [-\infty, \infty]$ to $n \in [-\infty, -1]$, $n = 0$ and $n \in [1, \infty]$. This results in,

$$\begin{aligned} f_r(t) = & \sum_{n=-\infty}^{-1} \frac{a_n}{2} \cos n\omega_0 t + a_0 \cos 0\omega_0 t + \sum_{n=1}^{\infty} \frac{a_n}{2} \cos n\omega_0 t + \\ & \sum_{n=-\infty}^{-1} \frac{b_n}{2} \sin n\omega_0 t + b_0 \sin 0\omega_0 t + \sum_{n=1}^{\infty} \frac{b_n}{2} \sin n\omega_0 t \end{aligned} \tag{65}$$

Or, equivalently,

$$\begin{aligned} f_r(t) = & \sum_{n=\infty}^{1} \frac{a_{-n}}{2} \cos(-n\omega_0 t) + a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} \cos n\omega_0 t + \\ & \sum_{n=\infty}^{1} \frac{b_{-n}}{2} \sin(-n\omega_0 t) + 0 + \sum_{n=1}^{\infty} \frac{b_n}{2} \sin n\omega_0 t \end{aligned} \tag{66}$$

Using (63) to substitute $a_n = a_{-n}$ and $b_n = -b_{-n}$, and considering that, $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, the relationship in (66) reduces to,

$$f_r(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{67}$$

Then, if $f(t)$ is real-valued in the interval $[t_1, t_2]$, we can write,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{68}$$

This is called the trigonometric Fourier series representation of the real function $f(t)$. Using (61), (62), it is easy to show that

$$\begin{aligned} a_0 &= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) \\ a_n &= \frac{2}{t_2 - t_1} \int_{t_1}^{t_2} f(t) \cos n\omega_0 t dt \\ b_n &= \frac{2}{t_2 - t_1} \int_{t_1}^{t_2} f(t) \sin n\omega_0 t dt \end{aligned} \tag{69}$$

Note that

$$F_n = \frac{1}{2}(a_n - jb_n) \tag{70}$$

In the following, we derive a different form for the Fourier representation of a real signal. We know that in general,

$$A \cos x + B \sin x = \sqrt{A^2 + B^2} \left[\frac{A}{\sqrt{A^2 + B^2}} \cos x - \frac{-B}{\sqrt{A^2 + B^2}} \sin x \right] \tag{71}$$

We define the angle ϕ such that,

$$\begin{aligned} \cos\phi &= \frac{A}{\sqrt{A^2 + B^2}} \\ \sin\phi &= \frac{-B}{\sqrt{A^2 + B^2}} \end{aligned} \tag{72}$$

(these two definitions are consistent because $\sin^2\phi + \cos^2\phi = 1$). We can equivalently define ϕ as, $\tan\phi = -B/A$. Substituting in (71) and using the identity, $\cos\phi\cos x - \sin\phi\sin x = \cos(x + \phi)$, we obtain,

$$A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x + \phi), \quad \text{where} \quad \phi = \tan^{-1}(-B/A) \tag{73}$$

Using these results to combine the sine and the cosine terms in (68), we obtain,

$$f(t) = \sum_{n=0}^{\infty} c_n \cos(n\omega_0 t + \phi_n) \tag{74}$$

where

$$c_n = \sqrt{a_n^2 + b_n^2}, \quad (c_0 = a_0), \quad \phi_n = \tan^{-1}(-b_n/a_n) \tag{75}$$

Using (62) and (75), it is easy to show that

$$\begin{aligned} c_n = 2|F_n| = 2\sqrt{F_n F_n^*}, \quad n \neq 0, \quad \text{and} \quad c_0 = F_0 \\ \phi_n = \tan^{-1} \frac{\mathcal{I}\{F_n\}}{\mathcal{R}\{F_n\}} \end{aligned} \tag{76}$$

3.1 Periodicity property of the Fourier series

We talked about the Fourier series representation of a time-function $f(t)$ in the interval $[t_1, t_2]$. We know that the value of the function and the value of the corresponding Fourier series are the same for $t_1 < t < t_2$. However, outside of this interval these two quantities are not necessarily equal.

Considering the Periodicity of the function $e^{jn\omega_0 t}$, we conclude that the Fourier series is a periodic time-function with period $T = t_2 - t_1$. So, the Fourier series representation of a function $f(t)$ will be valid for all the ranges of t if the function itself is periodic with period T, i.e., $f(t + T) = f(t)$.

3.2 Expansion of odd/even functions

Consider the trigonometric expansion of an even function over the interval $[-T/2, T/2]$. Using (69), we conclude that the b_n's are equal to zero and the expansion is composed of only cosine terms. Note that cosine is an even function over $[-T/2, T/2]$.

Consider the trigonometric expansion of an odd function over the interval $[-T/2, T/2]$. Using (69), we conclude that the a_n's are equal to zero and the expansion is composed of only sine terms. Note that sine is an odd function over $[-T/2, T/2]$.

In both of the above cases, considering the symmetry of the functions, the nonzero coefficients can be found by computing the corresponding integral over the range $[0, T/2]$ and multiplying the result by a factor of two.

We know that any function $f(t)$ can be written as the sum of an even part $f_e(t)$ and an odd part $f_o(t)$ where,

$$\begin{aligned} f(t) &= f_e(t) + f_o(t) \\ f_e(t) &= \frac{1}{2}[f(t) + f(-t)] \\ f_o(t) &= \frac{1}{2}[f(t) - f(-t)] \end{aligned} \tag{77}$$

It is easy to show that cosine terms in the trigonometric expansion of $f(t)$ are the expansion of $f_e(t)$ and the sine terms are the expansion of $f_o(t)$.

3.3 Parseval's theorem

Consider a period function $f(t)$ with period T. The power of the signal is equal to,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f(t)f^*(t)dt \quad \text{Watts} \tag{78}$$

Using the exponential Fourier series $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$, we obtain,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{m=-\infty}^{\infty} F_m e^{jm\omega_0 t} \sum_{n=-\infty}^{\infty} F_n^* e^{-jn\omega_0 t} dt \tag{79}$$

$$P = \sum_{m=-\infty}^{\infty} F_m \sum_{n=-\infty}^{\infty} F_n^* \frac{1}{T}\int_{-T/2}^{T/2} e^{j(m-n)\omega_0 t} dt \tag{80}$$

Using the orthogonality of the complex exponential functions, the integral in (80) is zero except for $m = n$. This results in:

$$P = \sum_{n=-\infty}^{\infty} F_n F_n^* = \sum_{n=-\infty}^{\infty} |F_n|^2 \tag{81}$$

We can say that the power of a period function is distributed over discrete frequencies (power $|F_n|^2$ at frequency $n\omega_0$). Generally, the plot of power versus frequency is called *power spectrum*. The power spectrum of a periodic signal is composed of a discrete set of lines.

3.4 Harmonic generation

We saw that a sinusoid of a given frequency at the input of a linear system results in a sinusoid of the same frequency at the output. What about non-linear systems? Does the frequency remains the same?

As a simple example of a non-linear system, consider an input-output relationship of the form

$$e_o(t) = a_1 e_i(t) + a_2 e_i^2(t) \tag{82}$$

where $e_i(t)$ is the input and $e_o(t)$ is the output. Assuming $e_i(t) = A\cos\omega_0 t$, we obtain,

$$e_0(t) = a_1 A\cos\omega_0 t + a_2 A^2 \cos^2\omega_0 t = \frac{1}{2}a_2 A^2 + a_1 A\cos\omega_0 t + \frac{1}{2}a_2 A^2 \cos 2\omega_0 t \tag{83}$$

It is seen that the nonlinear system has resulted in a second harmonic. Similarly, raising to the power of N results in generating harmonics up to the N'th order.
To measure the effect of the non-linearity, the factor THD (Total Harmonic Distortion) is defined as the ratio of the energy of the main harmonic to the total energy of the higher order harmonics. If we consider the output of the non-linear system in the form of a trigonometric series, i.e.,

$$f_o(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{84}$$

we obtain:

$$\text{THD} = \frac{\sum_{n=2}^{\infty}(a_n^2 + b_n^2)}{a_1^2 + b_1^2} \tag{85}$$

3.5 Fourier series of a periodic rectangular signal

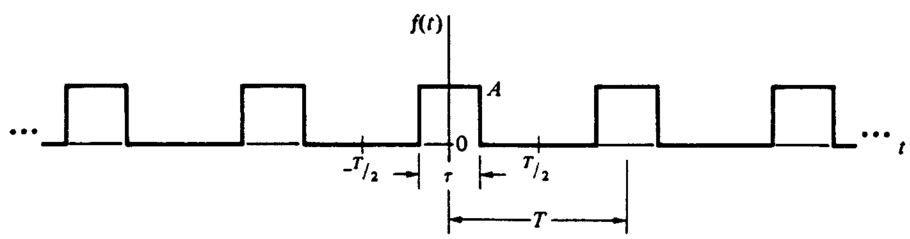

Figure 6: The periodic rectangular signal.

Consider the periodic time signal shown in Fig. 6. We have,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{86}$$

where,

$$
\begin{aligned}
F_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt, \\
&= \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-jn\omega_0 t}dt, \\
&= \frac{-A}{jn\omega_0 T}\left[e^{-jn\omega_0\tau/2} - e^{jn\omega_0\tau/2}\right], \quad n \neq 0 \qquad (87) \\
&= \frac{2A}{n\omega_0 T}\sin(n\omega_0\tau/2), \qquad n \neq 0 \\
&= \frac{A\tau}{T}\frac{\sin(n\omega_0\tau/2)}{(n\omega_0\tau/2)}, \qquad n \neq 0
\end{aligned}
$$

and

$$F_0 = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Adt = \frac{A\tau}{T} \qquad (88)$$

Comparing (87), (88) and considering that,

$$\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2} \to 1 \text{ as } n \to 0, \qquad (89)$$

we conclude that one can use (87) as the general term of the corresponding Fourier series. Define $x = n\omega_0\tau/2$, and $\mathrm{Sa}(x) = \sin x/x$, then

$$F_n = \frac{A\tau}{T}\left[\frac{\sin x}{x}\right] = \frac{A\tau}{T}\mathrm{Sa}(x) \qquad (90)$$

and,

$$f(t) = \frac{A\tau}{T}\sum_{n=-\infty}^{\infty}\mathrm{Sa}\left(\frac{n\omega_0\tau}{2}\right)e^{jn\omega_0 t}, \qquad (91)$$

or, substituting $\omega_0 = 2\pi/T$,

$$f(t) = \frac{A\tau}{T}\sum_{n=-\infty}^{\infty}\mathrm{Sa}\left(\frac{n\pi\tau}{T}\right)e^{j2\pi nt/T}. \qquad (92)$$

The function $\mathrm{Sa}(x)$ is shown in Fig. 7.

3.6 Effects of changing the parameters T and τ

Increasing T for fixed τ (refer to Fig. 8):

1. Height of the frequency envelope for the F_n's decreases as $1/T$.

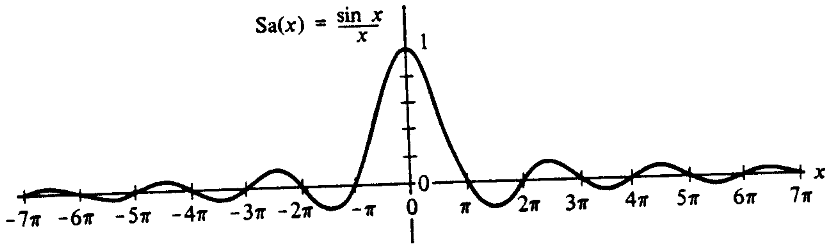

Figure 7: The function Sa(x).

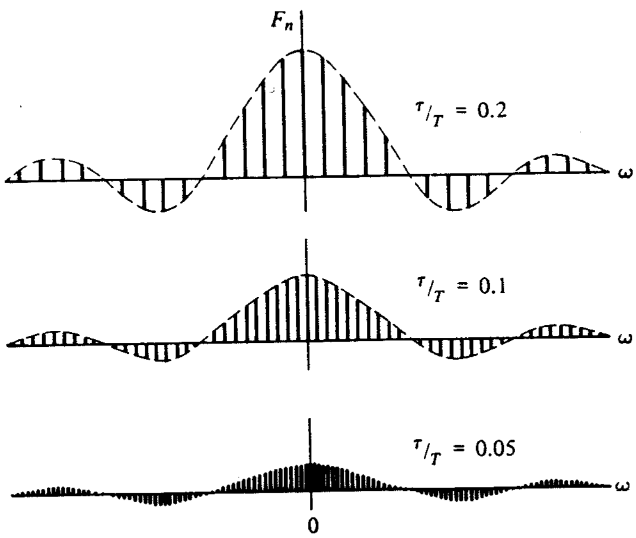

Figure 8: Amplitude spectra of the periodic rectangular signal for different values of τ/T, τ fixed.

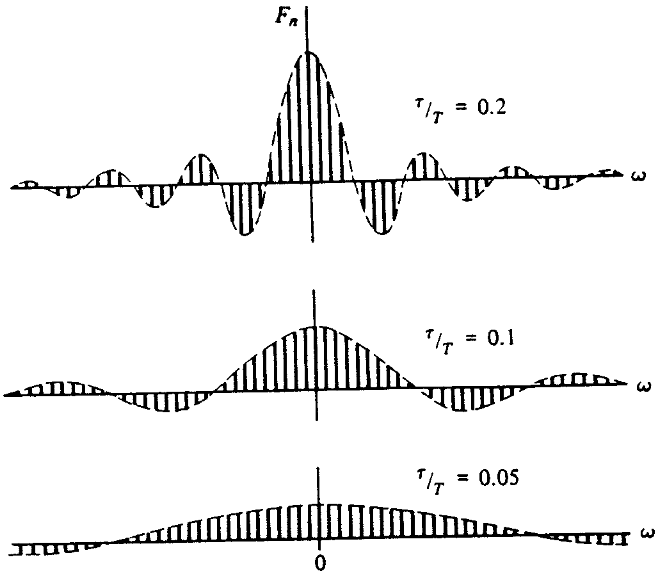

Figure 9: Amplitude spectra of the periodic rectangular signal for different values of τ/T, T fixed.

2. Spacing between the frequencies of two subsequent F_n's ($\Delta\omega = 2\pi/T$) decreases as $1/T$.

3. General shape of the envelope does not change.

Increasing τ for a fixed T ($T \geq \tau$) (refer to Fig. 9):

1. Height of the frequency envelope for the F_n's increases as τ.

2. Spacing between the frequencies of two subsequent F_n's ($\Delta\omega = 2\pi/T$) remains fixed.

3. Frequency envelope becomes narrower around zero. In general, there exists an inverse relationship between the width of a pulse and the spread of the corresponding frequency envelope.

3.7 Some special functions

3.7.1 Unit impulse, $\delta(t)$

The function $\delta(t)$ is defined such that:

$$\int_a^b f(t)\delta(t-t_0)dt = \begin{cases} f(t_0), & a < t_0 < b \\ 0 & \text{elsewhere} \end{cases} \tag{93}$$

Obviously,

$$\int_a^b \delta(t-t_0)dt = 1, \qquad a < t_0 < b \tag{94}$$

We have,

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{95}$$

To prove (95), we first prove that,

$$\int_{-\infty}^{\infty} f(t)\delta[a(t-t_0)]dt = \frac{1}{|a|}f(t_0). \tag{96}$$

We apply the change of variable $x = at$ to (96). For $a > 0$, we obtain,

$$\int_{-\infty}^{\infty} f\left(\frac{x}{a}\right)\delta(x-at_0)\frac{dx}{a} = \frac{1}{a}f(t_0) \tag{97}$$

For $a < 0$, we obtain,

$$\int_{-\infty}^{\infty} f(t)\delta[a(t-t_0)]dt = \int_{\infty}^{-\infty} f\left(\frac{x}{a}\right)\delta(x-at_0)\frac{dx}{a} = -\frac{1}{a}f(t_0) \tag{98}$$

Combining these two, we obtain,

$$\int_{-\infty}^{\infty} f(t)\delta[a(t-t_0)]dt = \frac{1}{|a|}f(t_0). \tag{99}$$

Comparing (93) with (96), and noting that $\delta(t-t_0)$ is defined only for a small neighborhood around t_0, we conclude that (95) is correct. Using $a = -1$ in (95), we obtain $\delta(t) = \delta(-t)$ which means that the $\delta(t)$ is an even function.

3.8 Unit step function

This function is defined as:

$$u(t-t_0) = \begin{cases} 1, & t > t_0 \\ 1/2, & t = t_0 \\ 0, & t < t_0 \end{cases} \tag{100}$$

It is easy to verify that

$$\int_{-\infty}^{t} \delta(\tau - t_0) d\tau = u(t - t_0) \tag{101}$$

and, consequently,

$$\delta(t - t_0) = \frac{d}{dt} u(t - t_0) \tag{102}$$

The function $\delta(t)$ can be written as the following limits:

Rectangular Pulse:

$$\lim_{\tau \to 0} \frac{1}{\tau} \left[u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \right] \tag{103}$$

Triangular Pulse:

$$\lim_{\tau \to 0} \frac{1}{\tau} \left[1 - \frac{|t|}{\tau} \right], \qquad |t| < \tau \tag{104}$$

Two-sided exponential:

$$\lim_{\tau \to 0} \frac{1}{\tau} e^{-|2t|/\tau} \tag{105}$$

Gaussian Pulse:

$$\lim_{\tau \to 0} \frac{1}{\tau} e^{-\pi (t/\tau)^2} \tag{106}$$

Sa(t) function:

$$\lim_{\tau \to 0} \frac{1}{\tau} \mathrm{Sa}(\pi t/\tau) \tag{107}$$

$\mathrm{Sa}^2(t)$ function:

$$\lim_{\tau \to 0} \frac{1}{\tau} \mathrm{Sa}^2(\pi t/\tau) \tag{108}$$