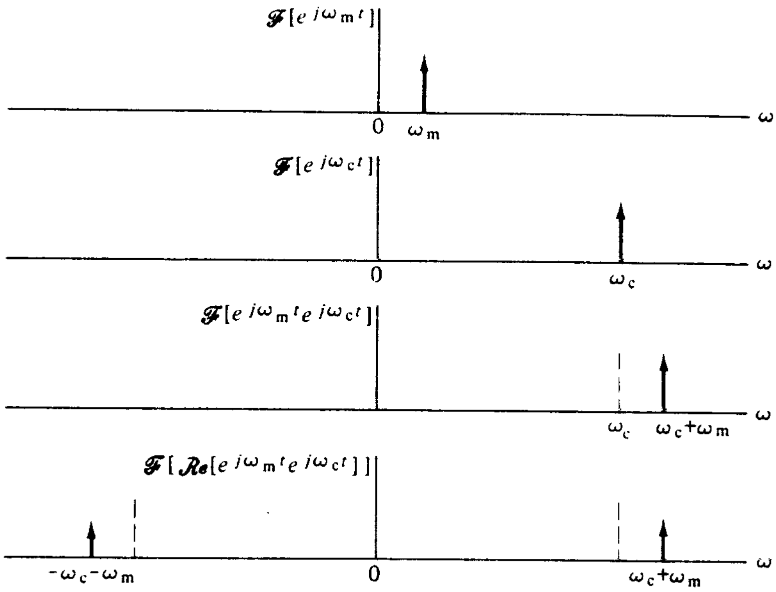

Figure 42: Spectra of SSB in modulating a sinusoid.

The corresponding spectrum are shown in Fig. 42.

Similarly, we have,

$$\phi_{SSB-}(t) = \cos\omega_m t \cos\omega_c t + \sin\omega_m t \sin\omega_c t \tag{348}$$

Considering a more complicated signal, say $f(t)$, as the sum of sinusoids, we have,

$$\phi_{SSB\mp}(t) = f(t)\cos\omega_c t \pm \hat{f}(t)\sin\omega_c t \tag{349}$$

where $\hat{f}(t)$ is the signal obtained by shifting the phase of $f(t)$ by $\pi/2$ at each frequency. The corresponding block diagram is shown in Fig. 43. In practice, it is very difficult to design a circuit which results in exactly $\pi/2$ phase shift for all the frequencies.

6.15 Analytic signals and Hilbert Transform

In general, any real valued signal can be expressed in terms of a complex signal with one sideband. Such a signal is called an analytic signal. Note that all the analytic signals are complex valued but the reverse in not necessarily true.

Assume that the real signal $f(t)$ corresponds to the analytic signal

$$z(t) = f(t) + j\hat{f}(t) \longrightarrow Z(\omega) = F(\omega) + j\hat{F}(\omega) \tag{350}$$

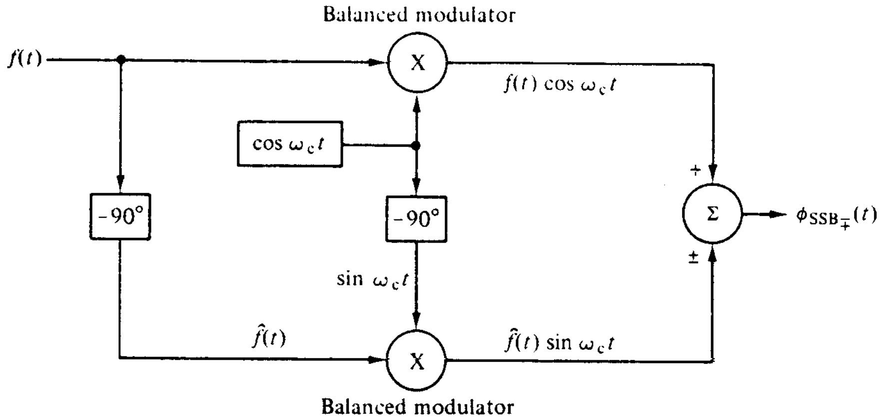

Figure 43: Phase shift method of generating SSB.

To delete one of the side bands (negative frequency), we should have,

$$\hat{F}(\omega) = \begin{cases} -jF(\omega), & \omega > 0 \\ jF(\omega), & \omega < 0 \end{cases} \tag{351}$$

or,

$$\hat{F}(\omega) = -jF(\omega)\text{sgn}(\omega) \tag{352}$$

This results in,

$$Z(\omega) = \begin{cases} 2F(\omega), & \omega > 0 \\ 0, & \omega < 0 \end{cases} \tag{353}$$

Taking the inverse Fourier Transform, we obtain,

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)e^{j\omega t}d\omega \tag{354}$$

The function $\hat{f}(t)$ is called the quadrature pair, or the Hilbert Transform, of $f(t)$ because each frequency component of $\hat{f}(t)$ is in phase quadrature ($\pi/2$ phase difference) with that of $f(t)$.

We know that: (i) The analytic signal $z(t)$ has a one-sided spectrum. (ii) The spectrum of $z(t)e^{j\omega_c t}$ is the same as the spectrum of $z(t)$ but centered around ω_c. (iii) Taking the real part of $z(t)e^{j\omega_c t}$ results in a spectrum which is symmetrical with respect to the origin. This spectrum corresponds to a SSB signal, i.e.,

$$\mathcal{R}\{z(t)e^{j\omega_c t}\} = \mathcal{R}\{[f(t) + j\hat{f}(t)]e^{j\omega_c t}\} = f(t)\cos\omega_c t - \hat{f}(t)\sin\omega_c t \tag{355}$$

This relationships is for the upper band case. Using $z^*(t)$ instead of $z(t)$ results in lower band case, i.e., $f(t)\cos\omega_c t + \hat{f}(t)\sin\omega_c t$.

To obtain the Hilbert relationship in the time domain, we have,

$$\text{sgn}(\omega) \Longrightarrow \frac{j}{\pi t} \tag{356}$$

resulting in,

$$\hat{f}(t) = f(t) * \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau \tag{357}$$

6.16 Demodulation of SSB signals

Demodulation is achieved using synchronous detection.

$$\phi_{SSB\mp}(t) = f(t)\cos\omega_c t \pm \hat{f}(t)\sin\omega_c t \tag{358}$$

Assume that the demodulation is achieved using the signal $\phi_d(t) = \cos[(\omega_c + \Delta\omega)t + \theta]$ where $\Delta\omega$ and θ are the frequency and the phase error, respectively. This results in,

$$\phi_{SSB\mp}(t)\phi_d(t) = [f(t)\cos\omega_c t \pm \hat{f}(t)\sin\omega_c t]\cos[(\omega_c + \Delta\omega)t + \theta] =$$

$$\frac{1}{2}f(t)\left\{\cos[(\Delta\omega)t + \theta] + \cos[(2\omega_c + \Delta\omega)t + \theta]\right\} \mp \frac{1}{2}\hat{f}(t)\left\{\sin[(\Delta\omega)t + \theta] - \sin[(2\omega_c + \Delta\omega)t + \theta]\right\} \quad (359)$$

Using a lowpass filter to eliminate the double-carrier frequency terms, we obtain,

$$e_o(t) = \frac{1}{2}f(t)\cos[(\Delta\omega)t + \theta] \mp \frac{1}{2}\hat{f}(t)\sin[(\Delta\omega)t + \theta] \quad (360)$$

For $\Delta\omega = 0$ and $\theta = 0$, we obtain,

$$e_o(t) = (1/2)f(t) \quad (361)$$

To study the effect of the phase error, for $\Delta\omega = 0$, we obtain,

$$e_o(t) = \frac{1}{2}[f(t)\cos(\theta) \mp \hat{f}(t)\sin(\theta)] \quad (362)$$

Note that for DSB-SC carrier, the demodulator output in the case of the phase error was equal to: $e_o(t) = \frac{1}{2}f(t)\cos(\theta)$. For a constant θ, the effect of the phase error acts just as a scale factor. However, for SSB-SC, the phase error is not just a scale factor and the degradation is more serious. To investigate this effect, let us write,

$$e_o(t) = \frac{1}{2}\mathcal{R}\{[f(t) \mp j\hat{f}(t)]e^{j\theta}\} \quad (363)$$

while, previously, we had,

$$e_o(t) = \frac{1}{2}\mathcal{R}\{[f(t) \mp j\hat{f}(t)]\} \quad (364)$$

This means that θ acts as a phase distortion. It turns out that the human ear is not very sensitive to phase distortion.

To study the effect of the frequency error, for $\theta = 0$, we obtain,

$$e_o(t) = \frac{1}{2}[f(t)\cos(\Delta\omega)t \mp \hat{f}(t)\sin(\Delta\omega)t] \quad (365)$$

$$e_o(t) = \frac{1}{2}\mathcal{R}\{[f(t) \mp j\hat{f}(t)]e^{j\Delta\omega t}\} \quad (366)$$

The frequency error results in a spectral shift in the demodulate output.

6.17 Single Sideband Large Carrier, SSB-LC

This is a signal of the form,

$$\phi(t) = A\cos\omega_c t + f(t)\cos\omega_c t \mp \hat{f}(t)\sin\omega_c t \tag{367}$$

The corresponding envelope is equal to,

$$r(t) = \sqrt{[A + f(t)]^2 + [\hat{f}(t)]^2} \tag{368}$$

Assuming that the carrier is much larger than the SSB-SC envelope, we obtain,

$$r(t) \simeq A\sqrt{1 + \frac{2f(t)}{A}} \tag{369}$$

Using binomial expansion, we obtain,

$$r(t) \simeq A\left[1 + \frac{f(t)}{A}\right] = A + f(t) \tag{370}$$

This means that such a signal can be detected using an envelope detector. However, the amount of carrier required for the envelope detection is substantially more than the case of the DSB-LC.

6.18 Vestigial-side band (VSB) modulation

The generation of SSB may be quite difficult when the modulating signal bandwidth is wide or when one can not disregard the low-frequency components of the signal. As an intermediate solution, VSB modulation is used which provides a compromise between DSB and SSB. In VSB modulation, one sideband and a portion of the other sideband is transmitted such that the demodulation process reproduces the original signal. If the vestigial filter is equal to $H_v(\omega)$, we obtain,

$$\Phi_{VSB}(\omega) = \left[\frac{1}{2}F(\omega - \omega_c) + \frac{1}{2}F(\omega + \omega_c)\right]H_v(\omega) \tag{371}$$

The output of a synchronous detector is equal to,

$$e_0(t) = [\phi_{VSB}(t)\cos\omega_c t]_{LP} \tag{372}$$

Or,

$$E_0(\omega) = \frac{1}{4}F(\omega)H_v(\omega + \omega_c) + \frac{1}{4}F(\omega)H_v(\omega - \omega_c) \tag{373}$$

For a faithful reproduction, we need,

$$[H_v(\omega - \omega_c) + H_v(\omega + \omega_c)]_{LP} = \text{constant}, \qquad |\omega| < \omega_m \tag{374}$$

Note that (374) is satisfied on a magnitude basis if $|H_v(\omega)|$ is antisymmetric with respect to ω_c. Motivated by this observation, we let the constant in (374) to be $2H_v(\omega_c)$, resulting in,

$$[H_v(\omega - \omega_c) - H_v(\omega_c)] = -[H_v(\omega + \omega_c) - H_v(\omega_c)] \tag{375}$$

The corresponding spectrums are shown in Fig. 44.

6.19 Noise is Amplitude Modulation

6.19.1 Bandpass noise

Consider a noise, $n(t)$, which has a power spectral density centered around the frequency ω_0. We can write,

$$n(t) = \mathcal{R}\{[n_c(t) + jn_s(t)]e^{j\omega_0 t}\} = n_c(t)\cos\omega_0 t - n_s(t)\sin\omega_0 t \tag{376}$$

where $n_c(t)$, $n_s(t)$ are low pass noise with a bandwidth equal to one-half of the bandwidth of $n(t)$.

If we apply the bandpass noise to a synchronous detector (multiplying it by $\cos\omega_0 t$), we obtain,

$$n(t)\cos\omega_0 t = n_c(t)\cos^2\omega_0 t - n_s(t)\sin\omega_0 t\cos\omega_0 t =$$
$$\frac{1}{2}n_c(t) + \frac{1}{2}n_c(t)\cos 2\omega_0 t - \frac{1}{2}n_s(t)\sin 2\omega_0 t \tag{377}$$

Retaining the low pass term, we obtain,

$$[n(t)\cos\omega_0 t]_{lp} = \frac{1}{2}n_c(t) \tag{378}$$

Using $\cos\omega_0 t = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2$, it is easy to show that the power spectral density of $n(t)\cos\omega_0 t$ is equal to: $[S_n(\omega + \omega_0) + S_n(\omega - \omega_0)]/4$. Substituting in (378), and noting

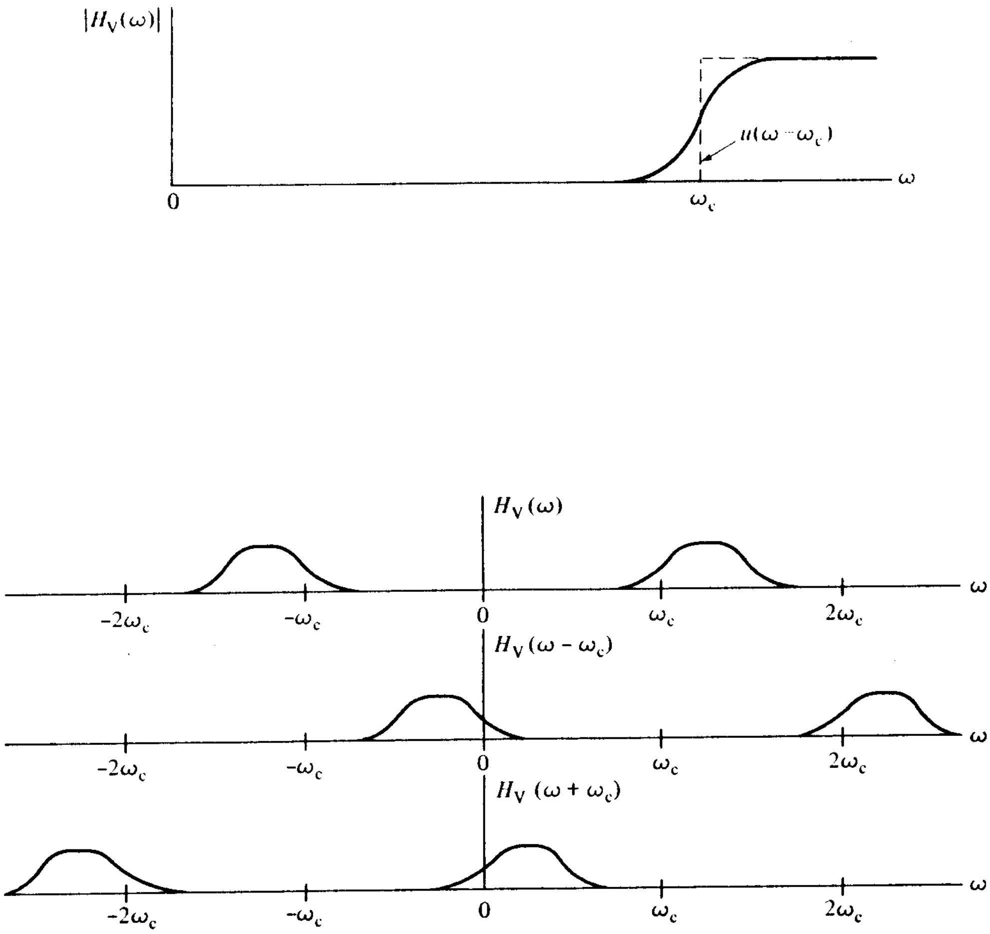

Figure 44: Different spectrums in VSB filtering.

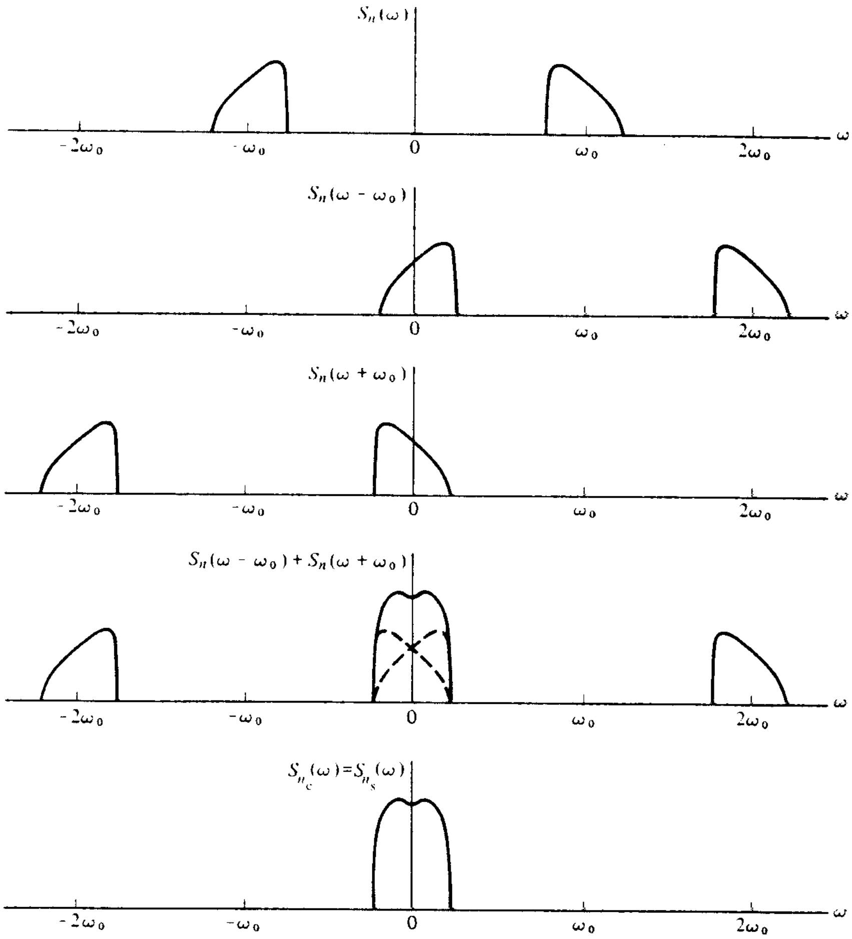

Figure 45: Spectrum of a bandpass noise.

that the power spectral density of $n_c(t)/2$ is equal to, $S_{n_c}(\omega)/4$, we obtain,

$$S_{n_c}(\omega) = [S_n(\omega + \omega_0) + S_n(\omega - \omega_0)]_{lP} \tag{379}$$

Similarly, by multiplying both sides of (376) with $\sin \omega_0 t$, we obtain,

$$S_{n_s}(\omega) = [S_n(\omega + \omega_0) + S_n(\omega - \omega_0)]_{lP} \tag{380}$$

An example is given in Fig. 45.

Referring to (379) and (380), we conclude that the power in the sine and cosine components of the noise are the same, and,

$$\overline{n^2(t)} = \overline{n_c^2(t)} = \overline{n_s^2(t)} \tag{381}$$

or, equivalently,

$$\overline{n^2(t)} = \frac{1}{2}\overline{n_c^2(t)} + \frac{1}{2}\overline{n_s^2(t)} \tag{382}$$

6.19.2 DSB-SC

The input signal is $f(t)\cos\omega_c t$ resulting in,

$$S_i = \overline{[f(t)\cos\omega_c t]^2} = \frac{1}{2}\overline{f^2(t)} \tag{383}$$

The useful output signal (after multiplication by cosine and low pass filtering) is equal to $(1/2)f(t)$ resulting in,

$$S_o = \overline{[(1/2)f(t)]^2} = (1/4)\overline{f^2(t)} = (1/2)S_i \tag{384}$$

Let $N_i = \overline{n_i^2(t)}$ denote the input noise power. The output noise (after multiplication by cosine and low pass filtering) is equal to $(1/2)n_c(t)$ resulting in the output noise power,

$$N_o = \overline{n_o^2(t)} = (1/4)\overline{n_c^2(t)} = (1/4)\overline{n_s^2(t)} = (1/4)\overline{n_i^2(t)} = (1/4)N_i \tag{385}$$

Combining these relationships, we obtain,

$$\frac{S_o}{N_o} = 2\frac{S_i}{N_i} \tag{386}$$

This means that the detector in DSB-SC improves the S/N by a factor of two. This improvement results from the fact that the synchronous detector rejects the quadrature components of the input noise, thereby reducing the noise power by a factor of two.

For the synchronous detection of DSB-LC, one should substitute $f(t)$ with $A+f(t)$. This results in,

$$S_i = \frac{1}{2}A^2 + \frac{1}{2}\overline{f^2(t)} \tag{387}$$

$$S_o = \overline{[(1/2)f(t)]^2} = (1/4)\overline{f^2(t)} \tag{388}$$

The output noise (after multiplication by cosine and low pass filtering) is equal to $(1/2)n_c(t)$ resulting in the output noise power,

$$N_o = \overline{n_o^2(t)} = (1/4)\overline{n_c^2(t)} = (1/4)\overline{n_s^2(t)} = (1/4)\overline{n_i^2(t)} = (1/4)N_i \tag{389}$$

and, consequently,

$$\frac{S_o}{N_o} = \frac{2\overline{f^2(t)}}{A^2 + \overline{f^2(t)}} \times \frac{S_i}{N_i} \tag{390}$$

6.19.3 SSB-SC

For a SSB-SC, we have,

$$\phi(t) = f(t)\cos\omega_c t \pm \hat{f}(t)\sin\omega_c t \tag{391}$$

$$S_i = \overline{\phi^2(t)} = \frac{1}{2}\overline{f^2(t)} + \frac{1}{2}\overline{\hat{f}^2(t)} \tag{392}$$

As $\hat{F}(\omega)$ has only a phase shift with respect to $F(\omega)$, we have,

$$|F(\omega)|^2 = |\hat{F}(\omega)|^2 \tag{393}$$

which using Parseval Theorem results in,

$$\overline{f^2(t)} = \overline{\hat{f}^2(t)} \tag{394}$$

and, finally,

$$S_i = \overline{f^2(t)} \tag{395}$$

The useful output is $(1/2)f(t)$, so that,

$$S_o = \overline{[(1/2)f(t)]^2} = (1/4)\overline{f^2(t)} = (1/4)S_i \tag{396}$$

The output noise (after multiplication by cosine and low pass filtering) is equal to $(1/2)n_c(t)$ resulting in the output noise power,

$$N_o = \overline{n_o^2(t)} = (1/4)\overline{n_c^2(t)} = (1/4)\overline{n_s^2(t)} = (1/4)\overline{n_i^2(t)} = (1/4)N_i \tag{397}$$

Combining these relations, we obtain,

$$\frac{S_o}{N_o} = \frac{S_i}{N_i} \tag{398}$$

Question: Is the noise performance of DSB-SC system superior to that of SSB-SC?

Answer: Not where the noise power is proportional to the bandwidth because the DSB-SC requires twice the bandwidth of the SSB-SC and therefore has twice the noise power.

DSB-LC: The Envelope Detector

The signal input and the noise can be written as: