5.5.4 Average value

Assume that:

$$f(t) = x(t) + m_1$$
$$g(t) = y(t) + m_2 \tag{269}$$

where the average values of $x(t)$ and $y(t)$ is zero. We have:

$$R_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} [x^*(t) + m_1][y(t+\tau) + m_2]dt \tag{270}$$

As the average value of $x(t)$ and $y(t)$ is zero, we obtain,

$$R_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)y(t+\tau)dt + m_1m_2 \tag{271}$$

This means that the average value of the cross correlation function is,

$$\overline{R_{fg}(\tau)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left[\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)y(t+\tau)dt + m_1m_2\right] d\tau \tag{272}$$

$$\overline{R_{fg}(\tau)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^*(t) \left[\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} y(t+\tau)d\tau\right] dt + m_1m_2 \tag{273}$$

In this case, as the average value of $y(t+\tau)$ is equal to zero, i.e., $\lim_{T\to\infty}(1/T)\int_{-T/2}^{T/2} y(t+\tau)d\tau = 0$, then we obtain,

$$\overline{R_{fg}(\tau)} = m_1m_2 \tag{274}$$

The average alue of the cross-correlation of two functions is equal to the product of their average values.

5.5.5 Maximum value

We want to show that,

$$\begin{aligned}\left|\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt\right|^2 &\leq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2dt \times \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |f(t+\tau)|^2dt \\ &\leq R_f(0)R_f(0)\end{aligned} \tag{275}$$

We know that for any two vectors $\mathbf{x}, \mathbf{y}$, we have,

$$|\mathbf{x}.\mathbf{y}|^2 = (|\mathbf{x}||\mathbf{y}|\cos\theta)^2 \leq |\mathbf{x}|^2|\mathbf{y}|^2 \tag{276}$$

In an analogy to vector spaces, if we consider the left-hand side of (275) as the inner product of the two functions $f(t+\tau)$ and $f(t)$, relationship (275) turns out to be equivalent to (276).

Relationship (275) results in,

$$R_f(\tau) \leq R_f(0) \tag{277}$$

5.5.6 Additivity

Assume that $z(t) = x(t) + y(t)$, we have,

$$R_z(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}[x^*(t) + y^*(t)][x(t+\tau) + y(t+\tau)]dt. \tag{278}$$

Multiplying the two brackets under the integral and computing the integral over each of the resulting fours terms, we obtain,

$$R_z(\tau) = R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau) \tag{279}$$

For $R_{xy}(\tau) = 0$, $R_{yx}(\tau) = 0$, we obtain,

$$R_z(\tau) = R_x(\tau) + R_y(\tau) \tag{280}$$

If $R_{xy}(\tau) = 0$, we say that x and y are uncorrelated.

It can be shown that $R_{yx}(\tau) = R^*_{xy}(-\tau)$, so that if $R_{xy}(\tau) = 0$, then $R_{yx}(\tau) = 0$. One conclusion is that if two signals are uncorrelated then their power spectral densities are additive. Note that power spectral density is the Fourier transform of the autocorrelation function and as for uncorrelated signals autocorrelation function is additive, then the power spectral density is also additive.

The concept of correlation can be extended to finite energy signals. Assuming $f(t)$, $g(t)$ are of finite energy, we define,

$$r_f(\tau) \equiv \int_{-\infty}^{\infty} f^*(t)f(t+\tau)dt \tag{281}$$

$$r_{fg}(\tau) = \int_{-\infty}^{\infty} f^*(t)g(t+\tau)dt \tag{282}$$

We have:

$$\mathcal{F}\{r_f(\tau)\} = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f^*(t)f(t+\tau)dt\right] e^{-j\omega\tau}d\tau \tag{283}$$

$$\mathcal{F}\{r_f(\tau)\} = \int_{-\infty}^{\infty} f^*(t) \int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega\tau}d\tau dt \tag{284}$$

$$\mathcal{F}\{r_f(\tau)\} = \int_{-\infty}^{\infty} f^*(t)e^{j\omega t}dt \int_{-\infty}^{\infty} f(t+\tau)e^{-j\omega(t+\tau)}d\tau \tag{285}$$

$$\mathcal{F}\{r_f(\tau)\} = F^*(\omega)F(\omega) = |F(\omega)|^2 \tag{286}$$

5.6 Band-limited white noise

White Noise has a flat power spectrum:

$$S_n(\omega) = \eta/2 \tag{287}$$

This can not exist in reality because the total power is infinity:

$$\overline{n^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\eta/2)d\omega \longrightarrow \infty \tag{288}$$

In practical cases, we deal with band-limited white noise. In other words, if the power spectral density of noise is flat over the bandwidth of the system under consideration (bandwidth of interest), the noise *appears* to be white. For a bandwidth $[-B, B]$, the noise power is:

$$P_n = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B}(\eta/2)d\omega = \eta B \qquad W. \tag{289}$$

If this noise is developed across a resistor R, the mean-square noise voltage is:

$$RP_n = \eta RB \qquad V^2 \tag{290}$$

and the mean-square noise current is,

$$P_n/R = \eta B/R = \eta BG \qquad \text{amp}^2 \tag{291}$$

Effect of linear systems on noise

Assume that $n_i(t)$ and $n_o(t)$ are the noise at input and output of a linear system $H(\omega)$. We have:

$$S_{n_o}(\omega) = S_{n_i}(\omega)|H(\omega)|^2 = \frac{\eta}{2}|H(\omega)|^2 \tag{292}$$

$$\overline{|n_o(t)|^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_o}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_i}(\omega)|H(\omega)|^2 d\omega \tag{293}$$

$$\overline{|n_o(t)|^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\eta}{2}|H(\omega)|^2 d\omega = \frac{\eta}{2\pi}\int_{0}^{\infty} |H(\omega)|^2 d\omega \tag{294}$$

5.6.1 Thermal Noise

Thermal noise is produced as a result of the thermaly excited random motion of free electrons in a conducting medium such as a resistor.

Power spectral density of thermal noise:

$$S_n(\omega) = \frac{h|\omega|}{\pi[\exp(h|\omega|/2\pi kT) - 1]} \quad \text{Watts per Hz} \tag{295}$$

$$S_n(\omega) \simeq 2kT \quad \text{for} \quad |\omega| \ll 2\pi kT/h \tag{296}$$

where

$$\begin{aligned} T &= \text{temperature of the conducting medium in Kelvin (K)} \\ k &= \text{Boltzman's constant} = 1.38 \times 10^{-23} \text{ joule/K}, \\ h &= \text{Planck's constant} = 6.625 \times 10^{-34} \text{ joul-sec} \end{aligned} \tag{297}$$

For very high frequencies, thermal noise is not really white however these frequencies are so high that we can essentially assume that it is white.

The mean-square (open-circuit) voltage generated by a resistor R in a bandwidth $[-B, B]$ is equal to:

$$\overline{v^2(t)} = RP_n = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} 2kTR d\omega = 4kTRB, \qquad V^2 \tag{298}$$

Similarly, the mean-square (short-circuit) current is:

$$\overline{i^2(t)} = P_n/R = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} 2kT/R d\omega = 4kTB/R = 4kTGB, \qquad V^2 \tag{299}$$

The equivalent circuit models for the thermal noise are shown in Fig. 21.

Effect of linear systems on thermal noise

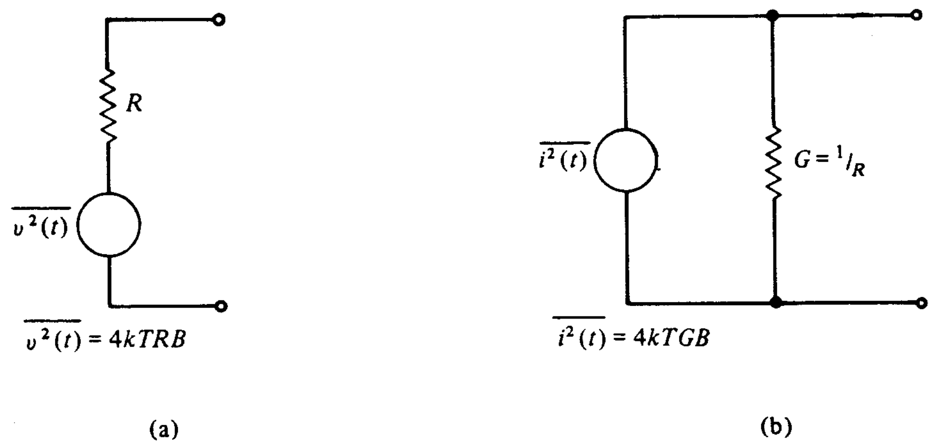

Figure 21: Equivalent circuit models for the thermal noise. (a) Voltage model, (b) Current model.

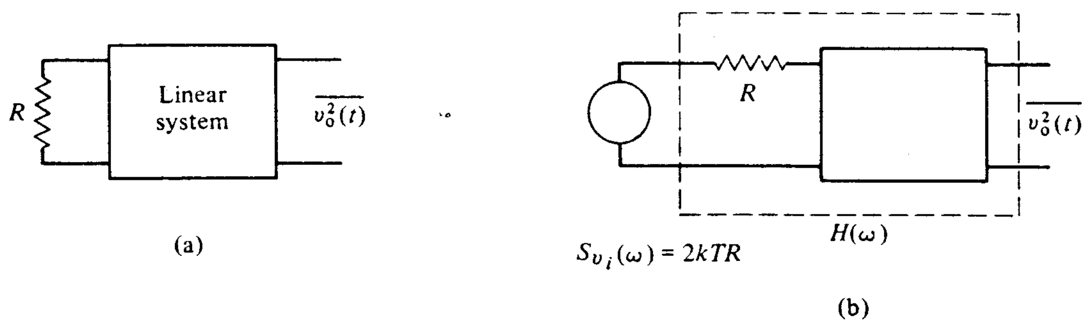

Figure 22: Transmission of thermal noise through a linear system.

For the system in Fig. 22, we have,

$$S_{v_i}(\omega) = 2kTR \tag{300}$$

$$S_{v_o}(\omega) = S_{v_i}(\omega)|H(\omega)|^2 \tag{301}$$

The mean-square output voltage is:

$$\overline{v_o^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{v_i}(\omega)|H(\omega)|^2 d\omega \tag{302}$$

Effective Noise resistance

Assume that the complex impedance at the input terminal of a circuit is equal to: $Z(\omega)$. Then, the effective noise resistance referred to those terminals is equal to:

$$R_{eq}(\omega) = \mathcal{R}\{Z(\omega)\} \tag{303}$$

resulting in a noise voltage of spectral density:

$$S_v(\omega) = 2kTR_{eq}(\omega) \tag{304}$$

Note that in general R_{eq} is a function of frequency and the noise is not white.

5.7 Equivalent noise bandwidth

Assume that a white noise of power spectral density $\eta/2$ is passed through a linear system $H(\omega)$. The mean-square output voltage (across a one-ohm resistance) is,

$$\overline{v_0^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\eta}{2}|H(\omega)|^2 d\omega = \frac{\eta}{4\pi}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega = \frac{\eta}{2\pi}\int_{0}^{\infty}|H(\omega)|^2 d\omega \tag{305}$$

Assume that the mid-band frequency of the system is equal to ω_0 (for a low pass filter we have $\omega_0 = 0$). The equivalent noise bandwidth, B_N, is defined as the bandwidth of an ideal filter with the mid-band gain of $H(\omega_0)$ and with the same noise power as the actual system. This definition of equivalent noise bandwidth allows us to discuss the noise behavior of the practical linear systems by using their idealized equivalents. For the case of the hypothetical ideal filter, we have,

$$\overline{v_0^2(t)} = \frac{1}{2\pi}\int_{-2\pi B_N}^{2\pi B_N}\frac{\eta}{2}|H(\omega_0)|^2 d\omega = \eta|H(\omega_0)|^2 B_N \tag{306}$$

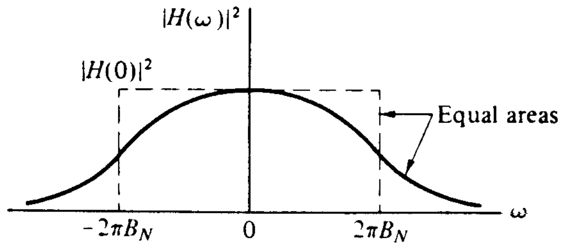

Figure 23: A graphic interpretation of noise equivalent band-width.

Comparing (305) and (306), we obtain,

$$B_N = \frac{1}{2\pi} \frac{\int_0^\infty |H(\omega)|^2 d\omega}{|H(\omega_0)|^2} \tag{307}$$

A graphic interpretation of noise equivalent band-width is shown in Fig. 23.

5.8 Available power and noise temperature

We know that the thermal noise power in a resistor R is,

$$P_n = 4kTB \tag{308}$$

Question: How much of this power can be extracted?

Using a matched resistor R (noise free), we can transfer one-half of the voltage resulting in one-fourth of the power. This means that the maximum available power, P_a, is,

$$P_a = kTB \tag{309}$$

The noise behavior of practical systems is specified in terms of their (i) *equivalent noise bandwidth*, (ii) *output resistance* and (iii) *noise temperature.* A system with the equivalent noise bandwidth B_N, output resistance R and the noise temperature T results in a noise power of kTB_N into a matched resistance (of value R). Note that in general the noise temperature may be less or greater than the ambient temperature.

5.9 Noise figure

The noise performance of systems is measure by the factor *Noise Figure*. This is a dimensionless quantity which can be used to compare the noise performance of different systems with each other.

Let $s_i(t), s_o(t)$ and the input, output signal voltages (or currents) in a system. Similarly, let $n_i(t), n_o(t)$ be the corresponding input, output noise voltages (or currents). The input Signal-to-Noise-Ratio $(S/N)_i$, is defined as:

$$\left(\frac{S}{N}\right)_i = \frac{\overline{s_i^2(t)}}{\overline{n_i^2(t)}} \tag{310}$$

The output Signal-to-Noise-Ratio $(S/N)_o$, is defined as:

$$\left(\frac{S}{N}\right)_o = \frac{\overline{s_o^2(t)}}{\overline{n_o^2(t)}} \tag{311}$$

Note that the Signal-to-Noise-Ratio is a dimensionless quantity.

The system always adds some noise to the input, so that the output Signal-to-Noise-Ratio is always higher that the input Signal-to-Noise-Ratio. This degradation is measured by the noise Figure which is defined as:

$$F \equiv \frac{(S/N)_o}{(S/N)_i}, \quad \text{or} \quad F_{\text{db}} = 10\log_{10}(F) \text{ in decibel} \tag{312}$$

Let us apply a signal of power S_i and a noise of temperature T_0 to the input of an amplifier with a power gain G_p, noise temperature T_e and noise bandwidth B. We have:

Maximum input noise power within the band-width of the amplifier: $N_i = kT_0B$

Equivalent input noise power generated by the amplifier : $N_e = kT_eB$

Total input noise power: $kT_0B + kT_eB$

Output noise power: $(kT_0B + kT_eB)G_p$

Output signal power: $S_o = S_iG_p$

$(S/N)_i = S_i/KT_0B$

$(S/N)_o = S_iG_p/(kT_0B + kT_eB)G_p = S_i/(kT_0B + kT_eB)$

This results in,

$$F = \frac{\left(\frac{S}{N}\right)_o}{\left(\frac{S}{N}\right)_i} = 1 + \frac{T_e}{T_0}$$

6 Modulation:

Modulation is a process in which a property or a parameter of a signal is varied in proportion to a second signal. For example, in amplitude modulation, the amplitude of a (high frequency) sine wave, whose frequency and phase are fixed, is varied in proportion to a given (low frequency) signal. This results in a translation of the frequency components of the given signal to higher frequencies.

Some examples for the application of modulation

1. The length of a transmitter antenna is determined by the wavelength of the signal to be transmitted (usually it is equal to a fraction, say 1/4, of the wavelength). The wavelength of low frequency signals, and consequently the length of the corresponding antenna, is very large (for example, the wavelength of voice is about 100 Km.). The frequency translating property of modulation can be used to solve this problem.

2. The frequency translating property of modulation can be used to match a given signal to the channel available for transmission (for example matching a low-pass signal to a band-pass channel).

3. Using modulation, a large number of signals can be transmitted at the same time without interference (frequency multiplexing).

4. Modulation can be used to facilitate the implementation of certain electrical systems. An example is given in Lab#1 in the design of a spectrum analyzer. More examples will be give later in the course.

General form of a modulated signal using a sinusoidal carrier

$$\phi(t) = a(t)\cos[\theta(t)] = a(t)\cos[\omega_c t + \gamma(t)] \tag{313}$$

where ω_c is called the carrier frequency. In general, we assume that $a(t)$ and $\gamma(t)$ are slowly varying with respect to $\omega_c t$.

In amplitude modulation, the phase $\gamma(t)$ in (313) is constant and the amplitude $a(t)$ is changed in proportion to the given signal.

In angle modulation, the amplitude $a(t)$ in (313) is constant and the phase $\gamma(t)$ is changed in proportion to the given signal.

6.1 Amplitude Modulation

General form:

$$\phi(t) = f(t)\cos\omega_c t \tag{314}$$

$\cos\omega_c t$ is called the carrier signal. $f(t)$ is called the modulating signal. The resulting $\phi(t)$ is called the modulated signal.

Using the Frequency shifting property of Fourier transform, we obtain,

$$\Phi(\omega) = \frac{1}{2}F(\omega + \omega_c) + \frac{1}{2}F(\omega - \omega_c) \tag{315}$$

This type of amplitude modulation is called *suppressed-carrier* because it has no identifiable carrier in it (although the spectrum is centered at the frequency $\omega_c \neq 0$). As both of the side-bands are present in the modulated signal, this type of modulation is called *double-sideband, suppressed-carrier*, DSB-SC, (refer to Fig. 24).

Recovery of the original signal from the DSB-SC requires another translation in frequency to shift the spectrum to its original position. This operation is called *demodulation*.

Demodulation:

$$\phi(t)\cos\omega_c t = f(t)\cos^2\omega_c t = \frac{1}{2}f(t) + \frac{1}{2}f(t)\cos 2\omega_c t \tag{316}$$

Computing the Fourier transform, we obtain:

$$\mathcal{F}\{\phi(t)\cos\omega_c t\} = \frac{1}{2}F(\omega) + \frac{1}{4}F(\omega + 2\omega_c) + \frac{1}{4}F(\omega - 2\omega_c) \tag{317}$$

A low-pass filter is required to separate out the unwanted terms at double-frequency (refer to Fig. 25). For proper recovery, we should have $\omega_c > W$.