Operation	$f(t)$ $\leftrightarrow$	$F(\omega)$
Linearity (superposition)	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
Complex conjugate	$f^*(t)$	$F^*(-\omega)$
Scaling	$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert} F\left(\frac{\omega}{\alpha}\right)$
Delay	$f(t - t_0)$	$e^{-j\omega t_0} F(\omega)$
Frequency translation	$e^{j\omega_0 t} f(t)$	$F(\omega - \omega_0)$
Amplitude modulation	$f(t) \cos \omega_0 t$	$\frac{1}{2}F(\omega + \omega_0) + \frac{1}{2}F(\omega - \omega_0)$
Time convolution	$\int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau)\, d\tau$	$F_1(\omega) F_2(\omega)$
Frequency convolution	$f_1(t) f_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(u) F_2(\omega - u)\, du$
Duality: time-frequency	$F(t)$	$2\pi f(-\omega)$
Symmetry: even-odd	$f_e(t)$	$F_e(\omega)$ [real]
	$f_o(t)$	$F_o(\omega)$ [imaginary]
Time differentiation	$\frac{d}{dt} f(t)$	$j\omega F(\omega)$
Time integration	$\int_{-\infty}^{t} f(\tau)\, d\tau$	$\frac{1}{j\omega} F(\omega) + \pi F(0)\, \delta(\omega)$, where $F(0) = \int_{-\infty}^{\infty} f(t)\, dt$

Table 2: Some important properties of the Fourier transform.

Table 2 summarizes some important properties of the Fourier transform.

4.8 Distortion-less transmission

For an input $f(t)$, we want the output to be equal to: $Kf(t - t_0)$, or $H(\omega) = Ke^{-j\omega t_0}$. This means constant magnitude response and linear phase in the frequency band of the input signal $f(t)$ (refer to Fig. 18).

Ideal low pass filter: The corresponding transfer function is,

$$H(\omega) = |H(\omega)| e^{j\Theta(\omega)} = \text{rect}\left(\frac{\omega}{2W}\right) e^{-j\omega t_0} \qquad (221)$$

This results in the following impulse response,

$$h(t) = \mathcal{F}^{-1}\{H(\omega)\} = \frac{W}{\pi} \text{Sa}[W(t - t_0)] \qquad (222)$$

(refer to Fig. 19).

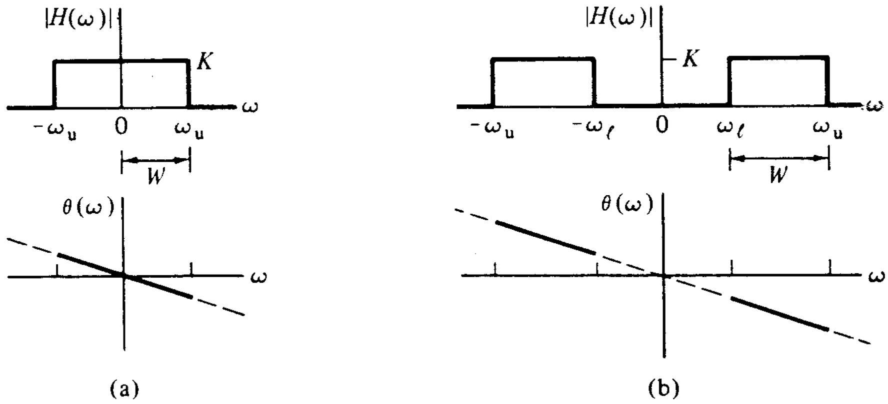

Figure 18: The ideal filter: (a) low-pass, (b) bandpass.

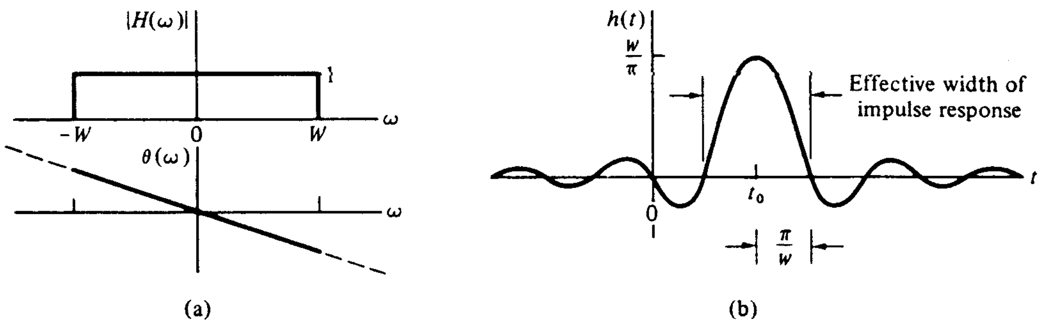

Figure 19: The ideal low-pass filter: (a) transfer function, (b) impulse response.

It is seen that the impulse response is non-causal (physically un-realizable). In general, for any $H(\omega)$ satisfying:

$$\int_{-\infty}^{\infty} |H(\omega)|^2 d\omega < \infty \tag{223}$$

a necessary and sufficient conditions for the filter to be realizable is the so-called Paley-Wiener conditions which is:

$$\int_{-\infty}^{\infty} |\frac{\ln |H(\omega)|}{1+\omega^2}| d\omega < \infty \tag{224}$$

The ideal filter is non-realizable because it is zero for a nonzero range of frequencies. In practice, it is also impossible to obtain infinitely steep cutoff in the frequency domain.

4.9 Sampling theorem

Question: Can we change an analog signal into a discrete one without loosing any information?

Basic Theorem: Any point of a real-valued band-limited signal having no spectral component above a given frequency of B Hz can be uniquely determined using $2B$ of its uniformly-spaced samples per second.

Proof: Optimum sampling of a given function $f(t)$ is achieved by multiplying $f(t)$ with $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$.

$$f(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} f(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) \tag{225}$$

In practice, we use a sampling function $P_T(t) = \sum_{n=-\infty}^{\infty} P(t - nT)$ for a proper $P(t)$, for example $P(t) = \text{rect}(t/\tau)$.

For a give $P(t)$, as $P_T(t)$ is periodic, we can write:

$$P_T(t) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t}, \qquad \omega_0 = 2\pi/T \tag{226}$$

Then,

$$f_s(t) = f(t)P_T(t) = f(t) \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t} \tag{227}$$

Taking the Fourier transform:

$$\begin{aligned}\mathcal{F}\{f_s(t)\} &= \mathcal{F}\left\{f(t)\sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t}\right\} \\ &= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} P_n f(t) e^{jn\omega_0 t}\right\} \\ &= \sum_{n=-\infty}^{\infty} P_n \mathcal{F}\{f(t)e^{jn\omega_0 t}\}\end{aligned} \tag{228}$$

We know that $\mathcal{F}\{f(t)e^{jn\omega_0 t}\} = F(\omega - n\omega_0)$. This results in:

$$\mathcal{F}\{f_s(t)\} = \sum_{n=-\infty}^{\infty} P_n F(\omega - n\omega_0) = P_0 F(\omega) + \sum_{n=-\infty, n\neq 0}^{\infty} P_n F(\omega - n\omega_0) \tag{229}$$

The spectrum of $F(\omega)$, Fourier transform of $f(t)$, can be recovered using a low pass filter (refer to Fig. 20). To avoid the overlap between the frequency replicas, we should have: $2\pi/T \geq 2W$, or since, $W = 2\pi B$, then,

$$T \leq \frac{1}{2B} \tag{230}$$

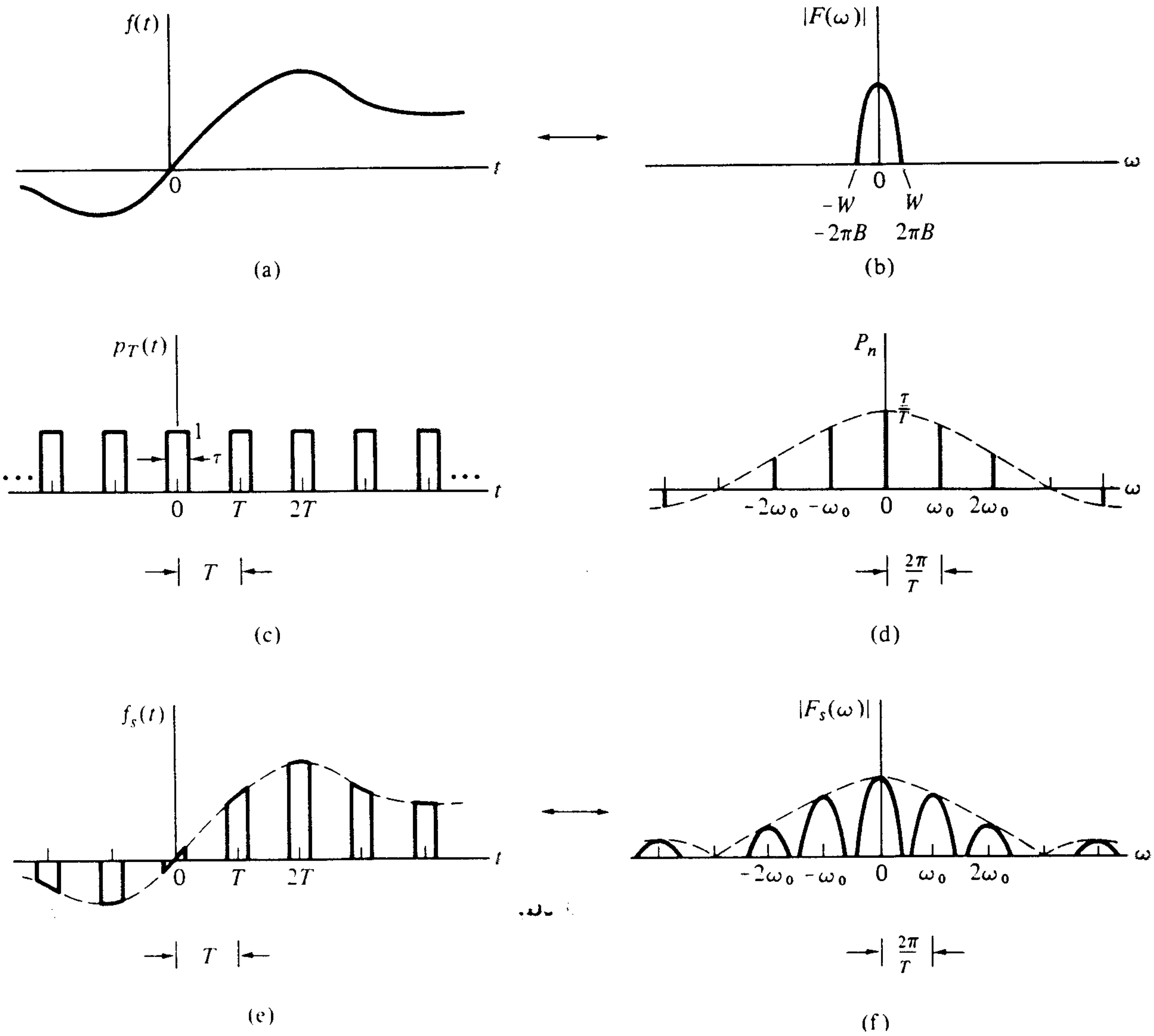

Figure 20: Sampling of a band-limited signal.

5 Spectral Density and Correlation

5.1 Energy spectral density

Parseval theorem for energy signals

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \tag{231}$$

The $|F(\omega)|^2$ is called the energy spectral density (or simply the energy density) of $f(t)$. The $(1/2\pi)|F(\omega)|^2 \Delta\omega$ is the amount of energy in the small frequency band of $\Delta\omega$ around frequency ω.

Effect of linear filtering on energy spectral density Consider a linear system H. Assume that F is the input to the linear system H, corresponding to the output G, we have,

$$G(\omega) = F(\omega)H(\omega) \tag{232}$$

The energy density of $g(t)$ is equal to:

$$|G(\omega)|^2 = |F(\omega)|^2 |H(\omega)|^2 \tag{233}$$

This results in the total energy,

$$E_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 |H(\omega)|^2 d\omega \tag{234}$$

Note that in computing the energy and the energy density, all the phase information (in the input signal and the system transfer function) is lost. In other words, the two signals $|F(\omega)|e^{j\phi_1(\omega)}$ and $|F(\omega)|e^{j\phi_2(\omega)}$ have the same energy and energy density.
For a real-valued signal, the magnitude of the Fourier transform, $|F(\omega)|$, is an even function of frequency and, consequently, the energy density is also an even function of frequency.

5.2 Power spectral density

Time-averaged power:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt \tag{235}$$

(for a periodic signal, T is taken as one period.).

Given a function $f(t)$ and a time interval $[-T/2, T/2]$, we define,

$$f_{\sqcap_T}(t) = f(t)\text{rect}(t/T)$$
$$F_{\sqcap_T}(\omega) = \mathcal{F}\{f(t)\text{rect}(t/T)\} \tag{236}$$

Then,

$$\lim_{T\to\infty} \int_{-T/2}^{T/2} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_{\sqcap_T}(\omega)|^2 d\omega \tag{237}$$

This results in:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} |F_{\sqcap_T}(\omega)|^2 d\omega \tag{238}$$

We define the power spectral density (or simply power spectrum) of $f(t)$ as,

$$S_f(\omega) = \lim_{T\to\infty} \frac{1}{T} |F_{\sqcap_T}(\omega)|^2 \tag{239}$$

The average power of $f(t)$ is equal to,

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega) d\omega \tag{240}$$

Periodic signals: If $f(t)$ is a periodic signal, we have,

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{241}$$

For periodic signals, the Parseval theorem is of the form,

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2 \tag{242}$$

This results in a line power spectrum of the form:

$$S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0) \tag{243}$$

Effect of linear filtering on power spectrum

Linear filtering of the truncated function:

$$G_{\sqcap_T}(\omega) = F_{\sqcap_T}(\omega) H(\omega) \tag{244}$$

The power spectral density of the output signal is:

$$S_g(\omega) = \lim_{T\to\infty} \frac{|F_{\sqcap_T}(\omega)H(\omega)|^2}{T} \tag{245}$$

$$S_g(\omega) = \lim_{T\to\infty} \frac{|F_{\sqcap_T}(\omega)|^2}{T}|H(\omega)|^2 \tag{246}$$

Substituting,

$$S_f(\omega) = \lim_{T\to\infty} \frac{|F_{\sqcap_T}(\omega)|^2}{T} \tag{247}$$

we obtain,

$$S_g(\omega) = S_f(\omega)|H(\omega)|^2 \tag{248}$$

5.3 Representation of noise

The amplitude of a noise signal $n(t)$ at a give time instant is a random variable. In representing such a variable we have to be content with the average values.

Mean Value (dc value):

$$\overline{n(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} n(t)dt \tag{249}$$

Mean Square Value (time-averaged power):

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |n(t)|^2 dt \tag{250}$$

AC component, $\sigma(t)$:

$$\sigma(t) = n(t) - \overline{n(t)} \tag{251}$$

We have:

$$\begin{aligned}
\overline{n^2(t)} &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\overline{n(t)} + \sigma(t)\right|^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |\sigma(t)|^2 dt + \\
&\quad \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \overline{n(t)}\sigma^*(t)dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left[\overline{n(t)}\right]^* \sigma(t)dt \\
&= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |\sigma(t)|^2 dt + \\
&\quad \lim_{T\to\infty} \frac{\overline{n(t)}}{T}\int_{-T/2}^{T/2} \sigma^*(t)dt + \lim_{T\to\infty} \frac{\left[\overline{n(t)}\right]^*}{T}\int_{-T/2}^{T/2} \sigma(t)dt
\end{aligned} \tag{252}$$

Recall that,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} n(t)dt = \overline{n(t)} \tag{253}$$

This results in,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \sigma(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \sigma^*(t)dt = 0 \tag{254}$$

Then, we have,

$$\overline{n^2(t)} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \left|\overline{n(t)}\right|^2 dt + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |\sigma(t)|^2 dt \tag{255}$$

5.4 Correlation function

Question: Is there some operation in time domain which is equivalent to finding the power spectrum in the frequency domain?

For the power spectral density, we had:

$$S_f(\omega) = \lim_{T\to\infty}\frac{1}{T}|F_{\sqcap_T}(\omega)|^2 \tag{256}$$

$$\mathcal{F}^{-1}\{S_f(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{T}|F_{\sqcap_T}(\omega)|^2 e^{j\omega\tau}d\omega \tag{257}$$

$$\begin{aligned}\mathcal{F}^{-1}\{S_f(\omega)\} = \; & \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty} F^*_{\sqcap_T}(\omega)F_{\sqcap_T}(\omega)e^{j\omega\tau}d\omega \\ & \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f^*(t)e^{j\omega t}dt\int_{-T/2}^{T/2} f(t_1)e^{-j\omega t_1}dt_1\right]e^{j\omega\tau}d\omega \\ & \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\int_{-T/2}^{T/2} f(t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t_1+\tau)}d\omega\right]dt_1 dt\end{aligned} \tag{258}$$

The integration over ω within the bracket is equal to $\delta(t-t_1+\tau)$, so that we obtain,

$$\begin{aligned}\mathcal{F}^{-1}\{S_f(\omega)\} = \; & \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\int_{-T/2}^{T/2} f(t_1)\delta(t-t_1+\tau)dt_1 dt \\ & \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt\end{aligned} \tag{259}$$

We define the $\boxed{\text{autocorrelation function}}$ of $f(t)$ as,

$$R_f(\tau) \equiv \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt \tag{260}$$

This results in:

$$S_f(\omega) = \mathcal{F}\{R_f(\tau)\} \tag{261}$$

Cross correlation function

$$R_{fg}(\tau) \equiv \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)g(t+\tau)dt \tag{262}$$

Correlation functions are either a measure of similarity of a function either with itself (in the case of the autocorrelation) or with another signal (in the case of cross-correlation) versus a relative shift by an amount of τ.

5.5 Some properties of correlation functions

5.5.1 Symmetry

$$R_f(-\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t-\tau)dt \tag{263}$$

Using the change of variable $t = \psi + \tau$, we obtain,

$$R_f(-\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(\psi+\tau)f(\psi)d\psi \tag{264}$$

resulting in,

$$R_f(-\tau) = R_f^*(\tau) \tag{265}$$

5.5.2 Mean square value

$$R_f(0) \equiv \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^*(t)f(t)dt \tag{266}$$

$$R_f(0) = \overline{f^2(t)} \tag{267}$$

5.5.3 Periodicity

If $f(t+T) = f(t)$ for all t, then,

$$R_f(\tau+T) = R_f(\tau) \quad \text{for all} \tau \tag{268}$$