Problem 4.6.2

Two pulse waveforms $x(t)$ and $y(t)$ are shown in Fig. P-4.6.2. Determine and sketch $r_x(\tau)$, $r_y(\tau)$, and $r_{xy}(\tau)$.

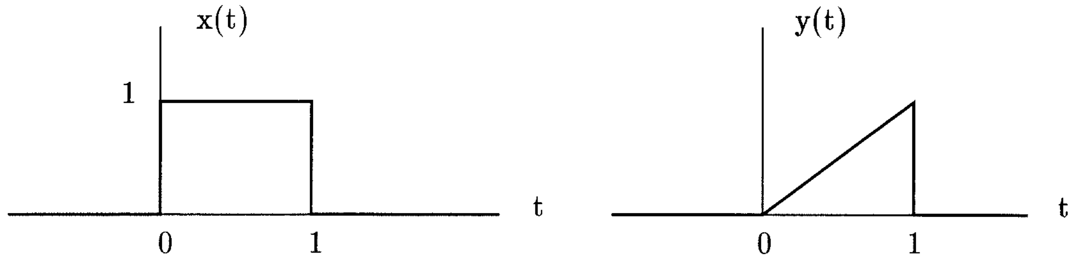


Figure 20: P-4.6.2

Solution:

The autocorrelation functions below are zero outside of the intervals given.

a)

$$r_x(\tau) = \int_{-\tau}^{1} dt = 1+\tau, \quad \text{for } -1<\tau<0, \tag{180}$$

$$= \int_{0}^{1-\tau} dt = 1-\tau, \quad \text{for } 0<\tau<1. \tag{181}$$

Combining the results, we have: $r_x(\tau) = \Lambda(\tau)$.

b)

$$r_y(\tau) = \int_{-\tau}^{1} t(t+\tau)dt = -\frac{1}{6}\tau^3 + \frac{1}{2}\tau + \frac{1}{3} \quad \text{for } -1<\tau<0, \tag{182}$$

$$= \int_{0}^{1-\tau} t(t+\tau)dt = \frac{1}{6}(\tau^3 - 3\tau + 2), \quad \text{for } 0<\tau<1. \tag{183}$$

c)

$$r_{xy}(\tau) = \int_{-\tau}^{1} (t+\tau)dt = (1+\tau)^2/2, \quad \text{for } -1<\tau<0, \tag{184}$$

$$= \int_{0}^{1-\tau} (t+\tau)dt = (1-\tau^2)/2, \quad \text{for } 0<\tau<1. \tag{185}$$

Problem 4.7.1

In a certain condition it is given that the rms thermal voltage developed across the series combination of two resistors R_1, R_2 is γ times that developed across the parallel combination of these two resistors.

a) Determine a lower bound on γ if both resistors are at the same temperature.

b) Find the relation required between R_1, R_2 to achieve this lower bound.

Solution:

a)

$$4kTB(R_1 + R_2) = \gamma^2\{4kTB[R_1R_2/(R_1 + R_2)]\} \quad (186)$$

$$R_1^2 - (\gamma^2 - 2)R_1R_2 + R_2^2 = 0 \quad (187)$$

$$R_1/R_2 = \frac{\gamma^2 - 2}{2} \pm \frac{1}{2}\sqrt{(\gamma^2 - 2)^2 - 4} \quad (188)$$

which implies $\gamma \geq 2$.

b) For $\gamma = 2$

$$R_1^2 - 2R_1R_2 + R_2^2 = 0 \Longrightarrow R_1 = R_2. \quad (189)$$

Problem 4.7.2

The frequency transfer function for the RC bandpass filter shown in Fig. P-4.7.2 can be approximated by that for two independent cascaded RC filter sections if $R_1 \ll R_2$. Using this approximation, determine an expression for the spectral density of the thermal noise voltage at the output terminals of the filter.

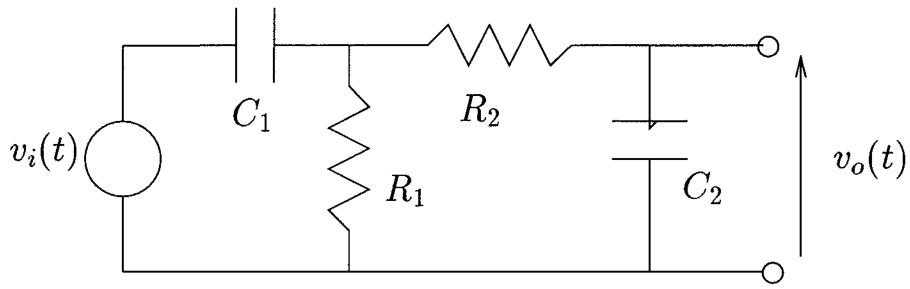


Figure 21: P-4.7.2

Solution:

To compute the effect of the noise, we assume that v_i is short circuit. In this case, as $R_2 \gg R_1$, we can assume that the parallel combination of R_1 and C_1 is short circuit and R_2 and C_2 are in parallel, then, the transfer function from the noise source of R_2 to the output terminal is equal to,

$$H(\omega) = \frac{1/jC_2\omega}{R_2 + (1/jC_2\omega)}$$

This results in the following power spectral density for the noise at output.

$$S_{n_o}(\omega) = 2KTR_2|H(\omega)|^2.$$

Problem 4.7.3

Find the rms thermal noise arising from the resistance bridge circuit in Fig. P-4.7.3 under the following limiting conditions:

a) $R_0 \to 0$,

b) $R_0 \to \infty$.

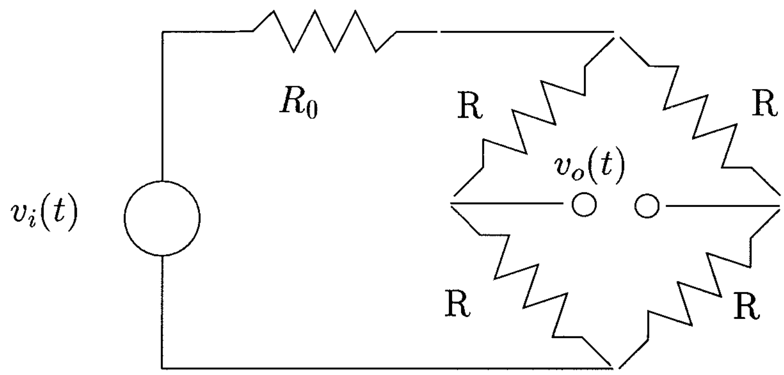


Figure 22: P-4.7.3

Solution:

We compute the output resistance referred to the output terminals. This results in,

a)

$$R_{eq} = (R \parallel R) + (R \parallel R) = R \tag{190}$$

b)

$$R_{eq} = (R + R) \parallel (R + R) = R \tag{191}$$

In both cases, assuming a bandwidth of B, we get: $\sqrt{\overline{v_o^2(t)}} = \sqrt{4kTR_{eq}B} = \sqrt{4kTRB}$.

Note that if we proceed using the superposition principle, in both cases of (a) and (b), considering the division of voltage in the corresponding circuit, the effect of each of the four noise sources is multiplied by a factor of (1/4) when it appears at the output terminals. Then, adding up the effect of the four sources, we get the same result.

Problem 4.7.4

The input of a voltage amplifier is connected to a $1 - K\Omega$ resistor and the output to a $100 - K\Omega$ resistor as shown in Fig. P-4.7.4. The voltage gain of the amplifier is 10, the input impedance is $1M\Omega$ (assumed noise-free), and the output impedance is 100Ω (noise-free). The bandwidth of the amplifier is 1MHz and the amplifier noise is $10\mu V$ rms, referred to the input. The temperature of the entire system is $300K$. compute the rms thermal noise at the output (point 2).

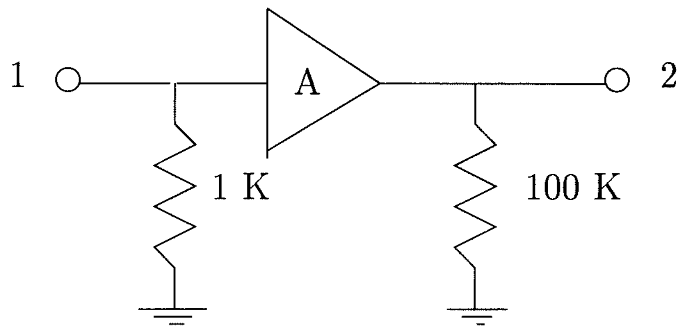


Figure 23: P-4.7.4

Solution:

Output noise arising from thermal noise in input circuitry:

$$\overline{n_1^2(t)} = 4kTB(1k\Omega)(1000/1001)^2(10)^2(1000/1001)^2 = 1.65 \times 10^{-12}\ V^2. \tag{192}$$

where $K = 1.38 \times 10^{-23}$ and $K = 300$. Note that the two $(1000/1001)^2$ terms arise from the voltage dividing at the input and output of the amplifier and $(10)^2$ is the power gain of the amplifier.

Output noise resulting from amplifier noise:

$$\overline{n_2^2(t)} = (10\mu V)^2(10)^2(1000/1001)^2 = 9.98 \times 10^{-9}\ V^2 \tag{193}$$

Output noise arising from thermal noise in output circuitry:

$$\overline{n_3^2(t)} = 4kTB(100k\Omega)(1/1001)^2 = 1.65 \times 10^{-15}\ V^2. \tag{194}$$

$$\overline{n_o^2(t)} = \overline{n_1^2(t)} + \overline{n_2^2(t)} + \overline{n_3^2(t)} = 9.9817 \times 10^{-9}\ V^2 \tag{195}$$

$$\sqrt{\overline{n_o^2(t)}} = 99.91\mu V. \tag{196}$$

Problem 4.7.5

A sinusoidal generator develops the waveform $v_i(t) = A \cos \omega_0 t$ as the input to the RC filter shown in fig. P-4.2.4.

a) Derive an equation for the rms value of the thermal noise at the output of the filter.

b) Determine the value of C which will yield the highest S/N ratio (i.e., ratio of average signal power average noise power) at the output.

Solution:

The frequency transfer function is:

$$H(\omega) = \frac{1/(R_1C)}{j\omega + 1/(R_pC)}, \quad \text{where } R_p = \frac{R_1R_2}{R_1 + R_2}. \tag{197}$$

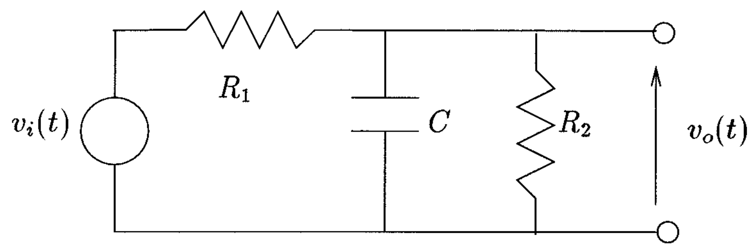


Figure 24: P-4.2.4

a) Using circuit analysis:

$$v_o(t) = \frac{A/(R_1C)}{\sqrt{\omega_0^2 + 1/(R_pC)^2}} \cos\left[\omega_0 t - \tan^{-1}(\omega_0 R_p C)\right], \tag{198}$$

The rms value of the output is:

$$v_{o_{rms}} = \frac{1}{\sqrt{2}} \frac{A/(R_1C)}{\sqrt{\omega_0^2 + 1/(R_pC)^2}}. \tag{199}$$

To compute the effect of the noise, we short circuit v_i. Then, R_1 and R_2 become parallel. In this case, referring to example 4.7.2 (page 193) of the text, we obtain $\sqrt{\overline{n_o^2(t)}} = \sqrt{KT/C}$.

The overall result is,

$$S/N = \frac{A/(R_1C)}{\sqrt{2KT/C}\sqrt{\omega_0^2 + 1/(R_pC)^2}} \tag{200}$$

Setting the derivative of S/N with respect to C equal to zero gives the optimum C.

Problem 4.7.7

Compute the noise equivalent bandwidth of the RC bandpass filter shown in Fig. P-4.7.2 for $R_1C_1 = R_2C_2$, and using the approximation described in Problem 4.7.2 for $R_1 \ll R_2$. Choose the midband frequency to be that frequency at which $|H(\omega)|$ is maximum.

Solution:

For the conditions specified, we can write:

$$H(\omega) = \frac{j\omega RC}{(j\omega RC + 1)^2} \tag{201}$$

and therefore

$$|H(\omega)|^2 = \frac{(\omega RC)^2}{[(\omega RC)^2 + 1]^2} \tag{202}$$

Setting $d|H(\omega)|^2/d\omega = 0$ and solving for ω_0, we get: $\omega_0 = 1/(RC)$.

Using Eq. (4.67) of the text and noting that $|H(\omega_0 = 1/RC)|^2 = 1/4$, we obtain:

$$B_n = \frac{4}{2\pi}\int_0^\infty \frac{(\omega RC)^2}{[(\omega RC)^2+1]^2}d\omega = \frac{1}{2RC} \tag{203}$$

Problem 4.7.8

A bandpass amplifier is to be designed to meet the following specifications:

$B_N = 200\text{kHz}$
$(S/N)_{output} \geq 20$ dB
Max. output = 1 V rms across 300 Ω
Estimated noise temperature of input stage = 1000 K

What is the maximum gain, in dB, for which you would design the amplifier to meet these specifications?

Solution:

On the basis of the output $S/N \geq 20$ dB or 100 and the maximum output, we can find the output noise power as:

$$S_o = 100N \tag{204}$$

$$S_o + N = v_o^2/R \tag{205}$$

$$100N + N = (1)^2/300 \rightarrow N = 3.30 \times 10^{-5}\ W. \tag{206}$$

Without any additional input thermal noise, the effect of the noise of the input stage at the output is equal to, $N = kT_eB_NG_p$. So that:

$$G_p \leq \frac{3.30\times 10^{-5}}{kT_eB_N} = 1.196\times 10^{10} \rightarrow G_p \leq 100.8\text{ dB}. \tag{207}$$

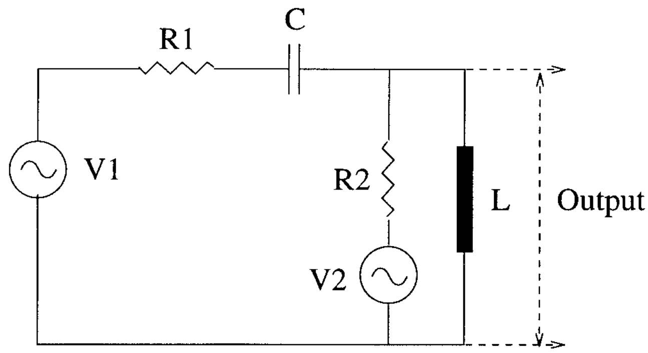
[If $T_0 = 290\ K$ were assumed at the input, this would change the answer only slightly to 99.7 dB].

Problem 4.7.9

A certain amplifier has input and output resistances of 50Ω and a noise-equivalent bandwidth of 140kHz. When connected to a matched source and a matched load, the net gain is 50dB. When a 50Ω resistor at $290K$ is connected to the input, the output rms noise voltage across 50Ω is $100\mu V$. Determine the equivalent noise temperature, T_e, of the amplifier.

Solution:

$$P_{out} = (100\ \mu V)^2/50 = kT_0B_NG_p + kT_eB_NG_p \tag{208}$$



from which

$$T_e = \frac{(100\ \mu V)^2/50}{kB_N G_p} - T_0 = 1035 - 290 = 745\ K. \tag{209}$$

A General Problem on Noise: Consider the following circuit where,

$$v_1(t) = \cos(\omega_0 t + \phi)$$

and

$$v_2(t) = \text{rect}_T(t/\tau), \quad \tau = T/2$$

find the power spectral density of the signal and the noise at the output terminal.

Solution:

We know that the power spectral density of a periodic signal with the Fourier series expansion $\sum_n F_n e^{jn\omega_0 t}$ is equal to $S_f(\omega) = 2\pi \sum_n |F_n|^2 \delta(\omega - n\omega_0)$. We have,

$$v_1(t) = \frac{1}{2}\left[e^{j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}\right] = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

This means that,

$$F_1 = e^{j\phi}/2, \qquad F_{-1} = e^{-j\phi}/2, \qquad F_n = 0,\ n \neq 1, -1$$

As $|F_1|^2 = |F_{-1}|^2 = (1/4)$, then,

$$S_{v_1}(\omega) = (\pi/2)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

Similarly, for v_2, we have,

$$F_n = (1/2)\text{Sa}(n\pi/2)$$

and,

$$S_{v_2}(\omega) = (\pi/2) \sum_n \text{Sa}^2(n\pi/2)\delta(\omega - n\omega_0).$$

The transfer function from $v_1(t)$ to output is equal to (short circuit v_2),

$$H_1(\omega) = \frac{R_2||(jL\omega)}{\left(R_1 + \frac{1}{jC\omega}\right) + [R_2||(jL\omega)]} = \frac{\frac{R_2 jL\omega}{R_2 + jL\omega}}{\left(R_1 + \frac{1}{jC\omega}\right) + \frac{R_2 jL\omega}{R_2 + jL\omega}}$$

The power spectrum of the output signal due to v_1 is equal to,

$$|H_1(\omega)|^2 S_{v_1}(\omega) = (2/\pi)|H_1(\omega_0)|^2\delta(\omega - \omega_0) + (2/\pi)H_1(-\omega_0)\delta(\omega + \omega_0)$$

Due to the symmetry of $H_1(\omega)$, we have,

$$|H_1(\omega)|^2 S_{v_1}(\omega) = (2/\pi)|H_1(\omega_0)|^2[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

The transfer function from $v_2(t)$ to output is equal to (short circuit v_1),

$$H_2(\omega) = \frac{(jL\omega)||\left(R_1 + \frac{1}{jC\omega}\right)}{R_2 + \left[(jL\omega)\,||\left(R_1 + \frac{1}{jC\omega}\right)\right]}$$

The power spectrum of the output signal due to v_2 is equal to,

$$|H_2(\omega)|^2 S_{v_2}(\omega) = (\pi/2)\sum_n |H_2(n\omega_0)|^2 \mathrm{Sa}^2(n\pi/2)\delta(\omega - n\omega_0).$$

As the noise sources are in series with the resistors, their effects can be computed using the same transfer functions. This means that, to compute the effect of the two noise sources at the output, we should replace S_{v_1} by $S_{n_1} = 2KTR_1$ and S_{v_2} by $S_{n_2} = 2KTR_2$ in the relationships used to compute the signal power at the output.

Problem 5.1.2 A modulating signal $f(t)$ [with Fourier transform $F(\omega)$] is applied to a double sideband suppressed carrier modulator operating at a carrier frequency of 200Hz. Sketch the spectral density of the resulting DSB -SC waveform, identifying the upper and lower sidebands, for each of the following cases.

a.) $f(t) = \cos 100\pi t$

b.)

$$F(\omega) = \begin{cases} [1 + \cos(\omega/200)]/2 & |\omega| < 200\pi \\ 0 & \text{elsewhere} \end{cases}$$

Solution:

a.)

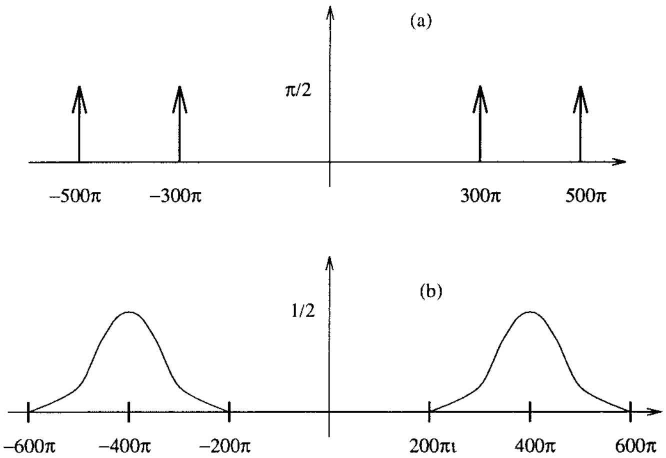
$$g(t) = \cos 100\pi t \cos 400\pi t = \frac{1}{2}\cos 300\pi t + \frac{1}{2}\cos 500\pi t$$

This gives the corresponding Fourier Transform:

$$G(\omega) = \frac{\pi}{2}[\delta(\omega - 300\pi) + \delta(\omega + 300\pi) + \delta(\omega - 500\pi) + \delta(\omega + 500\pi)]$$

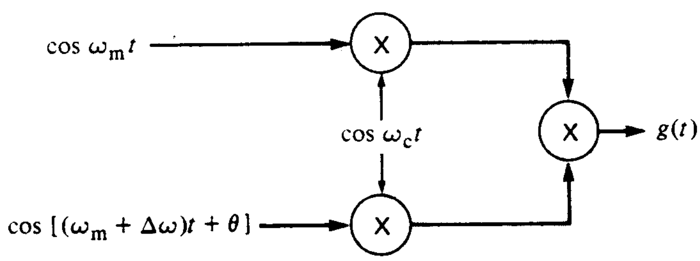
b.)

$$G(\omega) = \frac{1}{2}F(\omega - 400\pi) + \frac{1}{2}F(\omega + 400\pi);$$



Problem 5.1.3 We wish to examine the frequency and phase response of one low -frequency oscillator relative to the second. We decide to multiply the outputs of the oscillators to make the comparison. However, as a result of difficulties in building a low frequency multiplier we instead choose to use the DSB-SC modulators and bandpass multiplier, as shown in Fig. P-5.1.3.

a.) Determine the expression for the DSB-SC signals.
b.) Determine the expression for $g(t)$.
c.) Sketch a magnitude Spectrum of $g(t)$.
d.) Under what conditions does $g(t)$ represent the desired product of the two input waveforms?



Solution:

a.)

$$\Phi_1(t) = \cos(\omega_m t)\cos(\omega_c t)$$

$$\Phi_2(t) = \cos[(\omega_m + \Delta\omega)t + \theta]\cos(\omega_c t)$$

You can simplify/expand these if you wish.

b)

$$g(t) = \Phi_1(t) \times \Phi_2(t) = \frac{1}{4}\left[\cos(\Delta\omega t + \theta) + \cos(2\omega_m t + \Delta\omega t + \theta)\right][1 + \cos 2\omega_c t];$$

c.) We know that,

$$\cos(\Delta\omega t + \theta) + \cos(2\omega_m t + \Delta\omega t + \theta)$$

has the Fourier transform,

$$\pi e^{j\theta}\delta(\omega - \Delta\omega) + \pi e^{-j\theta}\delta(\omega + \Delta\omega) + \pi e^{j\theta}\delta(\omega - 2\omega_m - \Delta\omega) + \pi e^{-j\theta}\delta(\omega + 2\omega_m + \Delta\omega)$$

Let us show this Fourier transform by $F(\omega)$, then we have,

$$|F(\omega)| = \pi\delta(\omega - \Delta\omega) + \pi\delta(\omega + \Delta\omega) + \pi\delta(\omega - 2\omega_m - \Delta\omega) + \pi\delta(\omega + 2\omega_m + \Delta\omega)$$

Note that here we have simply added the magnitudes of different terms because we have impulses. However, in general, this is not possible.

The resulting magnitude spectrum after multiplication by $(1/4)[1 + \cos 2\omega_c t]$ is equal to,

$$\frac{1}{4}|F(\omega)| + \frac{1}{8}|F(\omega - 2\omega_c)| + \frac{1}{8}|F(\omega + 2\omega_c)|$$

d.) $g(t)$ will represent the product of the two waveform if $2\omega_c > (4\omega_m + 2\Delta\omega)$, or $\omega_c > (2\omega_m + \Delta\omega)$, and a low pass filter is used at the output.

Problem 5.1.4 A sinusoidal signal $f(t) = \cos 2000\pi t$ is multiplied by a periodic symmetric triangular waveform (c.f. Table 2.1) with unit peak amplitude and $T = 100\mu$sec. The output of the multiplier is applied to a low pass filter with a unity gain within the passband.
a.) Determine the minimum and maximum bandwidth of the LPF if the output is to be a DSB-SC waveform corresponding to $f(t)$.
b.) Determine an expression for the output of the LPF under the above conditions
c.) Can this system be represented by the DSB-SC modulator shown in Fig. 5.1.(a) under the above conditions ? If your answer is yes, the determine the two inputs to the modulator.

Solution:

a.) The signal has a bandwidth of 1 KHz. For the triangular waveform $f_0 = 1/T = 10$ KHz and it has odd harmonics only. Therefore, the LPF bandwidth must satisfy the condition $11\,\text{KHz} < B < 29\,\text{KHz}$ where $11 = 10$ (first harmonic) + 1 (bandwidth of the signal) and $29 = 30$ (third harmonics) − 1 (bandwidth of the signal). Use sketches to visualize this properly.
b.) Using Table 2.1, we represent the triangular signal by the corresponding Fourier series. The coefficients for the first harmonic are equal to $(2/\pi)^2$. Assuming unity scale factor for the multiplier and LPF, we have:

$$\begin{aligned} g(t) &= \left[\left(\frac{2}{\pi}\right)^2 e^{j\omega_0 t} + \left(\frac{2}{\pi}\right)^2 e^{-j\omega_0 t} + \ldots\right]\left[\frac{1}{2}e^{j\omega_1 t} + \frac{1}{2}e^{-j\omega_1 t}\right] \\ &= (2/\pi)^2 \cos 22000\pi t + (2/\pi)^2 \cos 18000\pi t \end{aligned}$$

Note that $\omega_0 = 20000\pi$ and $\omega_1 = 2000\pi$.

c.) Yes, the equivalent inputs are: $\cos 2000\pi t$ and $(2/\pi)^2 \cos 20000\pi t$.

Problem 5.1.6 When the input to a given audio amplifier is $(4\cos 800\pi t + \cos 1000\pi t)$ mV, the measured frequency component at 1000Hz is 1V and the ratio of the frequency component at 500Hz to that at 1000Hz is 0.002. Represent the amplifier output-input characteristic by

$$e_o(t) = a_1 e_i(t) + a_2[e_i(t)]^2.$$

a.) Evaluate the numerical values of a_1, a_2 from the test data given. (This type of test is known as the intermodulation distortion test.)

b.) What would you expect to be magnitudes of the frequency component at 800Hz and at 1600Hz?

Solution:

a.)

$$
\begin{aligned}
e_o(t) &= a_1(4\cos 800\pi t + \cos 1000\pi t) + a_2(4\cos 800\pi t + \cos 1000\pi t)^2 \\
&= 4a_1 \cos 800\pi t + a_1 \cos 1000\pi t + \\
&\quad 8.5a_2 + 8a_2 \cos 1600\pi t + 0.5a_2 \cos 2000\pi t + 4a_2 \cos 200\pi t + 4a_2 \cos 1800\pi t
\end{aligned}
$$

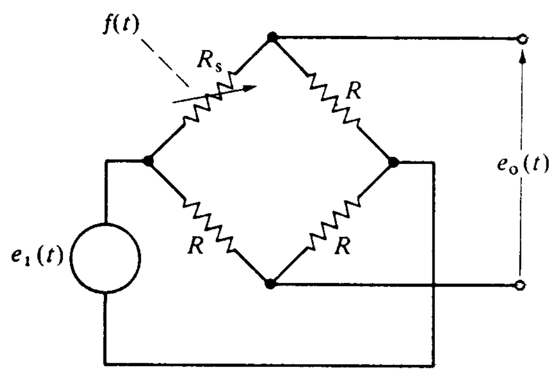
For the given values $0.5a_2 = 1000$ mV, then $a_2 = 2000$, and $2a_1/a_2 = 0.002, \Rightarrow a_1 = 2$

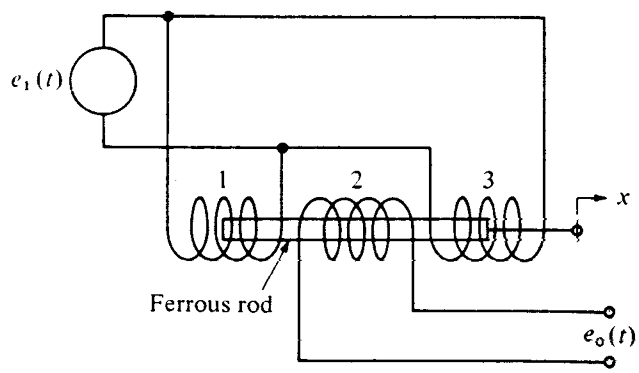
b.)

@400Hz : $4a_1$ mV

@800Hz : $8a_2$ mV

Problem 5.1.10 Two measurement systems are shown in Figure P-5.1.10. Find the output of each system if $e_i(t) = \cos\omega_1 t$. What type of modulation is present in each and how should it be detected ?





Solution:

a.) It can be shown that

$$e_0(t) = \left[\frac{R}{R+R} - \frac{R_s}{R+R_s}\right]\cos\omega_1 t = \left[\frac{1}{2} - \frac{R_s}{R+R_s}\right]\cos\omega_1 t$$

If we substitute the relation

$$R_s = R[1 + \alpha f(t)]$$

we obtain the relationship

$$e_0(t) = \left[\frac{-\alpha f(t)}{2(2 + \alpha f(t))}\right]\cos\omega_1 t$$

Given $|\alpha f(t)| \ll 2$, this can be simplified to give $e_0(t) \approx \frac{-\alpha}{4} f(t) \cos\omega_1 t$

b.) The voltage developed across the inductances is equal to,

$$\begin{aligned} e_0(t) &= e_{12}(t) - e_{23}(t) \\ &= k[1 + \alpha x(t)] \cos\omega_1 t - k[1 - \alpha x(t)] \cos\omega_1 t \\ &= 2\alpha k x(t) \cos\omega_1 t \end{aligned}$$

Both a.) and b.) are examples of DSB-SC modulation. They require synchronous detection.

Problem 5.2.4 For the sinusoidally modulated DSB-LC waveform shown in figure P-5.2.4

a.) Find the modulation index.

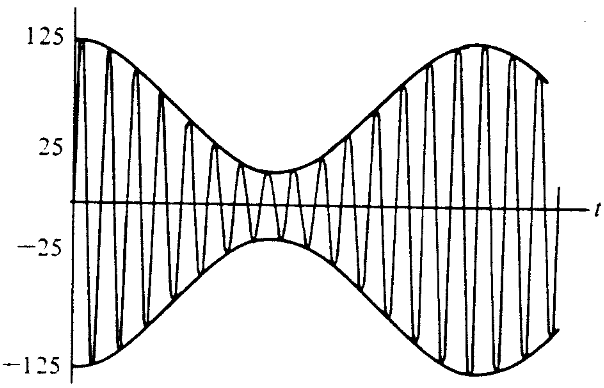
b.) Write an expression for the waveform in the form of equation (5.18).

c.) Sketch a line spectrum for the waveform.

d.) Show that the sum of the two sideband lines in part c.) , divided by the carrier line yields the modulation index.

e.) Determine the amplitude and phase of the additional carrier which must be added to make the waveform shown to attain a modulation index of 20%

f.) Repeat part e.) to attain a modulation index of 80%.



Solution:

a.) Consider the AM signal $\phi(t) = A(1 + m\cos\omega_m t)\cos\omega_c t$. Let $\psi(t) = A(1 + m\cos\omega_m t)$ denote the corresponding envelope. We have $\psi_{\max}/\psi_{\min} = (1 + m)/(1 - m)$, or,

$$m = \frac{\psi_{max} - \psi_{min}}{\psi_{max} + \psi_{min}}$$

In this case, we obtain,

$$m = \frac{125 - 25}{125 + 25} = \frac{100}{150} = 66.7\%$$

b.) We have $\psi_{\max} = A(1 + m) = 125$ resulting in $A = 75$. (The same result is obtained by using $\psi_{\min} = A(1 - m) = 25$.) Then,

$$\phi(t) = 75[1 + 0.667\cos\omega_m t]\cos\omega_c t$$

c.) Straightforward.

d.) Each spectrum line of the carrier has amplitude $75/2 = 37.5$ and each sideband has amplitude $75 \times 0.667/4 = 12.5$, then, $(12.5 + 12.5)/37.5 = 25/37.5 = 66.7\%$

e.)

$$\phi(t) = (75 + A)\cos\omega_c t + 50\cos\omega_m t\cos\omega_c t;$$

This gives the value of m as $m = 50/(75 + A) = 0.20$ resulting in, $A = 175$ V.

f.) Using the same procedure as above we get:

$$50/(75 + A) = 0.80;$$

Solving for a yields $A = -12.5V = 12.5e^{-j\pi}\ V$

Problem 5.2.6 A given AM (DSB-LC) transmitter develops an unmodulated power output of 1KW[2] across a 50-ohm resistive load . When a sinusoidal test tone with a peak amplitude of 5V is applied to the input of the modulator, it is found to the spectral line for each sideband in the magnitude spectrum for the output is 40% of the carrier line. Determine the following quantities in the output signal:

a.) The modulation index .

b.) The peak amplitude of the lower sideband.

c.) The ratio of the total sideband power to carrier power.

d.) The total power output.

e.) The total average power in the output if the peak amplitude of the modulation sinusoid is reduced to 4.0V.

Solution:

a.) Using the result of problem 5.2.4 (d), we conclude that the modulation index is equal to $m = 0.8$. This results in the following modulated signal,

$$A(1 + 0.8\cos\omega_m t)\cos\omega_c t = A\cos\omega_c t + 0.4A\cos(\omega_c - \omega_m)t + 0.4A\cos(\omega_c + \omega_m)t$$

b.) $A^2/2 = PR \Longrightarrow A = \sqrt{(2)(1000)(50)} = 316.22V \Longrightarrow$ Peak amplitude of sideband $= 0.4A = 126.5$.

c.) $P_s/P_c = m^2/2 = (0.8)^2/2 = 0.32$.

d.) Using the equation:

$$\overline{\Phi^2_{AM}(t)}/R = (1/R)\frac{1}{2}A^2 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)m^2A^2$$

and replacing $A^2/(2R) = 1000$, We get the result : $P_t = 1000 \times [1 + (m^2/2)] = 1.32kW$

[2]FCC power ratings for AM broadcast transmitters are for an average carrier (i.e. unmodulated) power.

e.) $P_t = 1000[1 + (4/5)^2(m^2/2)] = 1.025kW$

Problem 5.2.10 Let $f(t) = \cos\, \omega_m t$ in Eq. (5.28) and add the term $a_3e^3(t)$ to Eq. (5.27); then revise Eq. (5.29) and derive an expression for the modulation index m.

Solution:

$$i(t) = a_1 e(t) + a_2 e^2(t) + a_3 e^3(t), \tag{210}$$

where

$$e(t) = \cos\, \omega_m t + \cos\, \omega_c t \tag{211}$$

Expanding and collecting terms at the carrier frequency, we have

$$v_o(t) = kR[a_1 + a_3(3 + k^2)/2] \cos\, \omega_c t + 2kRa_2 \cos\, \omega_m t \cos\, \omega_c t + (3kRa_3/2) \cos\, 2\omega_m t \cos\, \omega_c t \tag{212}$$

The third term in the expression represents second-harmonic distortion in the output. Defining the modulation index for the fundamental,

$$m = \frac{4a_2}{2a_1 + (3 + k^2)a_3} \tag{213}$$

Thus it is desirable to keep a_3 very small, and not let k become large.

Problem 5.2.11

A sinusoidally modulated DSB-LC waveform $\phi(t)$ [cf. Eq. (5.18)] is applied to a square-law device, such that the output voltage $e_0(t)$ is $e_0(t) = [f(t)]^2$. Show that the ratio of the second harmonic to the first harmonic in $e_0(t)$ is equal to $m/4$.

Solution:

$$e_0(t) = A^2(1 + m \cos\, \omega_m t)^2 \cos^2\, \omega_c t, \tag{214}$$

$$e_0(t) = \frac{1}{2}A^2(1 + 2m \cos\, \omega_m t + \frac{m^2}{2} + \frac{m^2}{2} \cos 2\omega_m t)(1 + 2 \cos\, \omega_c t) \tag{215}$$

The ratio of the second harmonic to the fundamental is: $\frac{m^2/2}{2m} = \frac{m}{4}$.

Problem 5.4.2

A $SSB - LC_+$ transmitter is modulated with the input $f(t) = \cos\, 2000\pi t$. In complexed-valued notation, the output waveform can be written as

$$\Phi_{SSB-LC_+}(t) = \mathcal{R}\{A[1 + mf(t) + jm\hat{f}(t)]e^{j\omega_c t} \tag{216}$$

a) Evaluate $\hat{f}(t)$.

b) What is the maximum value of m if envelope detection is used?

c) Determine the numerical values of A, m if the unmodulated power across 50 ohms is 500 W, and the average power is found to increase 50

d) Find the peak envelope power (PEP; see Problem 5.4.1) under both the unmodulated and the modulated condition in part (c).

Solution:

a)

$$\hat{f}(t) = \cos(2000\pi t - \pi/2) = \sin 2000\pi t \quad \text{[cf. Fig. 5.31]} \tag{217}$$

b) The envelope is:

$$r(t) = \sqrt{(1 + m\cos\omega_m t)^2 + (m\sin\omega_m t)^2} \tag{218}$$

$$= \sqrt{1+m^2}\sqrt{1 + \frac{2m}{1+m^2}\cos\omega_m t} \tag{219}$$

Using a binomial series approximation, we obtain,

$$r(t) \approx \sqrt{1+m^2}\left(1 + \frac{m}{1+m^2}\right)\cos\omega_m t$$

The power of the $\cos\omega_m t$ term is equal to $0.5m^2/(1+m^2)$. To compute the proper value of m, we know that the total power of $r(t)$ is equal to, $1+m^2$ (AC power m^2) and we set the criterion of having $THD < 0.1$.

c)

$$\Phi(t) = A\cos\omega_c t + mA\cos(\omega_c + \omega_m)t \tag{220}$$

$$P_t = P_c + P_s = \frac{A^2}{2R} + \frac{m^2 a^2}{2R} \tag{221}$$

from which: $A = 223.6\text{V}$, $m = 0.707$.

d) Unmodulated:

$$PEP = \frac{A^2}{2R} = 500\ W, \tag{222}$$

Modulated at $m = 0.707$:

$$PEP = \frac{(1+0.707)^2 A^2}{2R} = 1457\ W \tag{223}$$

Problem 5.5.1

A vestigial sideband signal is generated from an input $f(t)$ by first generating a DSB-LC signal ($m = 0.8$) with a carrier frequency $f_c = 10$ kHz and then passing this signal through a filter whose magnitude frequency transfer function is

$$|H(\omega)| = \begin{cases} 1 + \cos\left[(\omega - \omega_0)/8000\right], & |\omega - \omega_0| < 8000\pi \\ 0 & \text{elsewhere} \end{cases} \tag{224}$$

where $\omega_0 = 24,000\pi$. Find an expression for the resulting VSB signal, sketch the spectral density, and calculate the peak envelope power (cf. Problem 5.2.3) to average power if

a) $f(t) = \cos 1000\pi t$.

b) $f(t) = \cos 2000\pi t$.

c) $f(t) = \cos 4000\pi t$.

Solution:

The general solution is

$$\phi(t) = [(1 + m\cos \omega_m t)\cos \omega_c t] \otimes h(t) \tag{225}$$

with the Fourier transform,

$$\Phi(\omega) = 2\pi\left\{\frac{1}{2}\delta(\omega \pm \omega_c) + \frac{m}{4}\delta[\omega \pm (\omega_c + \omega_m)] + \frac{m}{4}\delta[\omega \pm (\omega_c - \omega_m)]\right\} H(\omega) \tag{226}$$

a) $\omega_c = 20000\pi$, $\omega_m = 1000\pi$.

$$\phi(t) = \cos \omega_c t + 0.4[1 + \cos(3\pi/8)]\cos(\omega_c + \omega_m)t + 0.4[1 + \cos(5\pi/8)]\cos(\omega_c - \omega_m)t \tag{227}$$

$$= \cos \omega_c t + 0.553\cos(\omega_c + \omega_m)t + 0.247\cos(\omega_c - \omega_m)t \tag{228}$$

$$\frac{\text{PEP}}{P_{avg}} = \frac{(1 + 0.553 + 0.247)^2}{1 + (0.553)^2 + (0.247)^2} = 2.37 \tag{229}$$

b) $\omega_c = 20000\pi$, $\omega_m = 2000\pi$.

$$\phi(t) = \cos \omega_c t + 0.4[1 + \cos(2\pi/8)]\cos(\omega_c + \omega_m)t + 0.4[1 + \cos(6\pi/8)]\cos(\omega_c - \omega_m)t \tag{230}$$

$$= \cos \omega_c t + 0.683\cos(\omega_c + \omega_m)t + 0.117\cos(\omega_c - \omega_m)t \tag{231}$$

$$\frac{\text{PEP}}{P_{avg}} = \frac{(1 + 0.683 + 0.117)^2}{1 + (0.683)^2 + (0.117)^2} = 2.19 \tag{232}$$

c) $\omega_c = 20000\pi$, $\omega_m = 4000\pi$.

$$\phi(t) = \cos \omega_c t + 0.8\cos(\omega_c + \omega_m)t \tag{233}$$

$$\frac{\text{PEP}}{P_{avg}} = \frac{(1 + 0.8)^2}{1 + (0.8)^2} = 1.98 \tag{234}$$

Problem 5.6.1

A noise waveform with a power spectral density $S_n(\omega) = 10^{-7}e^{-|\omega|/\alpha}$ W/Hz, where $\alpha = 2\pi \times 10^7$/sec, is passed through an ideal BPF with unity gain, unity resistance levels, and a bandwidth of 200 kHz centered at 10 MHz.

a) Determine the mean-square values of the in-phase and quadrature components of the band-pass time representation given in Eq. (5.59).

b) Repeat part (a) assuming that $S_n(\omega)$is constant (at the center frequency value) across the bandwidth of the filter.

c) Repeat parts (a) and (b) for a bandwidth of 2 MHz.

Solution:

Let W be the low-pass bandwidth [i.e., 100 kHz for (a) and (b)].

a) Lowpass approach: using Eq. (5.65), we have

$$S_{n_c}(\omega) = S_{n_s}(\omega) = 10^{-7}[e^{-|\omega-\omega_0|/\alpha} + e^{-|\omega+\omega_0|/\alpha}]_{lp} \tag{235}$$

$$\overline{n_c^2(t)} = \overline{n_s^2(t)} = \frac{1}{2\pi}10^{-7}e^{-\omega_0/\alpha}\int_{-W}^{W}[e^{\omega/\alpha} + e^{-\omega/\alpha}]d\omega = 2e^{-1}[e^{W/\alpha} - e^{-W/\alpha}] \tag{236}$$

$$= 2e^{-1}[e^{0.01} - e^{-0.01}] = 0.014715 \ V^2 \tag{237}$$

Bandpass approach:

$$\overline{n^2(t)} = \frac{2}{2\pi}10^{-7}\int_{\omega_0-W}^{\omega_0+W} e^{-\omega/\alpha}d\omega = 2e^{-1}[e^{0.01} - e^{-0.01}] = 0.014715 \ V^2. \tag{238}$$

b) Lowpass approach:

$$\overline{n_c^2(t)} = \overline{n_s^2(t)} = \frac{1}{2\pi}10^{-7}e^{-\omega_0/\alpha}\int_{-W}^{W} 2d\omega = \frac{2W}{\pi}10^{-7}e^{-1} = 0.04e^{-1} = 0.014715 \ V^2 \tag{239}$$

Bandpass approach:

$$\overline{n^2(t)} = \frac{2}{2\pi}10^{-7}\int_{\omega_0-W}^{\omega_0+W} e^{-\omega/\alpha}d\omega = 0.04e^{-1} = 0.014715 \ V^2. \tag{240}$$

c) For exponential $S_n(\omega)$

$$\overline{n_c^2(t)} = \overline{n_s^2(t)} = 2e^{-1}[e^{0.1} - e^{-0.1}] = 0.1474 \ V^2 \tag{241}$$

For constant $S_n(\omega)$

$$\overline{n_c^2(t)} = \overline{n_s^2(t)} = 0.4e^{-1} = 0.1472 \ V^2 \tag{242}$$

Problem 5.7.1

A DSB-SC and a SSB-SC transmission are each sent at 1 MHz in the presence of additive noise. The modulating signal in each case is band-limited to 3 kHz. The received signal power in each case is 0.2 mW. The received noise is assumed to be white with a (two-sided) power spectral density of 10^{-3} μW/Hz. The receiver consists of a band-pass filter whose bandwidth matches the bandwidth of each transmission, followed by a synchronous detector.

a) Compare the signal-to-noise ratios at the detector inputs.

b) Compare the signal-to-noise ratios at the detector outputs.

c) Repeat part (a) if the (two-sided) power spectral density were $10^3/|f|$ μW/Hz. Would a "white noise" assumption be valid here?

Solution:

a)

$$\frac{S_i}{N_i} = \frac{0.2\,\text{mW}}{2(10^{-3}\,\mu\text{W/Hz})(6\text{ kHz})} = 16.7 \quad \text{for DSB,} \tag{243}$$

$$= \frac{0.2\,\text{mW}}{2(10^{-3}\,\mu\text{W/Hz})(3\text{ kHz})} = 33.3 \quad \text{for SSB,} \tag{244}$$

b) From Eqs. (5.73) and (5.80)

$$\frac{S_o}{N_o} = 2\frac{S_i}{N_i} = 33.3 \quad \text{for DSB,} \tag{245}$$

$$= \frac{S_i}{N_i} = 33.3 \quad \text{for SSB,} \tag{246}$$

Thus the net S_o/N_o ratio is the same for both systems in the presence of additive white noise.

c) For DSB

$$N_i = 2\int_{f_c-3000}^{f_c+3000} \frac{10^3}{f} df = 2000[\ln(10^6+10^3) - \ln(10^6-10^3)] = 12\ \mu\text{W} \tag{247}$$

For SSB_+, we have:

$$N_i = 2\int_{f_c}^{f_c+3000} \frac{10^3}{f} df = 2000[\ln(10^6+10^3) - \ln(10^6)] = 5.99\ \mu\text{W} \tag{248}$$

For SSB_-, we have:

$$N_i = 2\int_{f_c-3000}^{f_c} \frac{10^3}{f} df = 2000[\ln(10^6) - \ln(10^6-10^3)] = 6.009\ \mu\text{W} \tag{249}$$

Thus, the white noise assumption is good here as a result of narrow bandwidth, even though the noise power spectral density is not flat.

Problem 5.7.2

Show that, strictly on an average power basis, use of an envelope detector in a DSB-LC system results in a linear $(S/N)_o$ versus $(S/N)_i$ relationship. What makes this relationship nonlinear for commercial AM systems?

Solution:

For the DSB-LC signal, we have, (Eq. 5.81):

$$S_i(t) + n_i(t) = [A + f(t)]\cos\omega_c t + n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t \tag{250}$$

Assuming the signal and noise to be zero-mean, i.e.

$$\overline{f(t)} = \overline{n_c(t)} = \overline{n_s(t)} = 0 \tag{251}$$

and that the cross-correlations with the noise terms are zero, i.e.,

$$\overline{f(t)n_c(t)} = \overline{n_c(t)n_s(t)} = \overline{f(t)n_s(t)} = 0 \tag{252}$$

we get,

$$\overline{\phi(t)^2} = \frac{1}{2}\left[A^2 + \overline{f^2(t)} + \overline{n_c^2(t)} + \overline{n_s^2(t)}\right] \tag{253}$$

Because the signal and noise terms are additive in this result, the S/N dependence is a linear one in terms of average power. However, in AM detection the envelope is taken first to demodulate the signal, then the power is computed. In this case there is no longer a linear relation, as exhibited by Eq. (5.91). (Note that the subject of noise in AM using envelope detection, and in specific Eq. (5.91), has not been covered in the lectures and is not included in the exam.)

Problem 5.7.4

The DSB-LC signal

$$\phi(t) = 3\cos(10,000\pi t) + \cos(1000\pi t)\cos(10,000\pi t) \quad \text{V} \tag{254}$$

is present with additive white noise whose (two-sided) power spectral density is 1 μW/Hz. This signal-plus-noise is passed through an ideal low-pass filter with a bandwidth of 10 kHz. Assume all resistance levels are 1 ohm.

a) Compute the average S/N ratio at the output of the low-pass filter.

b) A synchronous detector is used to demodulate the above signal. Compute the average S/N ratio if the output of the detector is filtered to $0 < f < 1$ kHz.

c) An ideal square-law detector (i.e., the signal-plus-noise is multiplied by itself) is used to demodulate the above signal. Compute the average S/N ratio if the output of the detector is filtered to $0 < f < 1$ kHz. [*Hint:* Use frequency convolution, and signal-signal, signal-noise, noise-noise products.]

Solution:

a)

$$N_i = 2(\eta/2)B = 2\times10^{-6}(10^4) = 0.02\ \text{W} \quad (255)$$

$$S_i = \overline{\phi^2(t)} = (3)^2(1/2) + (1/2)(1/2) = 4.75\ \text{W} \quad (256)$$

$$\frac{S_i}{N_i} = 237.5\ \ (23.8\ \text{dB}) \quad (257)$$

b) After LPF

$$e_o(t) = (1/2)\cos(1000\pi t) \quad (258)$$

$$S_o = \overline{e_o^2(t)} = (1/2)^2(1/2) = 1/8 \quad (259)$$

$$(260)$$

Using Eq. (5.72), $N_o = N_i/4 = 0.005$ and $S_o/N_o = 25$ (14dB). Note that using Eq. (5.88) for $m = 1/3$ (which is the modulation index for the signal given in 254, gives $(2/19)(237.5) = 25$, checking this result.

c) After LPF

$$e_o(t) = \left\{[\phi(t) + n_c(t)\cos\omega_c t + n_s(t)\sin\omega_c t]^2\right\}_{lp} \quad (261)$$

$$= 3\cos\omega_m t + 3n_c(t) + n_c(t)\cos\omega_m t + \frac{1}{4}\cos 2\omega_m t + \frac{1}{2}n_c^2(t) + \frac{1}{2}n_s^2(t) \quad (262)$$

The first term in this result is the modulated signal, with the average power $(3)^2/2$, the second term is the primary noise term, with average power $(3)^2N_i = (3)^2(0.02)$, the ratio of these two terms is 25, the same as was obtained in (b) using synchronous detection. The third term represents the harmonic distortion, with average power $(1/4)^2(1/2)$. The forth term represents a bandpass in phase noise term around frequency ω_m. This noise term has the power $N_i/2 = 0.01$. The last two terms represent noise-noise interactions in the square-law detection. To compute the power of these terms, we neen to know the probability density of the noise. We assume that this is a Gaussian distribution. For a Gaussian distribution, we have $\overline{n^4} = 3\overline{n^2} = 3\times0.02$. Taking the effect of the (1/2) factor into account, the power of each these two noise terms is equal to 0.03/2. The overall result is,

$$\frac{S_o}{N_o} = \frac{(3)^2/2}{(3)^2(0.02) + (1/4)^2(1/2) + 2(8\times10^{-6})} = 21.3\ \ (13.3\ \text{dB}) \quad (263)$$

Note that the noise-noise term do not contribute appreciably here because the input S/N ratio is relatively high.

Problem 6.1.1

Determine the instantaneous frequency, in Hertz, of each of the following waveforms.

a) $10\cos(100\pi t + \pi/3)$

b) $10\cos(200\pi t + 10\sin \pi t)$

c) $2\exp[j200\pi t(1+\sqrt{t})]$

d) $\cos 200\pi t \cos(5\sin 2\pi t) + \sin 200\pi t \sin(5\sin 2\pi t)$

Solution: The instantaneous frequency, in Hertz, is

$$f_i = \frac{1}{2\pi}\frac{d\theta}{dt} \tag{264}$$

where θ is the instantaneous phase of the signal.

a)

$$f_i = \frac{1}{2\pi}\frac{d}{dt}(100\pi t + \pi/3) = \frac{100\pi}{2\pi} = 50 \text{ Hz} \tag{265}$$

b)

$$f_i = \frac{1}{2\pi}\frac{d}{dt}(200\pi t + 10\sin \pi t) = 100 + 5\cos \pi t \text{ Hz} \tag{266}$$

c)

$$f_i = \frac{1}{2\pi}\frac{d}{dt}[200\pi t(1+\sqrt{t})] = 100 + 150\sqrt{t} \text{ Hz} \tag{267}$$

d)

$$\cos 200\pi t \cos(5\sin 2\pi t) + \sin 200\pi t \sin(5\sin 2\pi t) = \cos(200\pi t - 5\sin 2\pi t) \tag{268}$$

$$f_i = \frac{1}{2\pi}\frac{d}{dt}(200\pi t - 5\sin 2\pi t) = 100 - 5\cos 2\pi t \text{ Hz} \tag{269}$$

Problem 6.1.3

a) Find an approximation to the Fourier series expansion of the angle-modulated waveform $\psi(t) = \Re[A\exp(j\omega_c t)\exp(j\beta \sin \omega_m t)]$ for small β by using the MacLaurin series expansion for $\exp(x)$ and retaining only the first two terms in the expansion.

b) Sketch the line spectrum of the approximation to $\psi(t)$, as determined in part (a).

c) Determine the Fourier transform (spectral density) of the approximation to $\psi(t)$, as determined in part (a).

Solution:

a)

$$\exp(j\beta \sin\omega_m t) \approx 1 + j\beta \sin\omega_m t = 1 + \frac{\beta}{2}e^{j\omega_m t} - \frac{\beta}{2}e^{-j\omega_m t} \tag{270}$$

$$\Re[A\exp(j\omega_c t)\exp(j\beta\sin\omega_m t)] \simeq A\cos\omega_c t + \frac{A\beta}{2}\cos(\omega_c+\omega_m)t - \frac{A\beta}{2}\cos(\omega_c-\omega_m)t = A\cos\omega_c t - \beta A\sin\omega_m t\sin\omega_c t$$

b) Easy.

c)

$$\Psi(\omega) = \pi A\delta(\omega \pm \omega_c) + \frac{\pi A\beta}{2}\delta[\omega \pm (\omega_c + \omega_m)] - \frac{\pi A\beta}{2}\delta[\omega \pm (\omega_c - \omega_m)] \tag{271}$$

Problem 6.2.1

A 1-GHz carrier is frequency-modulated by a ten-KHz sinusoid so that the peak frequency deviation is 100 Hz. Determine

a) the approximate bandwidth of the FM signal;

b) the bandwidth if the modulating signal amplitude were doubled;

c) the bandwidth if the modulating signal frequency were doubled;

d) the bandwidth if both the amplitude and the frequency of the modulating signal were doubled.

Solution:

a)

$$\beta = \frac{\Delta\omega}{\omega_m} = \frac{10^2}{10^4} = 0.01 \rightarrow \text{NBFM} \tag{272}$$

$$B \approx 2f_m = 20 \text{ KHz} \tag{273}$$

b)

$$\beta = \frac{\Delta\omega}{\omega_m} = \frac{2\times 10^2}{10^4} = 0.02 \rightarrow \text{NBFM} \tag{274}$$

$$B \approx 2f_m = 20 \text{ KHz} \tag{275}$$

c)

$$\beta = \frac{\Delta\omega}{\omega_m} = \frac{10^2}{2\times 10^4} = 0.005 \rightarrow \text{NBFM} \tag{276}$$

$$B \approx 2f_m = 40 \text{ KHz} \tag{277}$$

d)

$$\beta = \frac{\Delta\omega}{\omega_m} = \frac{2\times 10^2}{2\times 10^4} = 0.01 \rightarrow \text{NBFM} \tag{278}$$

$$B \approx 2f_m = 40 \text{ KHz} \tag{279}$$

Problem 6.2.3

The upper sideband of an AM waveform (DSB-LC) with sinusoidal modulation and modulation index m is multiplied by a factor α, where: $0 \leq \alpha \leq 1$. Derive a relation for the peak (i.e., maximum) phase deviation from the carrier as a function of m and α.

Solution:

$$\begin{aligned}\phi(t) &= A\cos\omega_c t + \frac{mA}{2}\cos(\omega_c - \omega_m)t + \frac{\alpha mA}{2}\cos(\omega_c + \omega_m)t \\ &= A\cos\omega_c t + \frac{mA}{2}\cos\omega_c t\cos\omega_m t + \frac{mA}{2}\sin\omega_c t\sin\omega_m t + \frac{\alpha mA}{2}\cos\omega_c t\cos\omega_m t - \frac{\alpha mA}{2}\sin\omega_c t\sin\omega_m t \\ &= A\left[1 + \frac{1+\alpha}{2}m\cos\omega_m t\right]\cos\omega_c t + A\left[\frac{1-\alpha}{2}m\sin\omega_m t\right]\sin\omega_c t \qquad (280)\end{aligned}$$

The phase angle from the carrier is, assuming $A\cos(\omega_c t + \gamma(t))$],

$$\gamma(t) = -\tan^{-1}\left[\frac{\frac{1-\alpha}{2}m\sin\omega_m t}{1+\frac{1+\alpha}{2}m\cos\omega_m t}\right] \qquad (281)$$

For $\omega_m t = \pi/2$, the denominator of $\gamma(t)$ is minimum and the numerator is maximum. Therefore γ_{max} occurs for $\omega_m t = \pi/2$

$$\gamma_{max} = -\tan^{-1}\left[m\left(\frac{1-\alpha}{2}\right)\right] \qquad (282)$$

Problem 6.3.5

A 1-KHz sinusoid is used to frequency modulate a 50-KHz carrier signal; the peak frequency deviation from carrier is 200 Hz. This FM signal is applied to a non-linear system whose input-output transfer characteristic is $e_o(t) = 2e_i^2(t)$. The output is filtered with an ideal bandpass filter (BPF) having a bandwidth of 20 KHz, centered at 100 KHz.

a) Sketch to scale the resulting line spectrum. b) Using a table of Bessel functions, estimate the modulation index β of the output signal.

Solution:

a) Easy.

b) $\beta = 200/1000 = 0.2 \Longrightarrow$ NBFM. Then, we have,

$$e_i(t) = J_0 \cos\omega_c t + J_1 \cos(\omega_c + \omega_m)t - J_1 \cos(\omega_c - \omega_m)t \tag{283}$$

We make use of the identity, $2\cos A \cos B = \cos(A+B) + \cos(A-B)$.

$$\begin{aligned} e_o(t) = 2e_i^2(t) = (J_0^2 - 2J_1^2)\cos 2\omega_c t + 2J_0J_1\cos(2\omega_c + \omega_m)t + J_1^2\cos(2\omega_c + 2\omega_m)t \\ -2J_0J_1\cos(2\omega_c - \omega_m)t + J_1^2\cos(2\omega_c - 2\omega_m)t \\ +(J_0^2 + 2J_1^2) - 2J_1^2\cos 2\omega_m t \end{aligned} \tag{284}$$

The last line does not pass through the LP-filter.

Now we compare these coefficients with the corresponding coefficients of an FM signal. We have

$$J_0^2 - 2J_1^2 = (0.99)^2 - 2(0.10)^2 = 0.960 \tag{285}$$

$$2J_0J_1 = 2(0.99)(0.10) = 0.198 \tag{286}$$

$$J_1^2 = (0.10)^2 = 0.01 \tag{287}$$

From tabulated values, the closest match is $\beta \approx 0.4$, which is the predicted result.

Problem 6.3.6

A carrier waveform is frequency-modulated by the sum of two sinusoids:

$$\phi(t) = 100\cos(\omega_c t + \sin\omega_m t + 2\sin 2\omega_m t), \tag{288}$$

where $f_c = 100$ KHz and $f_m = 1$ KHz.

a) What is the peak frequency deviation from the carrier?

b) Estimate the net bandwidth required for transmission of this FM signal.

c) Sketch to scale the resulting magnitude line spectrum (one-sided above carrier).

[*Hint:* Express $\phi(t)$ in complex-valued notation and use Bessel functions for series representations to recognize the required coefficients.]

Solution:

a)

$$\Delta f = \text{Max}\left(\frac{1}{2\pi}\frac{d\theta}{dt} - f_c\right) \tag{289}$$

$$= \text{Max}\left(f_m \cos\omega_m t + 4f_m \cos 2\omega_m t\right) \tag{290}$$

The maximum value occurs at $\omega_m t = 0$, so that

$$\Delta f = 5f_m = 5 \text{ KHz} \tag{291}$$

b)

$$\begin{aligned}
\phi(t) &= \Re\{Ae^{j\omega_c t}e^{j\beta_1 \sin\omega_1 t}e^{j\beta_2 \sin\omega_2 t}\} && (292)\\
&= \Re\left\{Ae^{j\omega_c t}\sum_{m=-\infty}^{\infty} J_m(\beta_1)e^{jm\omega_1 t}\sum_{n=-\infty}^{\infty} J_n(\beta_2)e^{jn\omega_2 t}\right\} && (293)\\
&= \Re\left\{Ae^{j\omega_c t}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} J_m(\beta_1)J_n(\beta_2)e^{j(m\omega_1+n\omega_2)t}\right\} && (294)\\
& && (295)
\end{aligned}$$

where $\beta_1 = 1$, $\beta_2 = 2$, $\omega_1 = \omega_m$, $\omega_2 = 2\omega_1 = 2\omega_m$.

To determine the bandwidth, using a 1% criterion, we find values for which $|J_m(\beta_1)J_n(\beta_2)| \geq 0.01$ and then determine the resulting bandwidth. Consider a matrix where m is the index of the row and n is the index of the column. We put $J_m(1) \times J_n(2)$ in location (m, n) of the matrix. The frequency corresponding to the (m, n) element is equal to $2n + m$ KHz.

$(.77)(.22) = .1694$	$(.77)(.58) = .4466$	$(.77)(.35) = .2695$	$(.77)(.13) = .1001$	$(.77)(.03) = .0231$
$(.44)(.22) = .0968$	$(.44)(.58) = .2552$	$(.44)(.35) = .1540$	$(.44)(.13) = .0572$	$*(.44)(.03) = .0132*$
$(.11)(.22) = .0242$	$(.11)(.58) = .0638$	$(.11)(.35) = .0385$	$(.11)(.13) = .0143$	$(.11)(.03) = .0033$
$(.02)(.22) = .0044$	$(.02)(.58) = .0116$	$(.02)(.35) = .0070$	$(.02)(.13) = .0026$	$(.02)(.03) = .0006$

(296)

The bandwidth is determined by the element with the $*$ sign because it has the maximum frequency. It corresponds to $m = 1$ and $n = 4$, resulting in a total bandwidth of: $2[4(2) + 1] = 18$ KHz.

Problem 6.3.9

The analytical method used to find the spectral density of FM with sinusoidal modulation can be used for more general periodic modulating signals with zero mean. Consider the case in which modulating signal is a periodic square wave of unit amplitude and period T. (For convenience, assume that the square wave has even symmetry about the origin). Let the peak frequency deviation from carrier be $\Delta\omega$, and define a modulation index $\beta = \Delta\omega/\omega_0$ where $\omega_0 = 2\pi/T$.

a) Sketch the instantaneous frequency and phase.

b) Derive an expression for the spectral density.

c) Sketch the magnitude spectrum for $\beta = 5$.

Solution:

a)

$$\theta(t) = \int_0^t \omega_i(t)dt = \omega_c t + \gamma(t) \tag{297}$$

where ω_i is the instantaneous frequency and

$$\gamma(t) = \begin{cases} \Delta\omega t & -T/4 < t < T/4 \\ \Delta\omega[(T/2) - t] & T/4 < t < 3T/4 \end{cases} \tag{298}$$

b) We can write

$$\phi(t) = \Re\{Ae^{j\theta(t)}\} = \Re\{Ae^{j\omega_c t}e^{j\gamma(t)}\} \tag{299}$$

The function $e^{j\gamma(t)}$ is periodic with period T and can be represented by a Fourier series with coefficients F_n

$$F_n = \frac{1}{T}\int_{-T/4}^{T/4} e^{j\Delta\omega t}e^{-j\omega_0 t}dt + \frac{1}{T}\int_{T/4}^{3T/4} e^{j\Delta\omega T/2}e^{-j\Delta\omega t}e^{-j\omega_0 t}dt \tag{300}$$

$$= \frac{1}{2}\{\text{Sa}[\pi(\beta - n)/2] + (-1)^n \text{Sa}[\pi(\beta + n)/2]\}, \quad \beta = \Delta\omega/\omega_0 \tag{301}$$

This result is real-valued, we can write it in terms of cosines

$$\phi_{FM}(t) = A\sum_{n=-\infty}^{\infty} F_n \cos(\omega_c + n\omega_0)t \tag{302}$$

The spectral density is

$$\Phi_{FM}(\omega) = \pi A \sum_{n=-\infty}^{\infty} F_n[\delta(\omega + \omega_c + n\omega_0) + \delta(\omega - \omega_c - \omega_0)] \tag{303}$$

c) Easy.

Problem 6.4.1

The sinusoidal signal $f(t) = a\cos 2\pi f_m t$ is applied to the input of a FM system. The corresponding modulated signal output (in volts) for $a = 1$ V, $f_m = 1$ KHz, is

$$\phi(t) = 100\cos(2\pi \times 10^7 t + 4\sin 2000\pi t) \tag{304}$$

across a 50-ohm resistive load.

a) What is the peak frequency deviation from the carrier?

b) What is the total average power developed by $\phi(t)$?

c) What percentage of the average power is at 10 MHz?

d) What is the approximate bandwidth, using Carson's rule?

e) Repeat (a)-(d) for the input parameters $a = 0.75$ V, $f_m = 2$ KHz; assume all other factors remain unchanged.

Solution:

a)

$$f_i = 10^7 + 4000\cos 2000\pi t \tag{305}$$

which leads to $f_c = 10$ MHz, $\Delta f = 4$ KHz.

b)

$$P_t = \frac{\overline{\phi^2(t)}}{R} = \frac{(100)^2}{2(50)} = 100 \text{ W} \tag{306}$$

c)

$$\beta = \frac{\Delta f}{f_m} = 4 \tag{307}$$

$$P_0 = P_t \frac{J_0^2(4)}{\sum_n J_n^2(4)} = 100(-0.40)^2 = 16 \text{ W} \tag{308}$$

Note that $J_0^2(4) = -0.4$ and $\sum_n J_n^2(4) = 1$.

d)

$$B \approx 2f_m + 2\Delta f = 2(1) + 2(4) = 10 \text{ KHz} \tag{309}$$

e)

$$\Delta f = 4(0.75/1) = 3 \text{ KHz} \Longrightarrow \beta = 3/2 \tag{310}$$

$$P_t = 100 \text{ W (unchanged)} \tag{311}$$

$$P_0 = P_t J_0^2(3/2) = 100(0.51)^2 = 26 \text{ W} \tag{312}$$

$$B \approx 2(2) + 2(3) = 10 \text{ KHz.} \tag{313}$$

Problem 6.4.2 A certain sinusoid at a frequency of f_m Hz is used as a modulating signal in both an AM (DSB-LC) and an FM system. When modulated, the peak frequency deviation of the FM system is set to three times the bandwidth of the AM system. The magnitudes of of those sidebands spaced at $\pm f_m$Hz from carrier in both systems are equal, and the total average powers are equal in both systems. Determine a.) The modulation index of the FM system, and b.) the modulation index of the AM system.

Solution: a.)

$$\Delta f = 3 \times (2f_m) \Rightarrow \beta = \frac{\Delta f}{f_m} = 6.$$

b.) Let A_1 and A_2 be the AM and FM carrier peak amplitudes, respectively.
We have total average power for FM = total average power for AM. Then,

$$A_2^2 = A_1^2(1 + m^2/2) \quad (314)$$

and also magnitudes of sidebands at $f_c \pm f_m$ are equal in both systems, i.e:

$$A_2|J_1(6)| = A_1 m/2 \quad (315)$$

From equations (314) and (315) above it can be seen that the modulation index m is given by:

$$m = \frac{2|J_1(6)|}{\sqrt{1 - 2J_1^2(6)}} \approx 0.61$$

Problem 6.4.3 The output of a given FM modulator with a sinusoidal input is: $\Phi(t) = A\cos(\omega_c t + \beta \sin \omega_m t)$. This output $\Phi(t)$ is applied to a synchronous detector and an RC lowpass filter, with $RC^{-1} = \omega_m$. Develop an expression for the average power at the filter output (for an arbitrary β) if it is given that the output is one watt when $\beta = 0$.

Solution: $\Phi(t)$ can be written as :

$$\Phi(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t.$$

After the synchronous demodulator, assuming that the frequency components around ω_c are filtered, the output is equal to:

$$e(t) = \frac{A}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \cos n\omega_m t$$

Using equation (6.43) of your book (Stremler) the power across a 1-ohm resistor is given by:

$$P = \overline{e^2(t)} = \frac{A^2}{8} \sum_{n=-\infty}^{\infty} J_n^2(\beta).$$

The magnitude squared of the RC low-pass filter transfer function is given by:

$$|H(\omega)|^2 = \frac{1}{1 + (\omega/RC)^2} = \frac{\omega_m^2}{\omega^2 + \omega_m^2}$$

The average output power of a linear system is given by equation (2.78) of your text (Stremler) as:

$$P_o = \sum_{n=-\infty}^{\infty} |H(n\omega_o)|^2 |F_n|^2.$$

In this case $\omega_o = \omega_m$. Using this relationship we get :

$$P_o = \frac{A^2}{8} \sum_{n=-\infty}^{\infty} J_n^2(\beta)|H(n\omega_m)|^2 = \frac{A^2}{8} \sum_{n=-\infty}^{\infty} \frac{J_n^2(\beta)}{n^2+1} \tag{316}$$

The case of $\beta = 0$ corresponds to having only the carrier. For $\beta = 0$, the total power is 1W. Or:

$$P_o = \frac{A^2}{8} J_0^2(0) = A^2/8 = 1W \tag{317}$$

replacing in equation (316), $A^2/8$ by 1W we get :

$$P_o = \sum_{n=-\infty}^{\infty} \frac{J_n^2(\beta)}{1+n^2} = J_0^2(\beta) + 2\sum_{n=1}^{\infty} \frac{J_n^2(\beta)}{1+n^2}$$

Problem 6.5.3 A carrier waveform is phase modulated by a sinusoidal signal $f(t)$. The peak phase deviation is 1 rad when the peak input amplitude is 0.5V. Find the bandwidth using Carson's rule, and the ratio of the average power in the carrier and the first-order sidebands for each of the following inputs.

a.) $f(t) = \cos 1000\pi t$

b.) $f(t) = 1.2\cos 300\pi t$

c.) $f(t) = 1.9\cos 200\pi t$

Solution: For Phase modulation (PM) the modulation index β is given by:

$$\beta = \frac{\Delta\omega}{\omega_m} = ak_p$$

In this case $k_p = \Delta\theta/a = 1$ rad/(0.5 volts) = 2 rad/V The bandwidth B is obtained using Carson's rule as:

$$B = 2f_m(1+\beta)$$

Let parameter Λ denote the desired ratio. We have,

$$\Lambda = \frac{J_0^2(\beta) + 2J_1^2(\beta)}{1 - J_0^2(\beta) - 2J_1^2(\beta)}$$

a.) $f_m = 500$Hz; $\beta = (1)(2) = 2$;

$B = 2 \times (500)(1+2)$Hz = 3KHz

$\Lambda = 2.59$. From tables of the Bessel functions

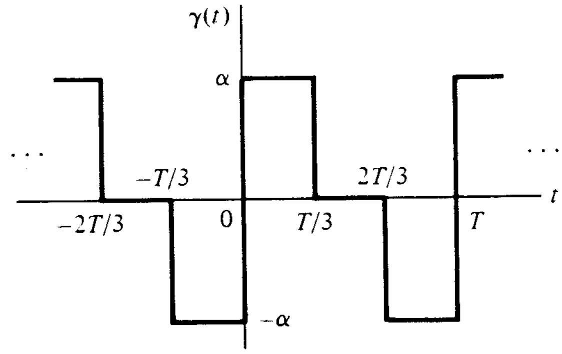
b.) $f_m = 150$Hz; $\beta = (1.2)(2) = 2.4$;

$B = 2 \times (150)(1+2.4)$Hz = 1.02KHz;

$\Lambda = 1.18$.

c.) $f_m = 100\text{Hz}$; $\beta = (1.9)(2) = 3.8$;
$B = 2 \times (100)(1 + 3.8)\text{Hz} = 0.96\text{KHz}$;
$\Lambda = 0.19$.

Problem 6.5.4 The testing procedure developed for a certain PM system designed to transmit remote sensing information to an orbiting satellite uses the staircase type periodic modulation signal $\gamma(t)$ shown in Fig. P-6.5.4
a.) Derive an expression for the spectral density, referred to the carrier frequency ω_c, for this modulation as a function of the peak phase deviation α. (The comments in problem 6.3.9 may be helpful here also.)
b.) Sketch the magnitude spectrum for $\alpha = \pi/3$ (60^o).



Solution: a.) We can write the PM signal as:

$$\Phi_{PM}(t) = \Re\{Ae^{j\gamma(t)}e^{j\omega_c t}\}$$

The function $\gamma(t)$ is periodic with period T and can be represented by the Fourier series with coefficients F_n given by:

$$\begin{aligned} F_n &= \frac{1}{T}\int_0^{T/3} e^{j\alpha}e^{-jn\omega_0 t}dt + \frac{1}{T}\int_{T/3}^{2T/3} e^0 e^{-jn\omega_0 t}dt + \frac{1}{T}\int_{2T/3}^{T} e^{-j\alpha}e^{-jn\omega_0 t}dt \\ &= \frac{(-1)^n}{3}\text{Sa}(n\pi/3)[1 + 2\cos(\alpha + 2n\pi/3)] \end{aligned}$$

We can also write $\Phi_{PM}(t)$ as :

$$\Phi_{PM}(t) = A\sum_{n=-\infty}^{\infty} F_n \cos(\omega_c + n\omega_0)t$$

where $\omega_0 = 2\pi/T$.
From this expression the spectral density can be obtained as:

$$\Phi_{PM}(\omega) = \pi A\sum_{n=-\infty}^{\infty} F_n[\delta(\omega + \omega_c + n\omega_0) + \delta(\omega - \omega_c - n\omega_0)]$$

b.) Easy.

Problem 6.8.3

Suppose that output of the FM discriminator in Fig 6.25 were applied to a lowpass filter with frequency transfer function $H(\omega)$, instead of the ideal LPF assumed in equation (6.110). Assuming sinusoidal modulation at $\omega = \omega_m$, show that Eq. (6.115) can be used except that N_c must be replaced by N_{eq}:

$$N_{eq} = 3\eta \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 d\omega}{\omega_m^2 |H(\omega_m)|^2}$$

Solution: For a filter with a transfer function $H(\omega)$, equations (6.109) and (6.110) of the text can be respectively written as :

$$\begin{aligned} S_{n_o}(\omega) &= \frac{\eta\omega^2}{A^2}|H(\omega)|^2 : \\ N_o &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (\eta/A^2)\omega^2|H(\omega)|^2 d\omega \end{aligned}$$

The output signal $s_o(t)$ [as in equation (6.100)] for sinusoidal modulation is given by:

$$s_o(t) = |H(\omega_m)| a k_f \cos(\omega_m t + \theta)$$

where $\theta = \angle H(\omega_m)$. Noting that $\Delta\omega = ak_f$, the output signal power is then :

$$S_o = |H(\omega_m)|^2 (\Delta\omega)^2/2$$

Then the output S/N ratio is then given by:

$$\begin{aligned} \frac{S_o}{N_o} &= \frac{|H(\omega_m)|^2(\Delta\omega)^2/2}{(\eta/A^2)\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2|H(\omega)|^2 d\omega} \\ &= 3\frac{(\Delta\omega)^2}{\omega_m^2}\frac{A^2/2}{N_{eq}} \\ &= 3\beta^2\frac{S_c}{N_{eq}} \end{aligned}$$

where:

$$N_{eq} = 3\eta \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 d\omega}{\omega_m^2 |H(\omega_m)|^2}$$

Problem 6.8.4

A communication system operates in the presence of white noise with a two sided power spectral density $S_a(\omega) = 0.25 \times 10^{-14}$ W/Hz and with total path losses (including antennas) of 100dB. The input bandwidth is 10-KHz. Calculate the minimum required carrier power of the transmitter for a 10-KHz sinusoidal input of and a 40dB output S/N ratio if the modulation is:

a.) AM (DSB-LC), with m=0.707 and m=1.0

b.) FM, with $\Delta f = 10$KHz and $\Delta f = 30$KHz.

c.) PM, with $\Delta\theta = 1$ rad and $\Delta\theta = 3$rad.

d.) Summarize your findings by marking these points on a graph of power required (in KW) versus bandwidth (in KHz).

Solution: a.)

$N_c = 2(0.25 \times 10^{-14})\text{W/Hz}(2 \times 10^4)\text{Hz} = 10^{-10}\text{W}$

Using equation (5.90) from your book (Stremler),

$$S_c = \left[\left(\frac{S_o}{N_o}\right)/m^2\right] N_c = 10^{-6}/m^2$$

The required transmitter power is then obtained (after taking into account the 100dB losses as):

$$P_t = 10^{10}(10^{-6}/m^2) = 10^4/m^2 = \begin{cases} 20\text{KW} & \text{m} = 0.707 \\ 10\text{KW} & \text{m} = 1 \end{cases}$$

b.) For $\Delta_f = 10$KHz, $\beta = \Delta f / f_m = 1$; $N_c = 10^{-10}$W from part (a).

Using equation (6.115) from your book (note that $S_o/N_o = 10^4$), we obtain,

$$S_c = \frac{N_c S_o}{3\beta^2 N_o} = \frac{10^{-10}}{3(1)^2}(10^4) = 3.33 \times 10^{-7}$$

From which we obtain the transmitter power as $P_t = 10^{10} S_c = 3.33$KW

For $\Delta_f = 30$KHz: $\beta = 3$ and P_t =370W

c.) Using the results of Example 6.8.1,

$$\left(\frac{S_o}{N_o}\right)_{PM} = (\Delta\theta)^2 \left(\frac{S_o}{N_o}\right)_{AM}$$

$$P_t = (10^{10})\frac{N_c S_o}{(\Delta\theta)^2 N_o} = 10\text{KW}, \quad (\Delta\theta = 1)$$

for $\Delta\theta = 3$, $P_t = 10 \times 10^3/(3)^2 = 1.11$KW.

d.) Easy.

Problem 6.8.6 A frequency -division multiplexing system uses SSB-SC subcarrier modulation and FM main carrier modulation. There are 20 equal-amplitude voice-input channels, each bandlimited to 3.3KHz. A 0.7-KHz guard band is allowed between channels and below the first channel.

a.) Determine the final transmission bandwidth if the peak frequency deviation is 400KHz.

b.) Compute the degradation in signal-to-noise of input No.20 when compared with input No.1. (Assume a white input noise spectral density to the discriminator and no De-emphasis.)

c.) Repeat part (b) if PM were used.

Solution: a.) Each channel requires (0.7 +3.3) = 4.0 KHz

$f_{max} = (20)(4.0) = 80\text{KHz};\ \Delta_f = 400\text{KHz};\ \beta_{\text{eff}} = 5.$

The final bandwidth is obtained using Carson's rule as:

$$B = 2(80\text{KHz})(1+5) = 960\text{KHz}.$$

b.)

Channel 1: 0.7 -4.0KHz

Channel 20 : 76.7-80KHz.

The FM discriminator noise output varies parabolically with frequency:

$$\frac{P_{20}}{P_1} = \left[\int_{76.7}^{80} f^2 df\right]\left[\int_{0.7}^{4} f^2 df\right]^{-1} = \frac{(80)^3 - (76.7)^3}{(4)^3 - (0.7)^3} = 954.8 \approx 29.8\text{dB}$$

c.) For PM, white input spectral noise density gives white output noise. (see example 6.8.1 of your book). Therefore $P_{20}/P_1 = 1$ and all channels are treated equally in terms of the noise.

Problem 6.10.1 A 10-KHz sinusoidal signal is to be transmitted using FM in the presence of additive white Gaussian Noise. If the S/N improvement at the demodulator output is required to to be 20dB, determine the required peak frequency deviation if a.) no pre-emphasis/de-emphasis is used; and b.) the standard pre-emphasis/de-emphasis is used (cf Fig.6.33) c.) Make a plot of the required peak frequency deviation versus f_m for the conditions in part (b) over the frequency range $5\text{KHz} \leq f_m \leq 15\text{KHz}$.

Solution a.) The input and output signal to noise ratio of an an FM discriminator are related by the equation:

$$\frac{S_o}{N_o} = 3\beta^2 \frac{S_c}{N_c}$$

in this case this relation translates into:

$3\beta^2 = 100$, or $\beta = 5.77$; $\Delta f = \beta f_m = (5.77)(10^4) = 57.7\text{KHz}$

b.) With standard preemphasis/deemphasis the signal to noise improvement is governed by the relationship:

$$\frac{S_o}{N_o} = 3\beta^2\Gamma\frac{S_c}{N_c}; \quad \text{or} \quad \beta^2\Gamma = 33.333$$

Where Γ is the S/N improvement due to preemphasis/deemphasis. Assuming $f_1 = 2.1$, in equation (6.124) of the book (Stremler), we obtain, $\Gamma(f_m = 10, f_1 = 2.1) = 10.59$, so that $\beta^2 = 33.333/10.59 = 3.147$ and finally $\Delta f = \beta f_m = 17.74$KHz

c.) Combining the steps in b.) into a more general solution, we get:

$$\Delta f = f_m\sqrt{\frac{100/3}{\Gamma(f_m)}}$$

where Γ is given in equation (6.124) of the text. A graph of this result can be easily drawn. Note that in this case the Δf is controlled by the highest modulating frequency (assuming equal weighting across the input bandwidth), in contrast to the conditions of (a) in which Δf is constant.