Problem 3.1.1

a) Determine the function $f(t)$ whose Fourier transform is shown in figure P-3.1.a.

b) Determine the function $f(t)$ whose Fourier transform is shown in figure P-3.1.b.

c) Sketch $f(t)$ and $g(t)$ near $t = 0$. What effects does the phase has on the symmetry of this waveform?

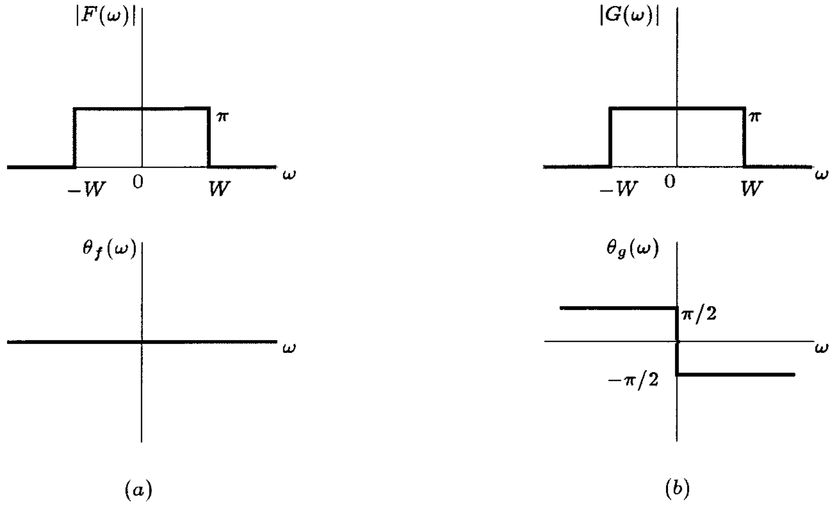

Figure 9: P-3.1.1

Solution: The function $f(t)$ can be obtained from $F(\omega)$ by doing an inverse Fourier transform, i.e.,

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}dt \tag{75}$$

a)

$$f(t) = \frac{1}{2\pi}\int_{-W}^{W} \pi e^{j\omega t}d\omega = W\frac{\sin Wt}{(Wt)}. \tag{76}$$

b)

$$f(t) = \frac{1}{2\pi}\int_{-W}^{0} \pi e^{j\pi/2}e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{W} \pi e^{-j\pi/2}e^{j\omega t}d\omega = W\frac{1-\cos Wt}{(Wt)} \tag{77}$$

c) The sketches for the time functions are given in the figures 10, 11. In the first case, the Fourier transform is real and even resulting in a real, even time function. In the second case, the Fourier transform is pure imaginary and odd resulting in a real, odd time function.

Problem 3.1.2 Show that if $F(\omega) = \mathcal{F}\{f(t)\}$, then

a)

$$F(0) = \int_{-\infty}^{\infty} f(t)dt; \tag{78}$$

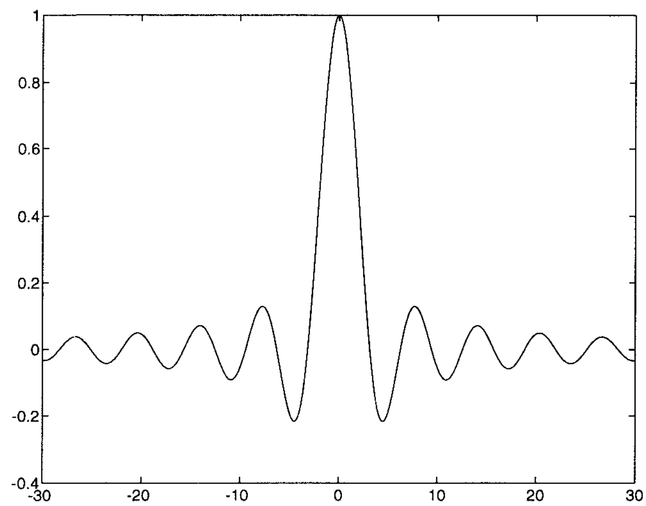

Figure 10: The function $f(t) = \sin(Wt)/t$ for $W = 1$, even symmetry.

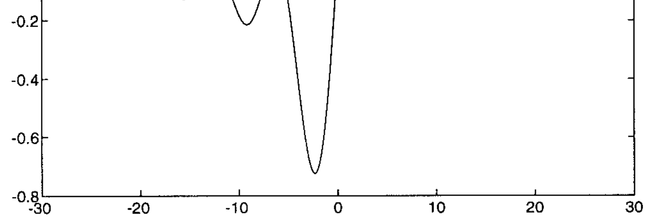

Figure 11: The function $f(t) = [1 - \cos(Wt)]/t$ for $W = 1$, odd symmetry.

b)

$$|F(\omega)| \leq \int_{-\infty}^{\infty} |f(t)|dt; \tag{79}$$

c)

$$\left|\frac{d^2 f}{dt^2}\right| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |F(\omega)|d\omega; \tag{80}$$

d)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega\tau}d\tau d\omega dt = 2\pi F(0). \tag{81}$$

Solution:

a)

$$F(0) = \left.\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right|_{\omega=0} = \int_{-\infty}^{\infty} f(t)e^{-j0}dt = \int_{-\infty}^{\infty} f(t)dt; \tag{82}$$

b) Using the triangle inequality, $|A+B| \leq |A| + |B|$, and considering integration as a limit of a summation, it can be shown that:

$$\left|\int_{\infty}^{\infty} f(t)e^{-j\omega t}dt\right| \leq \int_{-\infty}^{\infty} |f(t)| \left|e^{-j\omega t}\right| dt = \int_{-\infty}^{\infty} |f(t)|\, dt. \tag{83}$$

Note that $|e^{-j\omega t}| = 1$.

c)

$$\left|\frac{d^2 f}{dt^2}\right| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 F(\omega)e^{j\omega t}d\omega\right| \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2\, |F(\omega)|\, d\omega. \tag{84}$$

Note that the same inequality used in part b) was used.

d) Note that,

$$\int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega\tau}d\tau = e^{j\omega t}F(\omega) \tag{85}$$

then, we have,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega\tau}d\tau d\omega dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega dt = 2\pi\int_{-\infty}^{\infty} f(t)dt = 2\pi F(0) \tag{86}$$

Problem 3.2.1 a) Find the Fourier transform for of the raised cosine pulse signal defined by:

$$f(t) = \begin{cases} 1+\cos\pi t & \text{if} \quad -1 < t < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{87}$$

Express your answer in terms of $\mathrm{Sa}(\omega)$.

b) Use equation 3.15 (equation 3.15 is as follows: $F_n = F(n\omega_0)/T$) to find the exponential Fourier series coefficients for the following periodic pulse train for the case $T = 2$.

$$\sum_{k=-\infty}^{\infty} f(t-kT) \tag{88}$$

c) Sketch the time waveform of the periodic pulse train in part b) and then find the exponential Fourier series coefficients directly using Euler's identities.

Solution:

a)

$$\begin{aligned} F(\omega) &= \int_{-1}^{1} e^{-j\omega t} dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\omega-\pi)t} dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\omega+\pi)t} dt \\ &= 2\text{Sa}(\omega) + \text{Sa}(\omega-\pi) + \text{Sa}(\omega+\pi) \end{aligned}$$

Note that,

$$\frac{1}{2}\int_{-1}^{1} e^{-j\omega t} dt = \text{Sa}(\omega) \tag{89}$$

b) Recall that $F_n = (1/T)F(\omega)\,|_{\omega=n\omega_0}$. In this case, $\omega_0 = 2\pi/T = \pi$, therefore, from part a) above we get,

$$F_n = \frac{1}{2}\left\{2\text{Sa}[n\pi] + \text{Sa}[(n-1)\pi] + \text{Sa}[(n+1)\pi]\right\} \tag{90}$$

Since $\text{Sa}(k\pi) = 0$ for $k \neq 0$ and $\text{Sa}(k\pi) = 1$ for $k = 0$, it follows that:

$$F_1 = F_{-1} = \frac{1}{2}, \tag{91}$$

$$F_0 = 1, \tag{92}$$

and,

$$F_n = 0 \quad \text{for} \quad |n| > 1. \tag{93}$$

c) Using Euler's identity, we get directly an expansion of our function in exponential form as

$$f(t) = 1 + \cos(\pi t) = 1 + \frac{1}{2}(e^{j\pi t} + e^{-j\pi t}) \tag{94}$$

from which it is evident that $F_0 = 1$, $F_{-1} = F_1 = 1/2$ and $F_n = 0$ for $|n| > 1$.

Problem 3.2.2 a) Find the spectral density of the real-valued function:

$$f(t) = \begin{cases} a\exp(-at) & t > 0 \\ b\exp(at) & t < 0 \end{cases} \tag{95}$$

b) Examine your answer to part a) for the special cases $b = a$ and $b = -a$, particularly with respect to the following chart:

$f(t)$	$F(\omega)$	$F(\omega)$
even	even	real
odd	odd	imaginary

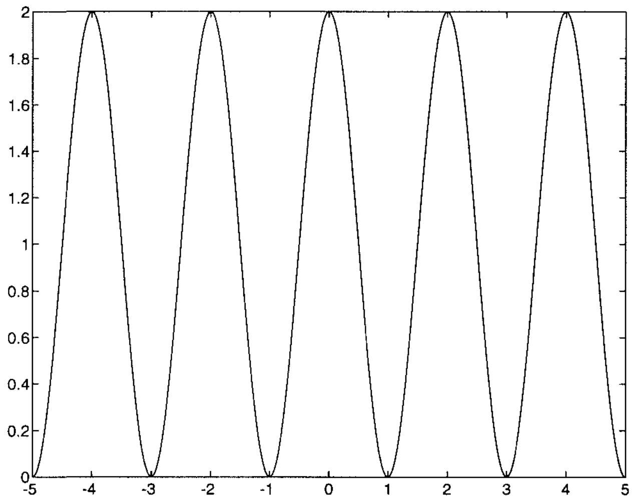

Figure 12: Raised cosine pulse.

Solution: a) The Fourier Transform can be written as:

$$F(\omega) = \int_{-\infty}^{0} be^{(a-j\omega)t}dt + \int_{0}^{\infty} ae^{-(a+j\omega)t}dt = \frac{a(b+a)+j\omega(b-a)}{a^2+\omega^2} \tag{96}$$

b) From the above expression we obtain the following cases:

$$F(\omega) = \begin{cases} \dfrac{2a^2}{a^2+\omega^2}, & \text{if } a=b \\ \\ \dfrac{-j2\omega a}{a^2+\omega^2}, & \text{if } a=-b \end{cases} \tag{97}$$

These conclusions are consistent with the chart given above, with $b=a$ corresponding to the first row and $b=-a$ to the second row.

Problem 3.2.3 A pulse signal described by $f(t) = \exp(-a|t|)\text{rect}(t/T)$ is repeated periodically with period T.

a) Find the exponential Fourier Series beginning with the Fourier transform of $\exp(-a|t|)$ and the converting to the series.

b) Compare your answer with problem 2.7.1.

c) Under what conditions can equation 3.15 (equation 3.15 is as follows: $F_n = \frac{1}{T}F(n\omega_0)$) be used to obtain the Fourier series coefficients ?

Solution: a) For one pulse ,

$$F(\omega) = \frac{2a}{\omega^2+a^2} \quad \text{(from table 3.1)} \tag{98}$$

Using equation 3.15:

$$\frac{1}{T}F(\omega)\mid_{\omega=n\omega_0} = \frac{2a/T}{(4\pi^2n^2/T^2)+a^2} \tag{99}$$

b) For $T=2$, the result in a) gives:

$$\frac{1}{T}F(\omega)\mid_{\omega=n\omega_0} = \frac{a}{a^2+\pi^2n^2}. \tag{100}$$

But, problem 2.7.1 gives:

$$F_n = \frac{a(1-e^{-a}\cos\pi n)}{a^2+n^2\pi^2}. \tag{101}$$

Relationship (100) is indeed the general term of the Fourier series representation of the periodic function obtained by repeating the function $\exp(-a|t|)$ while (101) is the general term of the Fourier series representation of the periodic function obtained by repeating the function $f(t) = \exp(-a|t|)\text{rect}(t/T)$.

c) For a give function $f(t)$, the formula $F_n = \frac{1}{T}F(n\omega_0)$ gives the Fourier coefficients of the periodic signal $\sum_{k=-\infty}^{\infty} f(t-kT)$. These are equal to the Fourier coefficients of $f(t)$ as long as $f(t)$ is zero outside of the time interval T. In this case, the replica of the original signal, namely $f(t-kT)$, do not have overlap and the resulting periodic signal, namely $\sum_{k=-\infty}^{\infty} f(t-kT)$, is equal to the original signal $f(t)$ over the time interval T.

Problem 3.2.4 The time function $f(t) = (1/\sigma\sqrt{2\pi})e^{-t^2/2\sigma^2}$ ($\sigma=$ constant) is known as the Gaussian function. This function has finite energy and thus is Fourier transformable. Find its Fourier transform. In carrying out your solution it will be helpful to combine exponents, complete the square in the exponent, and then use the finite integral $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$. Note that $f(t)$ and $F(\omega)$ have the same mathematical form; i.e., the Gaussian function is its own Fourier transform.

Solution:

$$F(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}e^{-t^2/2\sigma^2}e^{-j\omega t}dt = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-[(t^2+j2\sigma^2\omega t-\sigma^4\omega^2+\sigma^4\omega^2)/2\sigma^2]}dt. \tag{102}$$

Let $u = (t+j\sigma^2\omega)/\sqrt{2}$; then,

$$F(\omega) = \frac{1}{\sqrt{\pi}}e^{-(\sigma^2\omega^2/2)}\int_{-\infty}^{\infty} e^{-u^2}du = e^{-\sigma^2\omega^2/2}. \tag{103}$$

Problem 3.4.2 Show that a more general statement of Parseval's theorem for for energy signals than Eq. (3.21) is

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega. \tag{104}$$

Solution:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(x)e^{-jxt}dx\right]dt. \tag{105}$$

Allowing an interchange in the order of integration, this becomes:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)G^*(x)\int_{-\infty}^{\infty} e^{-j(x-\omega)t}d\omega dx dt. \tag{106}$$

Using Eq. (3.25) of the main text, this becomes:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)G^*(x)\delta(\omega - x)dx d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega. \tag{107}$$

Problem 3.4.3 Evaluate the following definite integrals using Parseval's theorem:

a) $\int_{-\infty}^{\infty} [\sin x/x]^2 dx$,

b) $\int_{-\infty}^{\infty} dx/(a^2 + x^2)$,

c) $\int_{-\infty}^{\infty} dx/(a^2 + x^2)^2$.

Solution: We know that,

a)

$$\text{rect}(t/2) \Longleftrightarrow 2\text{Sa}(\omega) \tag{108}$$

then,

$$\int_{-1}^{1} 1dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^2[\sin \omega/\omega]^2 d\omega \tag{109}$$

resulting in,

$$\int_{-\infty}^{\infty} [\sin \omega/\omega]^2 d\omega = \pi \tag{110}$$

b)

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a + j\omega} \tag{111}$$

$$\int_{-\infty}^{\infty} \left|\frac{1}{j\omega + a}\right|^2 d\omega = 2\pi\int_{0}^{\infty} (e^{-at})^2 dt = \pi/a, \quad a > 0. \tag{112}$$

c)

$$e^{-a|t|} \Longleftrightarrow \frac{2a}{a^2 + \omega^2} \tag{113}$$

$$\frac{1}{4a^2}\int_{-\infty}^{\infty}\left(\frac{2a}{a^2+\omega^2}\right)^2 d\omega = \frac{2\pi}{4a^2}\int_{-\infty}^{\infty}(e^{-a|t|})^2 dt = \frac{4\pi}{4a^2}\int_{0}^{\infty} e^{-2at}dt = \frac{\pi}{2a^3}, \quad a > 0. \tag{114}$$

Problem 3.4.4 Use the result of problem 3.4.2 to evaluate the following integrals, for $a > 0, b > 0$:

a) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$,

b) $\int_{-\infty}^{\infty} \frac{\text{Sa}(bx)}{x^2+a^2} dx$.

Solution:

a)

$$e^{-a|t|} \Longleftrightarrow \frac{2a}{a^2+\omega^2} \tag{115}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\omega^2+a^2)(\omega^2+b^2)} = 2\pi \int_{-\infty}^{\infty} \frac{1}{2a} e^{-a|t|} \frac{1}{2b} e^{-b|t|} dt = \frac{\pi}{ab} \int_0^{\infty} e^{-(a+b)t} dt = \frac{\pi}{ab(a+b)}. \tag{116}$$

b)

$$\frac{1}{\tau} \text{rect}(t/\tau) \Longleftrightarrow \text{Sa}(\omega\tau/2) \tag{117}$$

$$\int_{-\infty}^{\infty} \frac{\text{Sa}(b\omega)}{\omega^2+a^2} d\omega = 2\pi \int_{-\infty}^{\infty} \frac{1}{2b} \text{rect}\left(\frac{t}{2b}\right) \frac{1}{2a} e^{-a|t|} dt = \frac{\pi}{ab} \int_0^b e^{-at} dt = \frac{\pi}{a^2 b}(1-e^{-ab}). \tag{118}$$

Problem 3.5.1 Use Eq. (3.25) and an interchange in the order of integration to show that:

a) $\mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} = f(t)$,

b) $\mathcal{F}\{\mathcal{F}\{f(t)\}\} = f(-t)$.

Solution:

a)

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau\right] e^{j\omega t} d\omega \tag{119}$$

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} e^{-j\omega(t-\tau)} d\omega d\tau = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau) d\tau = f(t). \tag{120}$$

b)

$$\mathcal{F}\{\mathcal{F}\{f(t)\}\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau\right] e^{-j\omega t} d\omega = \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} e^{-j\omega(t+\tau)} d\omega\right] d\tau, \tag{121}$$

$$\mathcal{F}\{\mathcal{F}\{f(t)\}\} = \int_{-\infty}^{\infty} f(\tau)\delta(t+\tau) d\tau = f(-t). \tag{122}$$

Problem 3.6.1

a) Find $F(\omega)$ for the $f(t)$ shown in Fig. P-3.6.1 (a), (b).

b) Sketch $|F(\omega)|$ for $\tau \le T$ for both cases, and compare.

Solution:

a)

$$F_1(\omega) = A\tau e^{-j\omega T} \text{Sa}(\omega\tau/2), \tag{123}$$

$$F_2(\omega) = A\tau e^{j\omega T} \text{Sa}(\omega\tau/2) + A\tau e^{-j\omega T} \text{Sa}(\omega\tau/2) = 2A\cos(\omega T)\,\text{Sa}(\omega\tau/2). \tag{124}$$

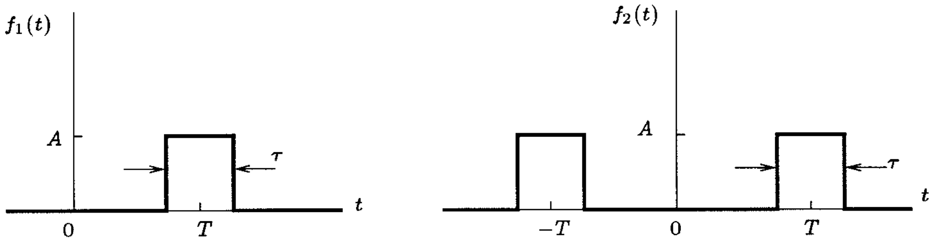

Figure 13: P-3.6.1

b) Graphs for $|F_1(\omega)|$ and $|F_2(\omega)|$ are shown if Figures (14) and (15), respectively.

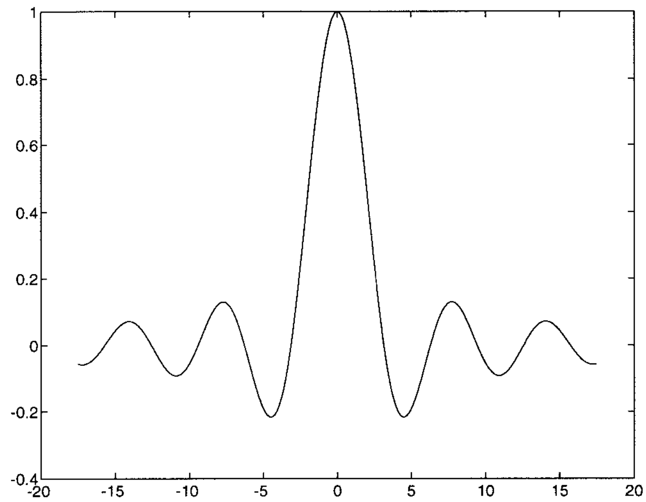

Figure 14: The function $A\text{Sa}(\omega\tau/2)$ for $A = 1$, $\tau = 1$.

Problem 3.6.2

Use the modulation property to find the function $f(t)$ whose Fourier transform is shown in Fig. P-3.6.2 for the conditions:

a) $B = A$;

b) $B = -A$.

Solution:

a) For $A = B$, we have,

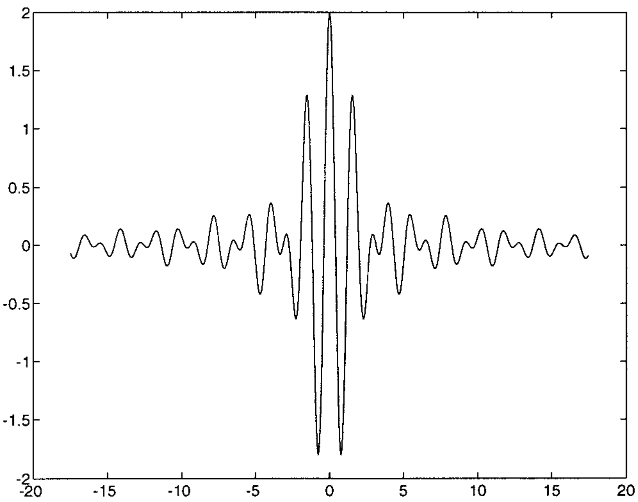

Figure 15: The function $2A\cos(\omega T)\mathrm{Sa}(\omega\tau/2)$ for $A=1$, $\tau=1$, $T=2$.

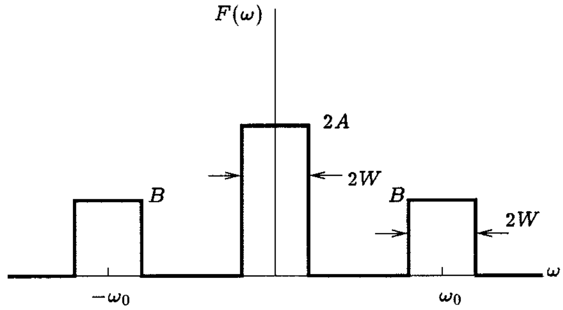

Figure 16: P-3.6.2

$$F(\omega) = 2A\,\mathrm{rect}[\omega/2W] + A\,\mathrm{rect}[(\omega-\omega_0)/2W] + A\,\mathrm{rect}[(\omega+\omega_0)/2W] \tag{125}$$

Using Table 3.1 and the modulation property of the Fourier transform,

$$f(t) = 2A\frac{W}{\pi}\mathrm{Sa}(Wt) + A\frac{W}{\pi}e^{j\omega_0 t}\mathrm{Sa}(Wt) + A\frac{W}{\pi}e^{-j\omega_0 t}\mathrm{Sa}(Wt) \tag{126}$$

$$= 2A\frac{W}{\pi}[1+\cos\,\omega_0 t]\mathrm{Sa}(Wt). \tag{127}$$

b) Similarly, for $A=-B$, we have,

$$F(\omega) = 2A\,\mathrm{rect}[\omega/2W] - A\,\mathrm{rect}[(\omega-\omega_0)/2W] - A\,\mathrm{rect}[(\omega+\omega_0)/2W] \tag{128}$$

and

$$f(t) = 2A\frac{W}{\pi}\mathrm{Sa}(Wt) - A\frac{W}{\pi}e^{j\omega_0 t}\mathrm{Sa}(Wt) - A\frac{W}{\pi}e^{-j\omega_0 t}\mathrm{Sa}(Wt) \tag{129}$$

$$= 2A\frac{W}{\pi}[1-\cos\,\omega_0 t]\mathrm{Sa}(Wt). \tag{130}$$

Problem 3.6.4

If $f(t) \leftrightarrow F(\omega)$, determine the Fourier transform of

a) $f(2-t)$,

b) $f[(t/2)-1]$,

c) $f(t)\cos\pi(t-1)$,

d) $\frac{d}{dt}[f(2t)]$.

Solution:

a) Assuming $\xi = 2-t$, we obtain,

$$\int_{-\infty}^{\infty} f(2-t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\xi)e^{-j\omega(2-\xi)}d\xi = e^{-j\omega 2}F(-\omega). \tag{131}$$

b) Assuming $\xi = (t/2)-1$, we obtain,

$$\int_{-\infty}^{\infty} f[(t/2)-1]e^{-j\omega t}dt = 2\int_{-\infty}^{\infty} f(\xi)e^{-j\omega(2\xi+2)}d\xi = 2e^{-j\omega 2}F(2\omega). \tag{132}$$

c) Substituting, $\cos(\pi t-\pi) = [e^{j(\pi t-\pi)} + e^{-j(\pi t-\pi)}]/2$, we obtain,

$$\frac{1}{2}e^{-j\pi}\int_{-\infty}^{\infty} f(t)e^{-j(\omega-\pi)t}dt + \frac{1}{2}e^{j\pi}\int_{-\infty}^{\infty} f(t)e^{-j(\omega+\pi)t}dt = -\frac{1}{2}[F(\omega-\pi)+F(\omega+\pi)]. \tag{133}$$

Note that $e^{j\pi} = e^{-j\pi} = -1$.

d) We have,

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} F(\xi)e^{j\xi t}d\xi \\ f(2t) &= \int_{-\infty}^{\infty} F(\xi)e^{j2\xi t}d\xi \\ \frac{d}{dt}[f(2t)] &= \int_{-\infty}^{\infty} (j2\xi)F(\xi)e^{j2\xi t}d\xi \end{aligned} \tag{134}$$

Assuming $\omega = 2\xi$, we obtain,

$$\frac{d}{dt}[f(2t)] = \int_{-\infty}^{\infty} j\omega F(\omega/2)e^{j\omega t}d\omega/2 \tag{135}$$

This results in,

$$\frac{d}{dt}[f(2t)] \Longleftrightarrow (j\omega/2)F(\omega/2) \tag{136}$$

Problem 3.6.5

Find the Fourier transform of the pulse waveform $f(t)$ shown in Fig. P-3.6.5 by differentiating to obtain impulse functions, then writing the transform using the delay and integration properties. [*Hint:* Consider use of superposition.]

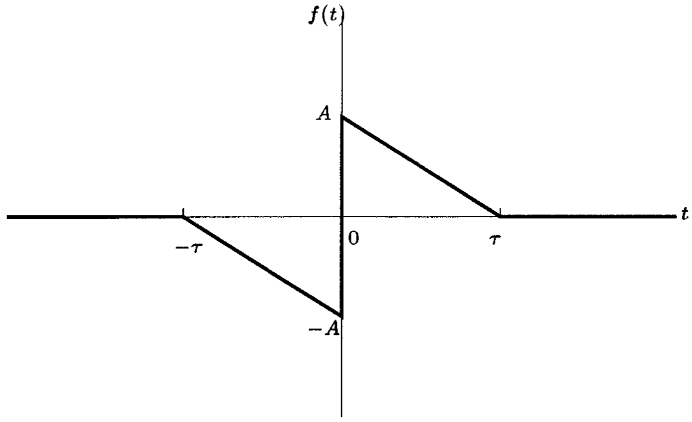

Figure 17: P-3.6.5

Solution:

Taking two derivatives of $f(t)$, we obtain $f''(t) = (-A/\tau)\delta(t+\tau) + (A/\tau)\delta(t-\tau) + 2A\delta'(t)$. Therefore, we can write,

$$f(t) = \int_{-\infty}^{t} \int_{-\infty}^{\xi} [(-A/\tau)\delta(\zeta+\tau) + (A/\tau)\delta(\zeta-\tau)]\, d\zeta\, d\xi + \int_{-\infty}^{t} 2A\delta(\xi)\, d\xi \qquad (137)$$

which has a Fourier transform

$$F(\omega) = \frac{(-A/\tau)e^{j\omega\tau} + (A/\tau)e^{-j\omega\tau}}{(j\omega)^2} + \frac{2A}{j\omega} = \frac{2A}{j\omega}[1 - \mathrm{Sa}(\omega\tau)]. \qquad (138)$$

Problem 3.6.8

Two functions of time, $f(t)$ and $g(t)$, are known to satisfy the following integral equation:

$$g(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau + \delta(t). \qquad (139)$$

a) If $f(t) = \exp(-at)u(t)$, find $g(t)$.

b) If $f(t) = \exp(-a|t|)$, find $g(t)$.

Solution: Computing the Fourier transform of the two sides of our main relationship, we obtain,

$$G(\omega) = G(\omega)F(\omega) + 1 \qquad (140)$$

or

$$G(\omega) = \frac{1}{[1 - F(\omega)]}. \qquad (141)$$

a) From Table 3.1, for $f(t) = \exp(-at)u(t)$, we have $F(\omega) = 1/(a + j\omega)$ and,

$$G(\omega) = \frac{j\omega + a}{j\omega + a - 1} = 1 + \frac{1}{j\omega + a - 1} \tag{142}$$

$$g(t) = \delta(t) + e^{-(a-1)t}u(t), \quad a > 1. \tag{143}$$

b) From Table 3.1, for $f(t) = \exp(-a|t|)$, we have,

$$F(\omega) = \frac{2a}{a^2 + \omega^2}, \tag{144}$$

$$G(\omega) = \frac{\omega^2 + a^2}{\omega^2 + a^2 - 2a} = 1 + \frac{2a}{\omega^2 + a^2 - 2a}, \tag{145}$$

$$g(t) = \delta(t) + \frac{1}{\sqrt{1 - 2/a}} e^{-\sqrt{a(a-2)}|t|}, \quad a > 2. \tag{146}$$

Problem 3.7.1

Consider the following (volume) integral:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)h(v)\delta[t - (u + v)]dudv. \tag{147}$$

a) Show that this integral describes the convolution integral, $f(t) * h(t)$.

b) Using this integral and the result of (a), show that $f(t) * h(t) = h(t) * f(t)$.

c) Use this integral to show that the area under the convolution result of two given functions is equal to the product of the areas under the two functions.

d) Repeat part (c) using the inverse Fourier transform of one function in $f(t) * h(t)$ and then interchanging the order of integration.

Solution:

a) Noting that the impulse function is at $v = t - u$, the integration over v yields:

$$\int_{-\infty}^{\infty} f(u)h(t - u)du = f(t) * h(t). \tag{148}$$

b) We could just as well have chosen the integration in (a) over $u = t - v$ to give:

$$\int_{-\infty}^{\infty} f(t - v)h(v)du = h(t) * f(t). \tag{149}$$

c)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)h(v)\delta[t-(u+v)]dudvdt \;=\; \tag{150}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)h(v)\int_{-\infty}^{\infty}\delta[t-(u+v)]dtdudv \;=\; \tag{151}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u)h(v)\,du\,dv \;=\; \tag{152}$$

$$\int_{-\infty}^{\infty} f(u)du\int_{-\infty}^{\infty} f(v)dv \tag{153}$$

d) We have,

$$\int_{-\infty}^{\infty} f(t)*h(t)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega\tau}d\omega h(t-\tau)\,d\tau\,dt \tag{154}$$

Assuming $\xi = t-\tau$, we obtain,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega\tau}d\omega h(t-\tau)\,d\tau\,dt \;=\; \tag{155}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} h(\xi)e^{-j\omega\xi}\,d\xi\right] e^{j\omega t}\,d\omega\,dt \;=\; \tag{156}$$

$$\int_{-\infty}^{\infty} F(\omega)H(\omega)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,dt\right]d\omega \;=\; \tag{157}$$

$$\int_{-\infty}^{\infty} F(\omega)H(\omega)\delta(\omega)\,d\omega = F(0)H(0) \tag{158}$$

Problem 3.7.3

Use frequency convolution to prove the following trigonometric identities (also see Fig. 3.3):

a) $2\cos^2\omega_0 t = 1+\cos\,2\omega_0 t$,

b) $2\sin^2\omega_0 t = 1-\cos\,2\omega_0 t$,

c) $2\cos\omega_1 t\,\cos\omega_2 t = \cos(\omega_1+\omega_2)t+\cos(\omega_1-\omega_2)t$.

Solution:

a) We know that $\cos(\omega_0 t) \Longleftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$, and, $f(t)g(t) \Longleftrightarrow [F(\omega)*G(\omega)]/2\pi$, then,

$$2\cos^2\omega_0 t \Longleftrightarrow 2\,\frac{1}{2\pi}\{\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]*\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\} \;=\; \tag{159}$$

$$\frac{2}{2\pi}\int_{-\infty}^{\infty}\pi[\delta(u-\omega_0)+\delta(u+\omega_0)]\pi[\delta(\omega-u-\omega_0)+\delta(\omega-u+\omega_0)]du \;=\; \tag{160}$$

$$\pi\delta(\omega-2\omega_0)+\pi\delta(\omega)+\pi\delta(\omega)+\pi\delta(\omega+2\omega_0) \Longleftrightarrow 1+\cos\,2\omega_0 t \tag{161}$$

b) We know that $\sin(\omega_0 t) \Longleftrightarrow -j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$, then,

$$2\sin^2\omega_0 t \Longleftrightarrow 2\,\frac{1}{2\pi}\{-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]*-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]\} \;=\; \tag{162}$$

$$\frac{2}{2\pi}\int_{-\infty}^{\infty} -\pi[\delta(u-\omega_0)-\delta(u+\omega_0)]\pi[\delta(\omega-u-\omega_0)-\delta(\omega-u+\omega_0)]du = \quad (163)$$

$$-\pi\delta(\omega-2\omega_0)+\pi\delta(\omega)+\pi\delta(\omega)-\pi\delta(\omega+2\omega_0) \Longleftrightarrow 1-\cos 2\omega_0 t \quad (164)$$

c)

$$2\cos\omega_1 t \cos\omega_2 t \Longleftrightarrow 2\frac{1}{2\pi}\{\pi[\delta(\omega-\omega_1)+\delta(\omega+\omega_1)] * \pi[\delta(\omega-\omega_2)+\delta(\omega+\omega_2)]\} = \quad (165)$$

$$\frac{2}{2\pi}\int_{-\infty}^{\infty} \pi[\delta(u-\omega_1)+\delta(u+\omega_1)]\pi[\delta(\omega-u-\omega_2)+\delta(\omega-u+\omega_2)]du = \quad (166)$$

$$\pi\delta(\omega-\omega_1-\omega_2)+\pi\delta(\omega+\omega_1-\omega_2)+]\pi\delta(\omega-\omega_1+\omega_2)+\pi\delta(\omega+\omega_1+\omega_2) \quad (167)$$

$$\Longleftrightarrow \cos(\omega_1+\omega_2)t+\cos(\omega_1-\omega_2)t. \quad (168)$$

Problem 3.8.1

Sketch the results of the following convolution operations (where $t_0 > 0$ in all cases). Check your result by writing a Fourier transform, of each function, multiplying and writing the corresponding time function.

a) $A\delta(t) * B\delta(t-t_0)$;
b) $A\delta(t+t_0) * B\delta(t-t_0)$;
c) $A\delta(t-t_1) * B\delta(t-t_0)$;
d) $A[\delta(t+t_0)+\delta(t-t_0] * B[\delta(t+t_0)+\delta(t-t_0)]$;
e) $A[\delta(t+t_1)+\delta(t-t_1] * B[\delta(t+t_0)+\delta(t-t_0)], \quad t_0 > t_1$;

Solution:

Recall that convolution of two functions $f(t)$ and $g(t)$ is given by:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau;$$

Using this, the convolution of our given functions becomes:

a)

$$A\delta(t) * B\delta(t-t_0) = \int_{-\infty}^{\infty} A\delta(\tau)B\delta(t-\tau-t_0)d\tau = AB\delta(t-t_0);$$

$$\mathcal{F}\{A\delta(t)\} = A; \quad \mathcal{F}\{B\delta(t-t_0) = Be^{-j\omega t_0};$$

The product is $ABe^{-j\omega t_0}$, and the inverse Fourier transform of the product is,

$$\mathcal{F}^{-1}\{ABe^{-j\omega t_0}\} = AB\delta(t-t_0).$$

which agrees with our equation obtained earlier.

b)

$$A\delta(t+t_0) * B\delta(t-t_0) = \int_{-\infty}^{\infty} A\delta(\tau+t_0)B\delta(t-\tau-t_0)d\tau = AB\delta(t);$$

$$\mathcal{F}\{A\delta(t+t_0)\} = Ae^{j\omega t_0}; \quad \mathcal{F}\{B\delta(t-t_0)\} = Be^{-j\omega t_0};$$

The inverse Fourier transform of their product is given by:

$$\mathcal{F}^{-1}\{Ae^{j\omega t_0}Be^{-j\omega t_0}\} = AB\delta(t)$$

Which agrees with our previous result.

c)

$$A\delta(t-t_1) * B\delta(t-t_0) = \int_{-\infty}^{\infty} A\delta(\tau - t_1)B\delta(t-\tau-t_0)d\tau = AB\delta(t-t_0-t_1);$$

$$\mathcal{F}\{A\delta(t-t_1)\} = Ae^{-j\omega t_1}; \quad \mathcal{F}\{B\delta(t-t_0)\} = Be^{-j\omega t_0};$$

The inverse Fourier transform of the product is given by:

$$\mathcal{F}^{-1}\{Ae^{-j\omega t_1}Be^{-j\omega t_0}\} = AB\delta(t-t_0-t_1)$$

Which agrees with our previous result.

d) The convolution expression given is equal to:

$$\int_{-\infty}^{\infty} A[\delta(\tau+t_0)+\delta(\tau-t_0)]B[\delta(t-\tau+t_0)+\delta(t-\tau-t_0)]d\tau = AB[\delta(t+2t_0)+2\delta(t)+\delta(t-2t_0)];$$

By evaluating the Fourier transform of the individual terms, multiplying and finding the inverse Fourier transform, as done in parts a)—c), it can be found that:

$$\mathcal{F}^{-1}\{A[e^{j\omega t_0}+e^{-j\omega t_0}]B[e^{j\omega t_0}+e^{-j\omega t_0}]\} = AB[\delta(t+2t_0)+2\delta(t)+\delta(t-2t_0)]$$

which agrees with our previous result.

e) The convolution of the given expression is equal to:

$$\int_{-\infty}^{\infty} A[\delta(\tau+t_1)+\delta(\tau-t_1)]B[\delta(t-\tau+t_0)+\delta(t-\tau-t_0)]d\tau =$$
$$AB[\delta(t+t_1+t_0)+\delta(t-t_1+t_0)+\delta(t+t_1-t_0)+\delta(t-t_1-t_0)];$$

Likewise ,

$$\mathcal{F}^{-1}\{A[e^{j\omega t_1}+e^{-j\omega t_1}]B[e^{j\omega t_0}+e^{-j\omega t_0}]\} = AB[\delta(t+t_1+t_0)+\delta(t+t_1-t_0)+\delta(t-t_1+t_0)+\delta(t-t_1-t_0)];$$

Which also agrees with the above result.

Problem 3.8.2 Evaluate the following convolution integrals; check your result by taking the Fourier transform of each function , multiplying, and finding the inverse Fourier transform.

a) $u(t) * e^{-t}u(t)$;

b) $e^{-at}u(t) * e^{-bt}u(t)$;

c) $e^{-a|t|} * \cos\omega_0 t$

Solution:

a)

$$u(t) * e^{-t}u(t) = \int_{-\infty}^{\infty} u(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = \int_0^t e^{-(t-\tau)}d\tau$$

The integral is zero if $t < 0$ and is equal to $(1 - e^{-t})$ if $t \geq 0$. Then, we have,

$$u(t) * e^{-t}u(t) = (1 - e^{-t})u(t).$$

The product of their Fourier transforms is given by:

$$\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]\left[\frac{1}{j\omega+1}\right] = \frac{1}{j\omega(j\omega+1)} + \frac{\pi\delta(\omega)}{j\omega+1} = \frac{1}{j\omega} - \frac{1}{j\omega+1} + \pi\delta(\omega)$$

The Inverse Fourier transform of the above expression is given by:

$$(1 - e^{-t})u(t)$$

b)

$$\begin{aligned} e^{-at}u(t) * e^{-bt}u(t) &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau \\ &= \int_0^t e^{-a\tau}e^{-b(t-\tau)}d\tau \\ &= \begin{cases} te^{-at}u(t) & a = b \\ \left[\left(e^{-bt} - e^{-at}\right)/(a-b)\right]u(t) & a \neq b \end{cases} \end{aligned}$$

c) Following the same procedure, it is seen that:

$$\begin{aligned} e^{-a|t|} * \cos\omega_0 t &= \int_{-\infty}^{\infty} e^{-a|t-\tau|}\cos(\omega_0\tau)d\tau \\ &= e^{-at}\int_{-\infty}^{t} e^{a\tau}\cos(\omega_0\tau)d\tau + e^{at}\int_t^{\infty} e^{-a\tau}\cos(\omega_0\tau)d\tau \\ &= \frac{2a}{a^2+\omega_0^2}\cos\omega_0 t. \end{aligned}$$

The Product of their Fourier transforms is given as:

$$\begin{aligned} &\left[\frac{2a}{a^2+\omega^2}\right][\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0] \\ =\ &\frac{2a}{a^2+\omega_0^2}[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)] \end{aligned}$$

The Inverse Fourier transform of this expression is given as.

$$\frac{2a}{a^2+\omega_0^2}\cos\omega_0 t.$$

This is also consistent with the earlier result.

Problem 3.8.3 Evaluate $f_1(t) * f_2(t)$, $f_1(t) * f_3(t)$, and $f_2(t) * f_3(t)$ for functions defined by waveforms shown in figure P- 3.8.3.

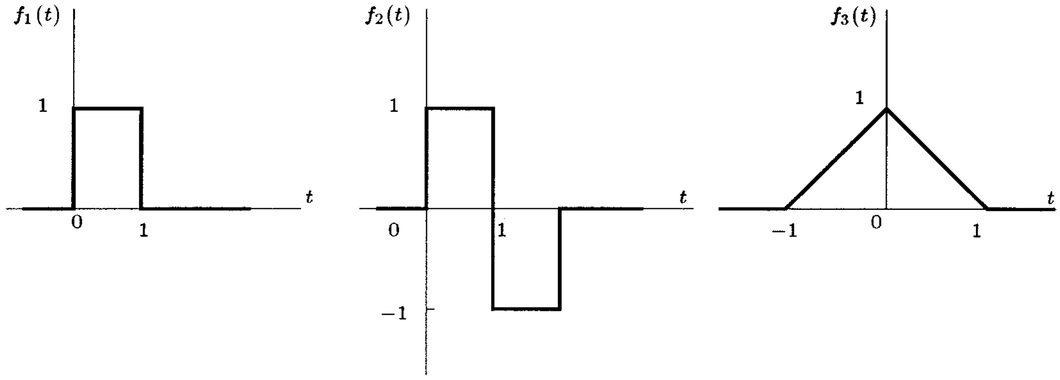

Figure 18: P-3.8.3

Solution:

a) We shift f_1 with respect to f_2, this results in,

$$\begin{aligned} f_1(t) * f_2(t) &= \int_0^t d\tau = t, && 0 < t < 1, \\ &= \int_{t-1}^1 d\tau - \int_1^t d\tau = 3 - 2t, && 1 < t < 2 \\ &= -\int_{t-1}^2 d\tau \ = \ t - 3, && 2 < t < 3. \end{aligned}$$

b) We shift f_1 with respect to f_3, this results in,

$$f_1(t) * f_3(t) = \begin{cases} \int_{-1}^t (1+\tau)d\tau = (t+1)^2/2, & -1 < t < 0, \\ \int_{t-1}^0 (1+\tau)d\tau + \int_0^t (1-\tau)d\tau = -(t^2 - t - 1/2), & 0 < t < 1, \\ \int_{t-1}^1 (1-\tau)d\tau = t^2/2 - 2t + 2, & 1 < t < 2 \\ 0 & \text{elsewhere} \end{cases}$$

c) We shift f_2 with respect to f_3, this results in,

$$\int_{-1}^t (1+\tau)d\tau = \frac{(t+1)^2}{2}, \quad -1 < t < 0,$$

$$-\int_{-1}^{t-1} (1+\tau)d\tau + \int_{t-1}^0 (1+\tau)d\tau + \int_0^t (1-\tau)d\tau = -\frac{3t^2}{2} + t + \frac{1}{2}, \quad 0 < t < 1$$

$$-\int_{t-2}^0 (1+\tau)d\tau - \int_0^{t-1} (1-\tau)d\tau + \int_{t-1}^1 (1-\tau)d\tau = \frac{3t^2}{2} - 5t + \frac{7}{2}, \quad 1 < t < 2$$

$$-\int_{t-2}^1 (1-\tau)d\tau = -\frac{t^2}{2} + 3t - \frac{9}{2}, \quad 2 < t < 3$$

Problem 3.8.4 Because the convolution of two impulse functions results in another impulse function, we can handle the convolution of two piecewise linear waveforms by (1) taking the derivatives until the impulse functions appear,(2) performing the convolution with the impulse functions, and (3) integrating the result as many times as the total number of differentiations. Use these methods to perform the convolution indicated in problem 3.8.3.

Solution:

It can be shown that the following are true:

$f_1'(t) = \delta(t) - \delta(t-1),$

$f_2'(t) = \delta(t) - 2\delta(t-1) + \delta(t-2),$

$f_3''(t) = \delta(t+1) - 2\delta(t) + \delta(t-1)$

The convolution results can therefore be expressed as:

a)

$$f_1(t) * f_2(t) = \int_{-\infty}^{t} \int_{-\infty}^{\tau} [\delta(\xi) - 3\delta(\xi-1) + 3\delta(\xi-2) - \delta(\xi-3)] d\xi d\tau.$$

b)

$$f_1(t) * f_3(t) = \int_{-\infty}^{t} \int_{-\infty}^{\tau} \int_{-\infty}^{\xi} [\delta(\gamma+1) - 3\delta(\gamma) + 3\delta(\gamma-1) - \delta(\gamma-2)] d\gamma d\xi d\tau$$

b)

$$f_2(t) * f_3(t) = \int_{-\infty}^{t} \int_{-\infty}^{\tau} \int_{-\infty}^{\xi} [\delta(\gamma+2) - 4\delta(\gamma+1) + 6\delta(\gamma) - 4\delta(\gamma-1) + \delta(\gamma-2)] d\gamma d\xi d\tau$$

Problem 3.8.7 Two functions of time, $f(t)$ and $g(t)$, are defined by:

$$f(t) = \begin{cases} e^{-j\omega_l t} & 0 \le t < T, \\ 0 & \text{elsewhere} \end{cases}$$

$$g(t) = \begin{cases} e^{j\omega_l t} & 0 \le t < T, \\ 0 & \text{elsewhere} \end{cases}$$

Determine and sketch $f(t) * g(t)$ for each of the following conditions:

a) $\omega_l = 0$;

b) $\omega_l = \pi/(2T)$.

Solution: a)

$$f(t) * g(t) = \begin{cases} \int_0^t d\tau = t, & 0 < t < T \\ \int_{t-T}^{T} d\tau = 2T - t, & T < t < 2T \\ 0 & \text{elsewhere} \end{cases}$$

b)

$$f(t)*g(t)=\begin{cases}\int_0^t e^{-j\omega\tau}e^{j\omega(t-\tau)}d\tau=[\sin(\omega_l t)]/\omega_l, & 0<t<T\\ \int_{t-T}^{T} e^{-j\omega\tau}e^{j\omega(t-\tau)}d\tau=\sin[\omega_l(2T-t)]/\omega_l, & T<t<2T\\ 0 & \text{elsewhere}\end{cases}$$

Problem 3.8.8 An averaging, or smoothing, operation used in signal analysis is defined by the integral operator:

$$g(t)=\int_{t-T}^{t} w(t-\tau)f(\tau)d\tau,$$

where t indicates present time, T is the averaging duration, $f(t)$ is the function being averaged, and $w(t)$ is a window function.

a) Express the $g(t)$ as the convolution of $f(t)$ with with a second function, $h(t)$, for the special case in which $w(t)=1$. Plot and dimension $h(t)$. [*Hint*: Express the limits of integration as in terms of step functions in the integrand so that the limits of integration can be extended.]

b) Take the Fourier transform of $g(t)$ to investigate the relative effects of of the various factors in the averaging operations in the Frequency domain. [*Hint*: Use an interchange in order of operations]

Solution:

a) If the function is given by $h(t)$, then:

$$g(t)=\int_{t-T}^{t} w(t-\tau)f(\tau)d\tau=\int_{-\infty}^{\infty} h(t-\tau)f(\tau)d\tau,$$

One of the possibility is to have $g(t)=[u(t)-u(t-T)]*f(t)$ and therefore $h(t)=u(t)-u(t-T)$.

b) $g(t)=h(t)*f(t)$, where $h(t)=w(t)[u(t)-u(t-T)]$

Taking the Fourier transform we obtain :

$$\begin{aligned} H(\omega) &= W(\omega)e^{-j\omega T/2}T\mathrm{Sa}(\omega T/2)\\ G(\omega) &= e^{-j\omega T/2}T\mathrm{Sa}(\omega T/2)W(\omega)F(\omega).\end{aligned}$$

It is seen that $G(\omega)$ gives a good representation of $W(\omega)F(\omega)$ if the term $e^{-j\omega T/2}T\mathrm{Sa}(\omega T/2)$ is approximately equal to 1. This happens when the frequencies of interest are within the restriction $\omega \ll 2\pi/T$.

Problem 4.1.3 The voltage $f(t)=10te^{-t}u(t)$ is developed across a 50-ohm resistor.

a.) Calculate the total energy developed in the resistor.

b.) What fraction of this energy is contained within a (low-pass) bandwidth of 1 radian/sec?

c.) What fraction of this energy is contained within a bandwidth of 2 rad/sec, with a center frequency of 4 rad/s?

Solution:

a.) Total dissipated energy is given by:

$$E = \left[\int_{-\infty}^{\infty} |f(t)|^2 dt\right] / R = 2\int_{-\infty}^{\infty} t^2 e^{-2t} u(t) dt = 2\int_{0}^{\infty} t^2 e^{-2t} dt = 0.5J.$$

b.) The fourier transform of the signal is given by:

$$F(\omega) = \frac{10}{(1+j\omega)^2}$$

Recall that the Energy of the voltage signal $f(t)$ with Fourier transform $F(\omega)$ in the frequency band $[\omega_1, \omega_2]$ when acting on a resistance R is given by:

$$E = \frac{1}{2\pi}\left[\int_{\omega_1}^{\omega_2} |F(\omega)|^2 d\omega\right] / R$$

In this case, the fraction of the total energy in the given bandwidth (frequency range $[-1, 1]$) is given by:

$$\left[\frac{2}{2\pi R}\int_0^1 \frac{(10)^2}{(1+\omega^2)^2} d\omega\right] / 0.5 = \frac{1}{\pi} + \frac{2}{\pi}\arctan(1) = 81.8\%$$

Note that,

$$\int_a^b \frac{2}{(1+\omega^2)^2} d\omega = \frac{b}{1+b^2} - \frac{a}{1+a^2} + \arctan(b) - \arctan(a) \tag{169}$$

c.) Similarly, the fraction of the energy for this case is given by:

$$\left[\frac{2}{2\pi R}\int_3^5 \frac{(10)^2}{(1+\omega^2)^2} d\omega\right] / 0.5 = \frac{2}{\pi}[\arctan(5) - \arctan(3)] - \frac{14}{65\pi} = 10.68\%$$

Problem 4.2.1 A certain signal $f(t)$ has the following power spectral density (assume a 1-Ohm resistive load):

$$S_f(\omega) = \left[\frac{1}{1+\omega^2} + \delta(\omega - 2) + \delta(\omega + 2)\right].$$

a.) What is the total mean power in $f(t)$?

b.) What is the mean power in $f(t)$ within the bandwidth 0.9 to 1.1 rad/sec?

c.) What is the mean power in $f(t)$ within the bandwidth 1.9 to 2.1 rad/sec?

d.) Find a signal $f(t)$, in terms of cosines and exponentials, that will satisfy this power spectral density. Are other solutions possible?

Solution:

Recall that the power of a signal with Spectral density function $S_f(\omega)$ in the frequency interval $[a, b]$ is given as:

$$P \triangleq \frac{1}{2\pi}\int_a^b S_f(\omega) d\omega.$$

a.)

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_f(\omega)d\omega = \frac{1}{2} + \frac{1}{\pi} = 0.818W$$

Note that,

$$\int_a^b \frac{1}{1+\omega^2}d\omega = \arctan(b) - \arctan(a), \quad \& \quad \arctan(\infty) = -\arctan(-\infty) = \pi/2. \tag{170}$$

b.)

$$P = \frac{2}{2\pi}\int_{0.9}^{1.1}\left[\frac{1}{1+\omega^2} + \delta(\omega-2)\right]d\omega = \frac{1}{\pi}[\arctan(1.1) - \arctan(0.9)] = 0.0319W$$

Note that the delta function is not in the range of the integration.

c.)

$$P = \frac{2}{2\pi}\int_{1.9}^{2.1}\left[\frac{1}{1+\omega^2} + \delta(\omega-2)\right]d\omega = \frac{1}{\pi}[\arctan(2.1) - \arctan(1.9) + 1] = 0.331W$$

Note that the delta function is in the range of the integration.

d.) The two impulse functions can be associated with a signal $\sqrt{2/\pi}\cos(2t)$ (refer to the example 4.21 of the text). The term $(\omega^2+1)^{-1}$ can be associated with a constant power spectral density through an RC low-pass filter with $RC = 1$. In general, because the phase information is not included, these results are not unique and other solutions are possible.

Problem 4.2.2 A given voltage is $f(t) = 4\cos 20\pi t + 2\cos 30\pi t$ across 1Ω. Note that this is a Fourier series.

a.) Determine and sketch the power spectral density of $f(t)$ and identify the component at the fundamental frequency.

b.) Calculate the average power, both in the time domain and the frequency domain dissipated by $f(t)$ across the 1Ω resistor.

c.) Determine and sketch the power spectral density of $f^2(t)$. (First perform the squaring operation, then write the result in the form of a Fourier series and from this determine the the power spectral density.)

Solution:

a.) We have,

$$f(t) = 2\left[e^{j20\pi t} + e^{-j20\pi t}\right] + \left[e^{j30\pi t} + e^{-j30\pi t}\right]$$

Using equation 4.20 of the text, we obtain

$$S_f(\omega) = 2\pi[\delta(\omega+30\pi) + (2)^2\delta(\omega+20\pi) + (2)^2\delta(\omega-20\pi) + \delta(\omega-30\pi)]$$

The component at the fundamental frequency (10π) is zero.

b.)

$$P = \frac{2}{2\pi}\int_0^\infty [8\pi\delta(\omega - 20\pi) + 2\pi\delta(\omega - 30\pi)]d\omega = 8 + 2 = 10W.$$

The power in the time domain can be computed by computing the integral $(1/T)\int_T |f(t)|^2 dt$ over one period, namely $T = 2/10$ (note that the fundamental frequency is equal to $\omega_0 = 10\pi$).

c.)

$$f^2(t) = (4)^2\cos^2 20\pi t + (2)^2\cos^2 30\pi t + 2(4)(2)\cos 20\pi t\cos 30\pi t$$

$$f^2(t) = 10 + 8\cos 10\pi t + 8\cos 40\pi t + 8\cos 50\pi t + 2\cos 60\pi t$$

$$S_{f^2}(\omega) = 2\pi[(10)^2\delta(\omega) + (4)^2\delta(\omega \pm 10\pi) + (4)^2\delta(\omega \pm 40\pi) + (4)^2\delta(\omega \pm 50\pi) + \delta(\omega \pm 60\pi)]$$

Problem 4.2.3 A symmetric square wave (i.e, zero average value) with a peak amplitude of 1V and period T is applied to the input of an amplifier whose magnitude transfer function is,

$$|H(\omega)| = \begin{cases} K(1+\cos\omega)/2 & |\omega| < 4\pi/T, \\ 0 & \text{elsewhere} \end{cases}$$

where K is the voltage gain. Assume that the input and output impedances are resistive and equal to 1-ohm. The voltage gain K is adjusted so that the amplifier output is equal to 1W.

a.) Determine the value of K.

b.) The square wave is replaced with a symmetric triangular wave (c.f. Table 2.1) with the same peak amplitude, and period T_1. What is the output power under the above conditions $T_1 = T$?

c.) Repeat part b.) if $T_1 = 2T$.

Solution:

a.) Recall that for the given input waveform, the Fourier Series coefficient are given by,

$$F_n = \text{Sa}(n\pi/2), \quad n \neq 0$$

Note that $F_n = (1/T)F(\omega)|_{\omega=n\omega_0}$ and refer to entry 13 of table 3.1, multiplied by two, $\tau = T/2$, zero dc value. You can also use directly table 2.2.

Looking at the frequency response of the given amplifier, it is seen that only the first harmonic ($n = 1$) will be passed (note that $F_0 = 0$ because the signal has zero dc value). We have $F_0 = 0$, $F_1 = F_{-1} = 2/\pi$. The output of the amplifier $g(t)$, can therefore be written as:

$$g(t) = \frac{K}{\pi}(1+\cos\omega_0)[e^{j\omega_0 t} + e^{-j\omega_0 t}] = \frac{2K}{\pi}(1+\cos\omega_0)\cos\omega_0 t$$

The rms value of this signal is,

$$\overline{g^2(t)} = (2K/\pi)^2(1+\cos\omega_0)^2\overline{\cos^2\omega_0 t} = 1$$

This gives the value of K as:

$$K = \frac{\pi/\sqrt{2}}{1+\cos\omega_0},$$

where $\omega_0 = 2\pi/T$. (Note that $\overline{\cos^2\omega_0 t} = 1/2$).

b.) In this case, $F_n = (1/T)F(\omega)|_{w=nw_0} = \text{Sa}^2(n\pi/2)$. (Refer to entry 16 of table 3.1, multiplied by two, $\tau = T/2$, zero dc value. You can also use directly table 2.2). We have $F_0 = 0$, $F_1 = F_{-1} = (2/\pi)^2$.

$$g(t) = (4K/\pi^2)(1+\cos\omega_0)\cos\omega_0 t$$

which, substituting for K, gives the value of $\overline{g^2(t)} = (2/\pi)^2$

c.) In this case, the first, and the third harmonic pass through the filter. Note that the triangular signal does not have the second harmonic. We have $F_0 = 0$, $F_1 = F_{-1} = (2/\pi)^2$, $F_3 = F_{-3} = (2/3\pi)^2$, resulting in,

$$g(t) = K\left[\frac{4}{\pi^2}(1+\cos\omega_1)\cos\omega_1 t + \frac{4}{9\pi^2}(1+\cos 3\omega_1)\cos 3\omega_1 t\right]$$

from which we obtain:

$$\overline{g^2(t)} = \frac{4}{\pi^2}\left[\frac{[1+\cos(\omega_0/2)]^2}{(1+\cos\omega_0)^2} + \frac{1}{9^2}\frac{[1+\cos(3\omega_0/2)]^2}{(1+\cos\omega_0)^2}\right]$$

Note that $\overline{\cos^2\omega_1 t} = \overline{\cos^2 3\omega_1 t} = 1/2$, $\overline{\cos\omega_1 t\cos 3\omega_1 t} = 0$ and $\omega_1 = \omega_0/2$.

Problem 4.2.4 A sinusoidal generator produces the waveform $v_i(t) = A\cos\omega_0 t$ as the input to the lowpass filter shown in figure P-4.2.4

a.) Using the the methods of ac circuit analysis, find an expression for the output waveform, $v_o(t)$.

b.) Find the average power in the $v_o(t)$ across R_2 from the results of part (a).

c.) Determine the spectral power density of $v_o(t)$, and from this find the average power in $v_o(t)$ across R_2.

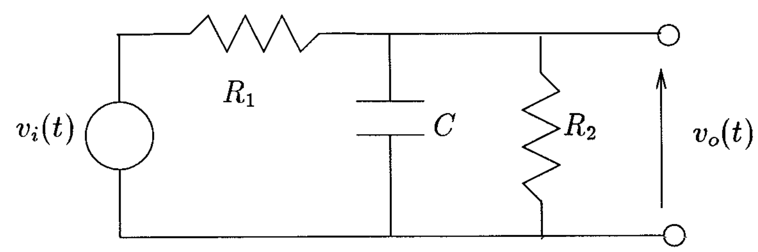

Figure 19: P-4.2.4

Solution:

a.)

$$H(\omega) = \frac{1/R_1 C}{j\omega + 1/(R_p C)}$$

where,

$$R_p = \frac{R_1 R_2}{R_1 + R_2}$$

It is easy to show that the output $v_o(t)$ can be written as

$$v_o(t) = \frac{A/(R_1 C)}{\sqrt{\omega_0^2 + 1/(R_p C)^2}} \cos\left[\omega_0 t - \arctan(\omega_o R_p C)\right]$$

b.)

$$P_{R_2} = \frac{\overline{v_o^2(t)}}{R_2} = \frac{1}{2R_2} \frac{A^2/(R_1 C)^2}{\omega_0^2 + 1/(R_p C)^2}$$

Note that $\overline{\cos^2(\omega_0 t + \phi)} = 1/2$.

c.) Using the method of example 4.2.1 of the text, it can be seen that,

$$S_{v_i}(\omega) = (\pi A^2/2)[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

d.)

$$S_{v_o}(\omega) = \left(\frac{\pi}{2}\right) \frac{A^2/(R_1 C)^2}{\omega_0^2 + 1/(R_p C)^2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

and the power is also given by

$$P_{R_2} = \frac{1}{2\pi R_2} \int_{-\infty}^{\infty} S_{v_o}(\omega) d\omega = \frac{1}{2R_2} \frac{A^2/(R_1 C)^2}{\omega_0^2 + 1/(R_p C)^2}$$

Problem 4.3.1 A full wave rectified sinusoid (c.f Table 2.1) has a peak amplitude A and period T.

a.) Determine its mean square value in time domain.

b.) Find the expression for the spectral power density (assume 1 ohm).

c.) Making use of the above results, show that

$$\sum_{\substack{n=-\infty \\ n\ even}}^{\infty} \frac{1}{(1 - n^2)^2} = \frac{\pi^2}{8}$$

d.) Do you reach the same conclusion as in part c.) when using a half wave rectified sinusoid?

Solution:

a.) Mean square value $= A^2/2$. This is the same as for the signal $A \sin \omega_0 t$.

b.) Using table 2.2 of the text, we obtain:

$$S_f(\omega) = 2\pi \sum_{\substack{n=-\infty \\ n\ even}}^{\infty} \frac{4A^2}{\pi^2(1 - n^2)^2} \delta(\omega - n\omega_0)$$

c.)

$$\overline{f^2(t)} = \left(\frac{1}{2\pi}\right) 2\pi \sum_{\substack{n=-\infty \\ n\ even}}^{\infty} \frac{4A^2}{\pi^2(1-n^2)^2} \int_{-\infty}^{\infty} \delta(\omega - n\omega_0) d\omega = \sum_{\substack{n=-\infty \\ n\ even}}^{\infty} \frac{4A^2}{\pi^2(1-n^2)^2}$$

Equating the above result and the result of part a.) the desired relationship is proved.

d.)

$$\overline{f^2(t)} = \frac{1}{T}\int_0^{T/2} A^2 \sin^2\omega_0 t dt = \frac{A^2}{4}$$

and also, referring to table 2.2, we have,

$$S_f(\omega) = 2\pi\left(\frac{A}{4}\right)^2 \delta(\omega+\omega_0) + 2\pi\left(\frac{A}{4}\right)^2 \delta(\omega-\omega_0) + 2\pi \sum_{\substack{n=-\infty \\ n\ even,\ n\neq 0}}^{\infty} \frac{A^2}{\pi^2(1-n^2)^2}\delta(\omega - n\omega_0)$$

Integrating $S_f(\omega)$ from $-\infty$ to ∞ and setting the result equal to $\overline{f^2(t)}$ yields the same answer as the one obtained in part (c).

Problem 4.3.2 White noise with a two-sided power spectral density of $\eta/2$ Watts per Hz (assume 1-ohm source), added to a signal described described by $\sqrt{2}\cos 30\pi t$ is applied to the input of an RC low-pass filter (c.f. Fig 2.16). The rms value of the signal at the filter output is 10mV, and the ratio of the mean square signal to the mean square noise at the output is 10dB.

a.) Determine the value of η in terms of the RC time constant of the filter.

b.) Determine the RC time constant of the filter.

Solution:

a.) In this case the output noise power is obtained in terms of the output SNR ($\text{SNR}_{dB} = 10\log_{10}\text{SNR}$) and the output signal power which is known.

$$\overline{n_0^2(t)} = [(10mV)/10]^2 = 1 \ \times 10^{-6}V^2.$$

Also, we can write (refer to example 4.7.2),

$$\overline{n_0^2(t)} = \frac{\eta}{2\pi}\int_0^{\infty} \frac{1/(RC)^2}{\omega^2 + 1/(RC)^2} d\omega = \frac{\eta}{4RC}$$

equating the above two expressions we get $\eta = 4RC \times 10^{-6}$W/Hz.

b.)

$$v_0(t) = \frac{1/(RC)}{\sqrt{(30\pi)^2 + 1/(RC)^2}}\sqrt{2}\cos[30\pi t - \arctan(30\pi RC)].$$

Form this we get

$$\overline{v_0^2(t)} = \frac{1/(RC)^2}{(30\pi)^2 + 1/(RC)^2} = (10mV)^2$$

Solving for RC, we get $RC = 10/3\pi = 1.061$ sec.

Problem 4.3.3 Repeat problem 4.3.2 for the case in which the power spectral density of input signal is $S_f(\omega) = 1/(1+\omega^2)$.

Solution:

a.) Same as in problem 4.3.2 a.) (no changes at all)

b.) The mean square value of $v_o(t)$ is computed in frequency domain as follows.

$$\overline{v_o^2(t)} = \frac{1}{\pi}\int_0^{\infty} \frac{1}{1+\omega^2}\frac{1/(RC)^2}{\omega^2+1/(RC)^2}d\omega = \frac{1}{2(RC+1)} = (10mV)^2$$

Solving; we get $RC = 5 \times 10^3$sec.

Problem 4.5.2 A white noise source, $\eta/2$ V^2/Hz, is connected to the input of the RC filter shown in figure P-4.2.4

a.) Determine the spectral density of the output.

b.) Determine the autocorrelation function of the output.

c.) Determine the rms value of the output using each of the two preceding results, and compare.

d.) What happens as $C \to 0$?

Solution:

a.)

$$H(\omega) = \frac{1/(R_1C)}{j\omega + 1/(R_pC)}$$

where $R_p = (R_1R_2)/(R_1+R_2)$

Using the relationship between the input and output power spectral densities of a linear system we get the result:

$$S_{v_o}(\omega) = \frac{\eta}{2}\frac{1/(R_1C)^2}{\omega^2+1/(R_pC)^2}$$

b.) Using table of Fourier transforms, we get,

$$R_{v_o}(\tau) = \frac{\eta R_p}{4R_1^2C}e^{-|\tau|/(R_pC)}$$

c.) The rms value of the output is given by:

$$\overline{v_o^2(t)} = R_{v_o}(0) = \frac{\eta R_p}{4R_1^2C};$$

$$\sqrt{\overline{v_o^2(t)}} = \frac{1}{2R_1}\sqrt{\frac{\eta R_p}{C}}$$

Integrating $S_{v_o}(\omega)$ results in the same value.

d.) As $C \to 0$, $H(\omega)$ becomes a resistive divider and because the input source is assumed to be white, the mean square output $\to \infty$.

Problem 4.4.1 a.) Determine the power spectral density of $F_1 \exp(j\omega_0 t)$ by first finding the autocorrelation function, then taking the Fourier Transform of the autocorrelation function.
b.) Repeat part a.) for $[F_1 \exp(j\omega_0 t) + F_2 \exp(j2\omega_0 t)]$.
c.) Extend your result in part (b) to $\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$.

Solution:
a.)

$$R_f(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} F_1^* e^{-j\omega_0 t} F_1 e^{j\omega_0 (t+\tau)} dt = |F_1|^2 e^{j\omega_0 \tau};$$
$$S_f(\omega) = \mathcal{F}\{R_f(\tau)\} = 2\pi |F_1|^2 \delta(\omega - \omega_0)$$

b.)

$$\begin{aligned} R_f(\tau) &= \frac{1}{T}\int_{-T/2}^{T/2} [F_1^* e^{-j\omega_0 t} + F_2^* e^{-j2\omega_0 t}][F_1 e^{j\omega_0 (t+\tau)} + F_2 e^{j2\omega_0 (t+\tau)}] dt \\ &= |F_1|^2 e^{j\omega_0 \tau} + |F_2|^2 e^{j2\omega_0 \tau} \end{aligned}$$

¿From this we get:

$$S_f(\omega) = 2\pi |F_1|^2 \delta(\omega - \omega_0) + 2\pi |F_2|^2 \delta(\omega - 2\omega_0)$$

c.) The cross-terms integrate to zero and we get: $R_f(\tau) = \sum_{n=-\infty}^{\infty} |F_n|^2 e^{jn\omega_0 \tau}$
The power spectral density is then found to be given by:

$$S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0).$$

Problem 4.4.2 The complex pulse signal

$$f(t) = \begin{cases} e^{j\omega_c t} & 0 \le t < t_1 \\ 0 & t_1 \le t < T \end{cases}$$

is represented periodically with period T, $T > 2t_1$.
a.) Find the autocorrelation function $R_f(\tau)$, and sketch the real part.
b.) Find and sketch the power spectral density $S_f(\omega)$.

Solution:
a.) We want to compute the integral,

$$R(\tau) = \frac{1}{T}\int_T f^*(t) f(t+\tau)\, dt$$

We select the range of the integration as $[0, T]$. We have,

$$f^*(t) = \begin{cases} e^{-j\omega_c t} & 0 \le t < t_1 \\ 0 & t_1 \le t < T \end{cases}$$

Note that for $\tau < 0$, the function $f(t+\tau)$ is obtained by shifting $f(t)$ towards positive t axis and vice versa. For $-t_1 < \tau < 0$, we have,

$$f(t+\tau) = \begin{cases} 0 & 0 \le t < -\tau \\ e^{j\omega_c(t+\tau)} & -\tau \le t < t_1 - \tau \\ 0 & t_1 - \tau \le t < T \end{cases}$$

$$R_f(\tau) = \frac{1}{T}\int_{-\tau}^{t_1} e^{-j\omega_c t} e^{j\omega_c(t+\tau)}dt = e^{j\omega_c\tau}\frac{t_1}{T}\left(1+\frac{\tau}{t_1}\right)$$

Note that the lower limit of the integral is determined by $f(t+\tau)$, because $f(t+\tau) = 0$ for $t < -\tau$, and the upper limit by $f(t)$, because $f(t) = 0$ for $t > t_1$.

Similarly, for $0 < \tau < t_1$, we have,

$$f(t+\tau) = \begin{cases} e^{j\omega_c(t+\tau)} & 0 \le t < t_1 - \tau \\ 0 & t_1 - \tau \le t < T \end{cases}$$

and,

$$R_f(\tau) = \frac{1}{T}\int_{0}^{t_1-\tau} e^{-j\omega_c t} e^{j\omega_c(t+\tau)}dt = e^{j\omega_c\tau}\frac{t_1}{T}\left(1-\frac{\tau}{t_1}\right)$$

Combining the results and taking the real part,

$$\mathcal{R}\{R_f(\tau)\} = \frac{t_1}{T}\Lambda\left(\frac{\tau}{t_1}\right)\cos\omega_c\tau$$

Note that in general, in computing autocorrelation function, as we know $R(\tau) = R^*(-\tau)$, we can consider only positive (or negative) τ and use the above symmetry to compute the rest.

b.) Using tables of Fourier transforms and properties, we get:

$$S_f(\omega) = \frac{t_1^2}{T}\mathrm{Sa}^2[(\omega-\omega_c)t_1/2]$$

Problem 4.5.3

A sinusoidal signal is transmitted; on reception, the signal is present together with additive noise. Assume that the signal and noise are uncorrelated. The autocorrelation function of observable quantity at the receiver input is $R(\tau) = a\cos\omega_0\tau + b\exp(-c|\tau|)$. The receiver input contains a bandpass filter (BPF-assume ideal) with bandwidth B Hz, centered at $\pm\omega_0$. Find an expression for the S/N ratio at the output of the BPF.

Solution:

First of all, referring to example 4.4.2 of the text, it is easy to show that the autocorrelation function for for the signal $f(t) = A\cos(\omega_0 t + \theta)$ is equal to, $R(\tau) = (A^2/2)\cos\omega_0\tau$. Since the signal and noise are uncorrelated, $R(\tau) = R_s(\tau) + R_n(\tau)$ in which $R_s(\tau) = a \cos \omega_0 \tau$ and $R_n(\tau) = b \exp(-c|\tau|)$. Then

$$S_i = R_s(0) = a, \tag{171}$$
$$N_i = R_n(0) = b, \tag{172}$$

This means that, $S_f(\omega) = \mathcal{F}^{-1}\{R_f(\tau)\} = (a/2)(2\pi)[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$, and $S_n(\omega) = \mathcal{F}^{-1}\{R_n(\tau)\} = 2bc/(\omega^2 + c^2)$. The output signal and noise powers are equal to,

$$S_o = \frac{2}{2\pi}\int_{\omega_0-\pi B}^{\omega_0+\pi B} a\pi\delta(\omega - \omega_0)d\omega = a. \tag{173}$$
$$N_o = \frac{2}{2\pi}\int_{\omega_0-\pi B}^{\omega_0+\pi B} \frac{2bc}{\omega^2 + c^2}d\omega = \frac{2b}{\pi}\left\{\tan^{-1}[(\omega_0 + \pi B)/c] - \tan^{-1}[(\omega_0 - \pi B)/c]\right\}. \tag{174}$$

Note that,

$$\int \frac{c}{\omega^2 + c^2}d\omega = \tan^{-1}\left(\frac{\omega}{c}\right). \tag{175}$$

The desired ratio is:

$$\frac{S_0}{N_0} = \frac{a\pi}{2b\tan^{-1}[(\omega_0 + \pi B)/c] - \tan^{-1}[(\omega_0 - \pi B)/c]}. \tag{176}$$

Problem 4.6.1

Determine the autocorrelation function of each of the following pulse waveforms by first taking a Fourier transform, next taking the magnitude squared, and then taking an inverse Fourier transform.

a) $e^{-at}u(t)$,
b) $\text{rect}(t/t_1)$,
c) $\text{Sa}(Wt)$.

Solution:

a)

$$e^{-at}u(t) \Longleftrightarrow \frac{1}{j\omega + a} \Longrightarrow \frac{1}{\omega^2 + a^2} \Longleftrightarrow \frac{1}{2a}e^{-a|\tau|}. \tag{177}$$

b)

$$\text{rect}(t/t_1) \Longleftrightarrow t_1\text{Sa}(\omega t_1/2) \Longrightarrow t_1^2\text{Sa}^2(\omega t_1/2) \Longleftrightarrow t_1\Lambda(\tau/t_1). \tag{178}$$

c)

$$\text{Sa}(Wt) \Longleftrightarrow \frac{\pi}{W}\text{rect}[\omega/(2W)] \Longrightarrow \left(\frac{\pi^2}{W}\right)^2 \text{rect}[\omega/(2W)] \Longleftrightarrow \frac{\pi}{W}\text{Sa}(W\tau). \tag{179}$$