Figure 42: Spectra of SSB in modulating a sinusoid.

The corresponding spectrum are shown in Fig. 42.

Similarly, we have,

$$\phi_{SSB-}(t) = \cos \omega_m t \cos \omega_c t + \sin \omega_m t \sin \omega_c t$$  \hspace{1cm} (348)

Considering a more complicated signal, say \( f(t) \), as the sum of sinusoids, we have,

$$\phi_{SSB\mp}(t) = f(t) \cos \omega_c t \pm \dot{f}(t) \sin \omega_c t$$  \hspace{1cm} (349)

where \( \dot{f}(t) \) is the signal obtained by shifting the phase of \( f(t) \) by \( \pi/2 \) at each frequency.

The corresponding block diagram is shown in Fig. 43. In practice, it is very difficult to design a circuit which results in exactly \( \pi/2 \) phase shift for all the frequencies.

6.15 Analytic signals and Hilbert Transform

In general, any real valued signal can be expressed in terms of a complex signal with one sideband. Such a signal is called an analytic signal. Note that all the analytic signals are complex valued but the reverse in not necessarily true.

Assume that the real signal \( f(t) \) corresponds to the analytic signal

$$z(t) = f(t) + j\dot{f}(t) \rightarrow Z(\omega) = F(\omega) + j\dot{F}(\omega)$$  \hspace{1cm} (350)
To delete one of the sidebands (negative frequency), we should have,

\[
\hat{F}(\omega) = \begin{cases} 
-jF(\omega), & \omega > 0 \\
jF(\omega), & \omega < 0 
\end{cases} \tag{351}
\]

or,

\[
\hat{F}(\omega) = -jF(\omega)\text{sgn}(\omega) \tag{352}
\]

This results in,

\[
Z(\omega) = \begin{cases} 
2F(\omega), & \omega > 0 \\
0, & \omega < 0 
\end{cases} \tag{353}
\]

Taking the inverse Fourier Transform, we obtain,

\[
\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\omega)e^{j\omega t}d\omega \tag{354}
\]

The function \(\hat{f}(t)\) is called the quadrature pair, or the Hilbert Transform, of \(f(t)\) because each frequency component of \(\hat{f}(t)\) is in phase quadrature (\(\pi/2\) phase difference) with that of \(f(t)\).
We know that: (i) The analytic signal $z(t)$ has a one-sided spectrum. (ii) The spectrum of $z(t)e^{j\omega t}$ is the same as the spectrum of $z(t)$ but centered around $\omega_c$. (iii) Taking the real part of $z(t)e^{j\omega t}$ results in a spectrum which is symmetrical with respect to the origin. This spectrum corresponds to a SSB signal, i.e.,

$$\mathcal{R}\{z(t)e^{j\omega t}\} = \mathcal{R}\{[f(t) + j\hat{f}(t)]e^{j\omega t}\} = f(t)\cos\omega_c t - \hat{f}(t)\sin\omega_c t$$  \hspace{1cm} (355)

This relationships is for the upper band case. Using $z^*(t)$ instead of $z(t)$ results in lower band case, i.e., $f(t)\cos\omega_c t + \hat{f}(t)\sin\omega_c t$.

To obtain the Hilbert relationship in the time domain, we have,

$$\text{sgn}(\omega) \implies \frac{j}{\pi t}$$  \hspace{1cm} (356)

resulting in,

$$\hat{f}(t) = f(t)*\frac{1}{\pi t} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t-\tau} d\tau$$  \hspace{1cm} (357)

### 6.16 Demodulation of SSB signals

Demodulation is achieved using synchronous detection.

$$\phi_{\text{SSB}_\pi}(t) = f(t)\cos\omega_c t \pm \hat{f}(t)\sin\omega_c t$$  \hspace{1cm} (358)

Assume that the demodulation is achieved using the signal $\phi_d(t) = \cos[(\omega_c + \Delta\omega)t + \theta]$ where $\Delta\omega$ and $\theta$ are the frequency and the phase error, respectively. This results in,
\[ \phi_{SSB}(t)\phi_d(t) = [f(t) \cos \omega_c t \pm \hat{f}(t) \sin \omega_c t] \cos[(\omega_c + \Delta \omega)t + \theta] = \]
\[ \frac{1}{2} f(t) \{ \cos((\Delta \omega)t + \theta) + \cos((2\omega_c + \Delta \omega)t + \theta) \} \mp \frac{1}{2} \hat{f}(t) \{ \sin((\Delta \omega)t + \theta) - \sin((2\omega_c + \Delta \omega)t + \theta) \} \]
(359)

Using a lowpass filter to eliminate the double-carrier frequency terms, we obtain,
\[ e_o(t) = \frac{1}{2} f(t) \cos((\Delta \omega)t + \theta) \mp \frac{1}{2} \hat{f}(t) \sin((\Delta \omega)t + \theta) \]
(360)

For \(\Delta \omega = 0\) and \(\theta = 0\), we obtain,
\[ e_o(t) = (1/2)f(t) \]
(361)

To study the effect of the phase error, for \(\Delta \omega = 0\), we obtain,
\[ e_o(t) = \frac{1}{2} [f(t) \cos(\theta) \mp \hat{f}(t) \sin(\theta)] \]
(362)

Note that for DSB-SC carrier, the demodulator output in the case of the phase error was equal to: \(e_o(t) = \frac{1}{2} f(t) \cos(\theta)\). For a constant \(\theta\), the effect of the phase error acts just as a scale factor. However, for SSB-SC, the phase error is not just a scale factor and the degradation is more serious. To investigate this effect, let us write,
\[ e_o(t) = \frac{1}{2} \mathcal{R}\{[f(t) \mp j\hat{f}(t)]e^{j\theta}\} \]
(363)

while, previously, we had,
\[ e_o(t) = \frac{1}{2} \mathcal{R}\{[f(t) \mp j\hat{f}(t)]\} \]
(364)

This means that \(\theta\) acts as a phase distortion. It turns out that the human ear is not very sensitive to phase distortion.

To study the effect of the frequency error, for \(\theta = 0\), we obtain,
\[ e_o(t) = \frac{1}{2} [f(t) \cos(\Delta \omega)t \mp \hat{f}(t) \sin(\Delta \omega)t] \]
(365)

\[ e_o(t) = \frac{1}{2} \mathcal{R}\{[f(t) \mp j\hat{f}(t)]e^{j\Delta \omega t}\} \]
(366)

The frequency error results in a spectral shift in the demodulate output.
6.17 Single Sideband Large Carrier, SSB-LC

This is a signal of the form,

\[ \phi(t) = A \cos \omega_c t + f(t) \cos \omega_c t = \hat{f}(t) \sin \omega_c t \]  

(367)

The corresponding envelope is equal to,

\[ r(t) = \sqrt{[A + f(t)]^2 + [\hat{f}(t)]^2} \]  

(368)

Assuming that the carrier is much larger than the SSB-SC envelope, we obtain,

\[ r(t) \approx A \sqrt{1 + \frac{2f(t)}{A}} \]  

(369)

Using binomial expansion, we obtain,

\[ r(t) \approx A \left[ 1 + \frac{f(t)}{A} \right] = A + f(t) \]  

(370)

This means that such a signal can be detected using an envelope detector. However, the amount of carrier required for the envelope detection is substantially more than the case of the DSB-LC.

6.18 Vestigial-side band (VSB) modulation

The generation of SSB may be quite difficult when the modulating signal bandwidth is wide or when one can not disregard the low-frequency components of the signal. As an intermediate solution, VSB modulation is used which provides a compromise between DSB and SSB. In VSB modulation, one sideband and a portion of the other sideband is transmitted such that the demodulation process reproduces the original signal. If the vestigial filter is equal to \( H_v(\omega) \), we obtain,

\[ \Phi_{VSB}(\omega) = \left[ \frac{1}{2} F(\omega - \omega_c) + \frac{1}{2} F(\omega + \omega_c) \right] H_v(\omega) \]  

(371)

The output of a synchronous detector is equal to,

\[ e_0(t) = [\phi_{VSB}(t) \cos \omega_c t]_{LP} \]  

(372)
Or,
\[ E_0(\omega) = \frac{1}{4} F(\omega) H_v(\omega + \omega_c) + \frac{1}{4} F(\omega) H_v(\omega - \omega_c) \]  
(373)

For a faithful reproduction, we need,
\[ [H_v(\omega - \omega_c) + H_v(\omega + \omega_c)]_{LP} = \text{constant}, \quad |\omega| < \omega_m \]  
(374)

Note that (374) is satisfied on a magnitude basis if \( |H_v(\omega)| \) is antisymmetric with respect to \( \omega_c \). Motivated by this observation, we let the constant in (374) to be \( 2H_v(\omega_c) \), resulting in,
\[ [H_v(\omega - \omega_c) - H_v(\omega_c)] = -[H_v(\omega + \omega_c) - H_v(\omega_c)] \]  
(375)

The corresponding spectrums are shown in Fig. 44.

### 6.19 Noise is Amplitude Modulation

#### 6.19.1 Bandpass noise

Consider a noise, \( n(t) \), which has a power spectral density centered around the frequency \( \omega_0 \). We can write,
\[ n(t) = \mathcal{R}\{[n_c(t) + jn_s(t)]e^{j\omega_0 t}\} = n_c(t) \cos \omega_0 t - n_s(t) \sin \omega_0 t \]  
(376)

where \( n_c(t), n_s(t) \) are low pass noise with a bandwidth equal to one-half of the bandwidth of \( n(t) \).

If we apply the bandpass noise to a synchronous detector (multiplying it by \( \cos \omega_0 t \)), we obtain,
\[ n(t) \cos \omega_0 t = n_c(t) \cos^2 \omega_0 t - n_s(t) \sin \omega_0 t \cos \omega_0 t = \]
\[ \frac{1}{2} n_c(t) + \frac{1}{2} n_c(t) \cos 2\omega_0 t - \frac{1}{2} n_s(t) \sin 2\omega_0 t \]  
(377)

Retaining the low pass term, we obtain,
\[ [n(t) \cos \omega_0 t]_{LP} = \frac{1}{2} n_c(t) \]  
(378)

Using \( \cos \omega_0 t = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2 \), it is easy to show that the power spectral density of \( n(t) \cos \omega_0 t \) is equal to: \([S_n(\omega + \omega_0) + S_n(\omega - \omega_0)]/4 \). Substituting in (378), and noting
Figure 44: Different spectrums in VSB filtering.
Figure 45: Spectrum of a bandpass noise.

that the power spectral density of $n_c(t)/2$ is equal to, $S_{n_c}(\omega)/4$, we obtain,

$$S_{n_c}(\omega) = [S_n(\omega + \omega_0) + S_n(\omega - \omega_0)]_{1P} \quad (379)$$

Similarly, by multiplying both sides of (376) with $\sin \omega_0 t$, we obtain,

$$S_{n_s}(\omega) = [S_n(\omega + \omega_0) + S_n(\omega - \omega_0)]_{1P} \quad (380)$$

An example is given in Fig. 45.

Referring to (379) and (380), we conclude that the power in the sine and cosine components of the noise are the same, and,

$$\overline{n_c^2(t)} = \overline{n^2(t)} = \overline{n_s^2(t)} \quad (381)$$

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or, equivalently,

\[ \bar{n}^2(t) = \frac{1}{2} n_x^2(t) + \frac{1}{2} n_y^2(t) \]  

(382)

6.19.2 DSB-SC

The input signal is \( f(t) \cos \omega_c t \) resulting in,

\[ S_i = \left[ f(t) \cos \omega_c t \right]^2 = \frac{1}{2} f^2(t) \]  

(383)

The useful output signal (after multiplication by cosine and low pass filtering) is equal to \((1/2)f(t)\) resulting in,

\[ S_o = \left[ (1/2)f(t) \right]^2 = (1/4)f^2(t) = (1/2)S_i \]  

(384)

Let \( N_i = \bar{n}^2(t) \) denote the input noise power. The output noise (after multiplication by cosine and low pass filtering) is equal to \((1/2)n_c(t)\) resulting in the output noise power,

\[ N_o = \bar{n}_o^2(t) = (1/4)\bar{n}_x^2(t) = (1/4)\bar{n}_y^2(t) = (1/4)n_i^2(t) = (1/4)N_i \]  

(385)

Combining these relationships, we obtain,

\[ \frac{S_o}{N_o} = 2 \frac{S_i}{N_i} \]  

(386)

This means that the detector in DSB-SC improves the \( S/N \) by a factor of two. This improvement results from the fact that the synchronous detector rejects the quadrature components of the input noise, thereby reducing the noise power by a factor of two.

For the synchronous detection of DSB-LC, one should substitute \( f(t) \) with \( A + f(t) \). This results in,

\[ S_i = \frac{1}{2} A^2 + \frac{1}{2} f^2(t) \]  

(387)

\[ S_o = \left[ (1/2)f(t) \right]^2 = (1/4)f^2(t) \]  

(388)

The output noise (after multiplication by cosine and low pass filtering) is equal to \((1/2)n_c(t)\) resulting in the output noise power,

\[ N_o = \bar{n}_o^2(t) = (1/4)\bar{n}_x^2(t) = (1/4)\bar{n}_y^2(t) = (1/4)n_i^2(t) = (1/4)N_i \]  

(389)

and, consequently,

\[ \frac{S_o}{N_o} = \frac{2f^2(t)}{A^2 + f^2(t)} \times \frac{S_i}{N_i} \]  

(390)
6.19.3 SSB-SC

For a SSB-SC, we have,

\[ \phi(t) = f(t) \cos \omega_c t \pm \tilde{f}(t) \sin \omega_c t \]  
\[ S_i = \overline{\phi^2(t)} = \frac{1}{2} \overline{f^2(t)} + \frac{1}{2} \overline{\tilde{f}^2(t)} \]  
\[ |F(\omega)|^2 = |\hat{F}(\omega)|^2 \]

which using Parseval Theorem results in,

\[ \overline{f^2(t)} = \overline{\tilde{f}^2(t)} \]

and, finally,

\[ S_i = \overline{f^2(t)} \]

The useful output is \((1/2)f(t)\), so that,

\[ S_o = \overline{[(1/2)f(t)]^2} = (1/4) \overline{f^2(t)} = (1/4) S_i \]

The output noise (after multiplication by cosine and low pass filtering) is equal to \((1/2)n_e(t)\) resulting in the output noise power,

\[ N_o = \overline{n^2_e(t)} = (1/4) \overline{n^2_e(t)} = (1/4) \overline{n^2_e(t)} = (1/4) n_i \]

Combining these relations, we obtain,

\[ \frac{S_o}{N_o} = \frac{S_i}{N_i} \]

**Question:** Is the noise performance of DSB-SC system superior to that of SSB-SC?

**Answer:** Not where the noise power is proportional to the bandwidth because the DSB-SC requires twice the bandwidth of the SSB-SC and therefore has twice the noise power.

[DSP-LC: The Envelope Detector]

The signal input and the noise can be written as: