- **Duality**

We have,

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$
$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \tag{165}$$

Replacing t by $-t$ in the second relationship of (165), we obtain,

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega \tag{166}$$

By interchanging the role of t and ω in (166) and comparing the result with the first relationship in (165), we obtain,

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega) \tag{167}$$

An example of such duality property is shown in Fig. 13.

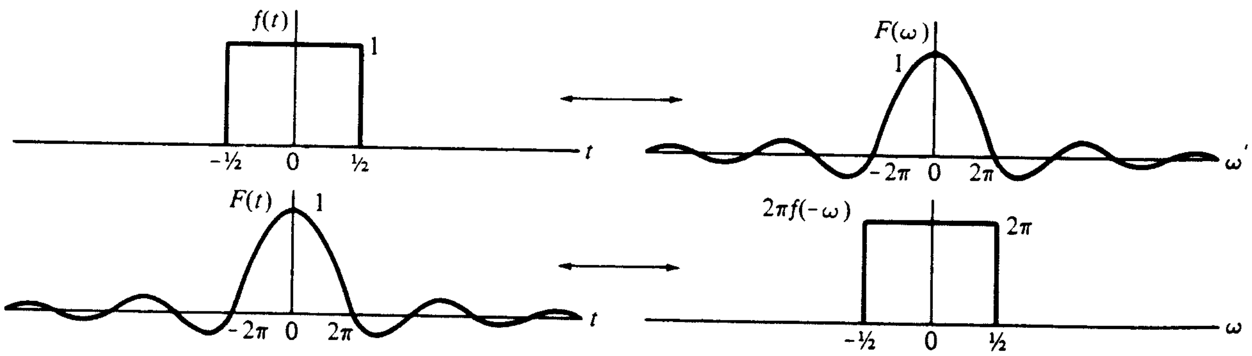


Figure 13: Duality of the Fourier transformation.

- Time scaling

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt \tag{168}$$

Using the change of variable $x = at$, we obtain,

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(x)e^{-j\omega x/a}dx/a = \frac{1}{a}F\left(\frac{\omega}{a}\right) \qquad a > 0 \tag{169}$$

When a is negative, the limits of the integral are reversed when we apply the change of the variable so that,

$$\mathcal{F}\{f(at)\} = -\frac{1}{a}F\left(\frac{\omega}{a}\right) \qquad a < 0 \tag{170}$$

or in general, we have,

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right) \tag{171}$$

If a is positive and greater than unity, $f(at)$ is a compressed version of $f(t)$ and its spectral density is expanded in frequency by $1/a$. In this case, the magnitude of the spectral density decreases and this is necessary to keep the total energy constant.

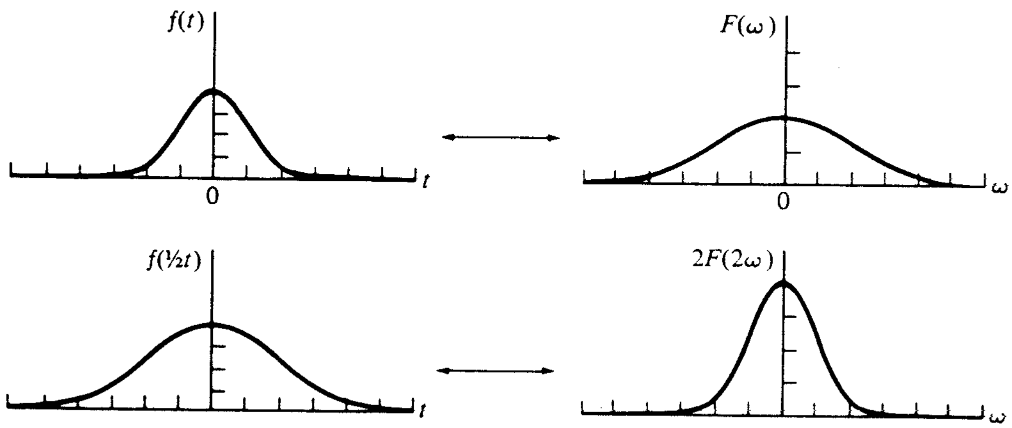


Figure 14: Effect of time scaling on Fourier transformation.

- Time shift

$$\mathcal{F}\{f(t-t_0)\} = ? \tag{172}$$

$$\mathcal{F}\{f(t-t_0)\} = \int_{-\infty}^{\infty} f(t-t_0)e^{-j\omega t}dt \quad (173)$$

Let $x = t - t_0$.

$$\begin{aligned}\mathcal{F}\{f(t-t_0)\} &= \int_{-\infty}^{\infty} f(x)e^{-j\omega(x+t_0)}dx \\ &= e^{-j\omega t_0}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx \\ &= e^{-j\omega t_0}F(\omega)\end{aligned} \quad (174)$$

- Frequency shift

$$\mathcal{F}\{f(t)e^{j\omega_0 t}\} = ? \quad (175)$$

$$\begin{aligned}\mathcal{F}\{f(t)e^{j\omega_0 t}\} &= \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt \\ &= F(\omega-\omega_0)\end{aligned} \quad (176)$$

$$f(t)\cos\omega_0 t = \frac{1}{2}f(t)[e^{j\omega_0 t} + e^{-j\omega_0 t}] \quad (177)$$

$$\mathcal{F}\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)] \quad (178)$$

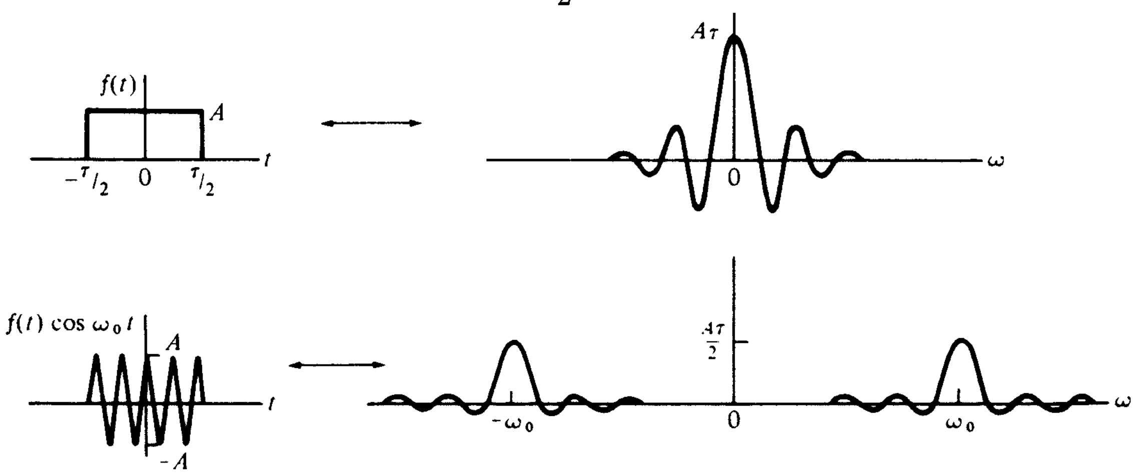


Figure 15: Frequency shifting property of multiplication by a sinusoid.

- Differentiation and Integration

$$\mathcal{F}\{\frac{d}{dt}f(t)\} = ? \tag{179}$$

We have,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \tag{180}$$

$$\begin{aligned}\frac{d}{dt}f(t) &= \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}[F(\omega)e^{j\omega t}]d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(\omega)e^{j\omega t}d\omega\end{aligned} \tag{181}$$

Comparing with the general definition of the Fourier transform, we obtain,

$$\mathcal{F}\left\{\frac{d}{dt}f(t)\right\} = j\omega F(\omega) \tag{182}$$

It is seen that the time differentiation enhances the high frequency components of a signal.

Similarly,

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega) \tag{183}$$

where

$$F(0) = \int_{-\infty}^{\infty} f(t)dt \tag{184}$$

It is seen that the time integration suppresses the high frequency components of a signal.

Example: We want to compute the Fourier transform of a Trapezoidal pulse using the differentiation property (refer to Fig. 16).

Computing the Fourier transform of th second derivative, we have,

$$(j\omega)^2F(\omega) = \frac{A}{\tau}\left(e^{j2\omega\tau} - e^{j\omega\tau} - e^{-j\omega\tau} + e^{j2\omega\tau}\right) \tag{185}$$

or,

$$F(\omega) = \frac{A}{(j\omega)^2\tau}\left(e^{j2\omega\tau} - e^{j\omega\tau} - e^{-j\omega\tau} + e^{j2\omega\tau}\right) \tag{186}$$

After simplifying, we get,

$$F(\omega) = A\tau\mathrm{Sa}^2(\omega\tau/2)[1 + 2\cos\omega\tau] \tag{187}$$

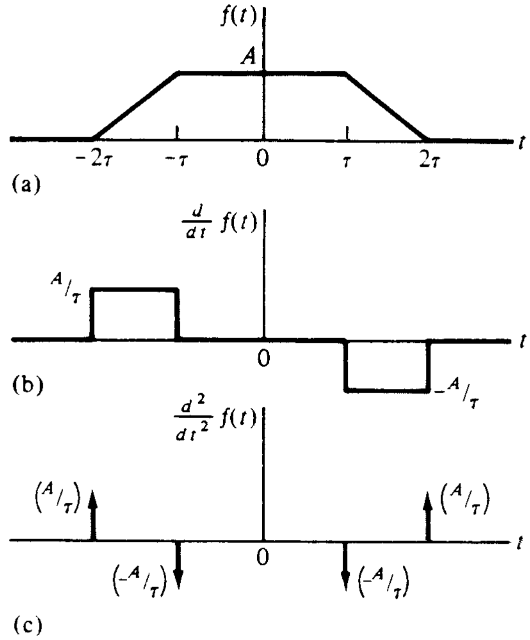


Figure 16: Trapezoidal signal and its derivatives.

4.5 Time convolution

Consider a system $\mathcal{T}$. The impulse response of $\mathcal{T}$ is defined as,

$$\mathcal{T}\{\delta(t-\tau)\} = h(t,\tau) \tag{188}$$

Time-invariant system:

$$\mathcal{T}\{\delta(t-\tau)\} = h(t-\tau) \tag{189}$$

We know that a function $f(t)$ can be written as,

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \tag{190}$$

The output of the system to this input is,

$$\begin{aligned} g(t) &= \mathcal{T}\left\{\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau\right\} \\ &= \int_{-\infty}^{\infty} f(\tau)\mathcal{T}\{\delta(t-\tau)\}d\tau \\ &= \int_{-\infty}^{\infty} f(\tau)h(t,\tau)d\tau \end{aligned} \tag{191}$$

Time invariant system:

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \tag{192}$$

Definition of Convolution Integral:

$$f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau \tag{193}$$

Step response of linear, time-invariant system:

$$g(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau \tag{194}$$

Basic result:

If,

$$\mathcal{F}\{f(t)\} = F(\omega) \quad \text{and} \quad \mathcal{F}\{h(t)\} = H(\omega) \tag{195}$$

then,

$$\mathcal{F}\{f(t) * h(t)\} = F(\omega)H(\omega) \tag{196}$$

Proof:

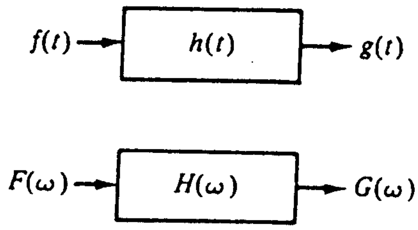
$$\begin{aligned}
\mathcal{F}\{f(t) * h(t)\} &= \int_{-\infty}^{\infty} [f(t) * h(t)]e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right] e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} f(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right] d\tau \\
&= \int_{-\infty}^{\infty} f(\tau)\mathcal{F}\{h(t-\tau)\}d\tau
\end{aligned} \tag{197}$$

We know that, $\mathcal{F}\{h(t-\tau)\} = H(\omega)e^{-j\omega\tau}$. Substituting, we get,

$$\begin{aligned}
\mathcal{F}\{f(t) * h(t)\} &= \int_{-\infty}^{\infty} f(\tau)H(\omega)e^{-j\omega\tau}d\tau \\
&= H(\omega)\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau \\
&= H(\omega)F(\omega)
\end{aligned} \tag{198}$$

Linear systems:

$$\begin{aligned}
g(t) &= f(t) * h(t) \\
G(\omega) &= F(\omega)H(\omega)
\end{aligned} \tag{199}$$



Assume that,

$$F(\omega) = |F(\omega)|e^{j\Theta_F(\omega)}$$
$$H(\omega) = |H(\omega)|e^{j\Theta_H(\omega)} \quad (200)$$
$$G(\omega) = |G(\omega)|e^{j\Theta_G(\omega)}$$

Then, we have,

$$|G(\omega)|e^{j\Theta_G(\omega)} = |F(\omega)|e^{j\Theta_F(\omega)}|H(\omega)|e^{j\Theta_H(\omega)} \quad (201)$$

This results in,

$$|G(\omega)| = |F(\omega)||H(\omega)|$$
$$\Theta_G(\omega) = \Theta_F(\omega) + \Theta_H(\omega) \quad (202)$$

- Frequency Convolution

If,

$$\mathcal{F}\{f_1(t)\} = F_1(\omega), \quad \mathcal{F}\{f_2(t)\} = F_2(\omega). \quad (203)$$

then,

$$\mathcal{F}\{f_1(t)f_2(t)\} = \frac{1}{2\pi}[F_1(\omega) * F_2(\omega)] \quad (204)$$

$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(u)F_2(\omega - u)du \quad (205)$$

Proof: Similar to the case of the time convolution shown earlier.

4.6 Some convolutional relationships

4.6.1 Graphic interpretation of convolution

The convolution,

$$g(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \tag{206}$$

can be computed using the following steps:

1. Replace t by τ in $f_1(t)$ resulting in $f_1(\tau)$.

2. Replace t by $-\tau$ in $f_2(t)$ resulting in $f_2(-\tau)$. This folds the function f_2 about the vertical axis.

3. Translate the entire frame of $f_2(-\tau)$ by an amount of t. For negative t the shift is towards the negative τ axis (and for positive t towards the positive τ axis). This results in the function $f_2(t-\tau)$.

4. At any given relative shift t, compute the integral,

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \tag{207}$$

An example of these steps is shown in Fig. 17

4.6.2 Causality

For $h(t)$ causal:

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \tag{208}$$

Note that $h(t-\tau) = 0$, if $t - \tau < 0$ $(\tau > t)$. Reflecting this fact in (208), we obtain,

$$g(t) = f(t) * h(t) = \int_{-\infty}^{t} f(\tau) h(t-\tau) d\tau \tag{209}$$

For $f(t)$, $h(t)$ causal, we have $h(t-\tau) = 0$, if $t - \tau < 0$ $(\tau > t)$ and $f(t) = 0$, if $t < 0$. Reflecting these facts in (208), we obtain,

$$g(t) = f(t) * h(t) = \int_{0}^{t} f(\tau) h(t-\tau) d\tau \tag{210}$$

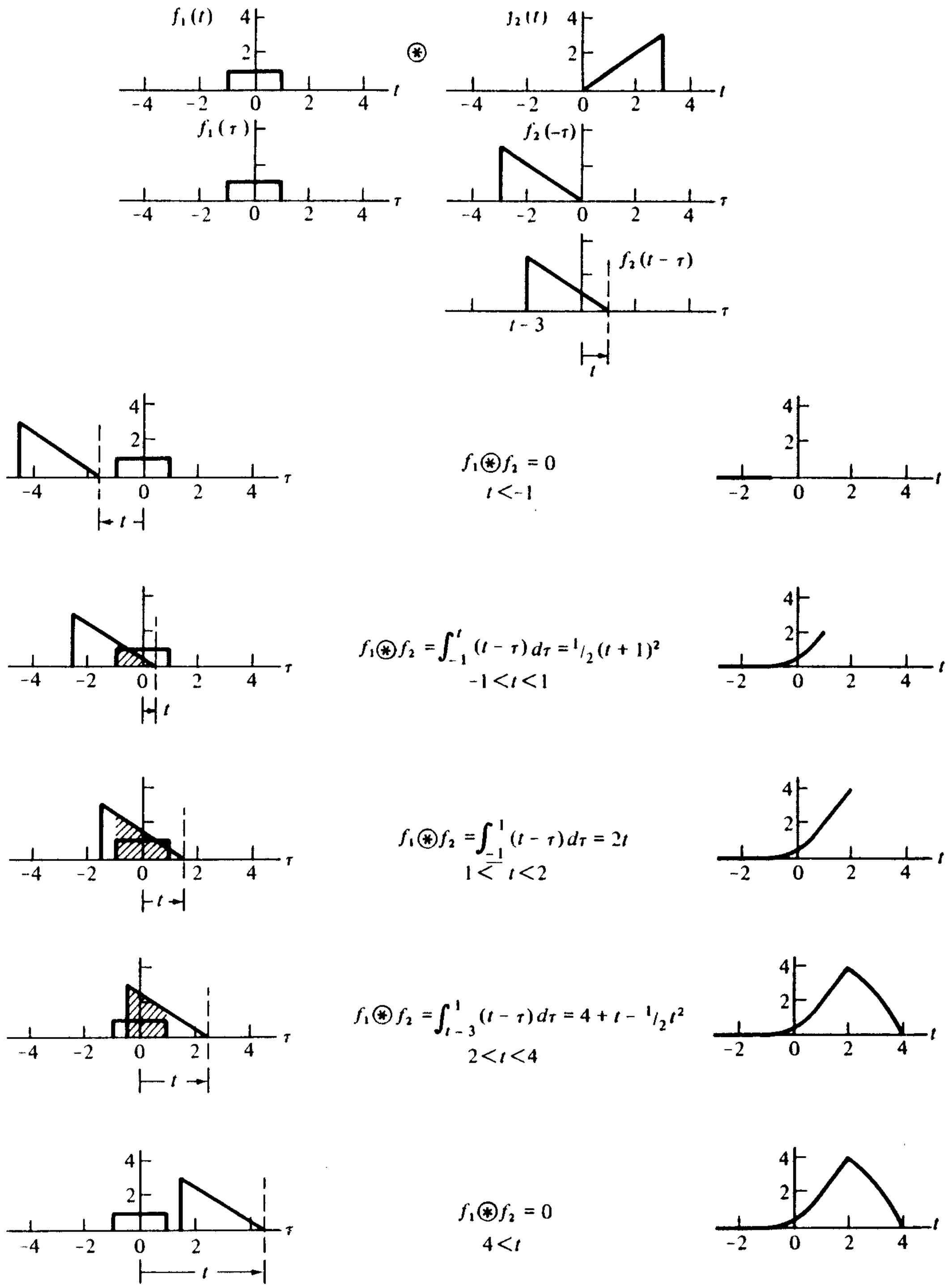


Figure 17: Graphic interpretation of the convolution of a rectangular and a triangular pulse.

4.6.3 Commutative law

$$f_1(t) * f_2(t) = f_2(t) * f_1(t) \tag{211}$$

Proof:

$$F_1(\omega)F_2(\omega) = F_2(\omega)F_1(\omega) \tag{212}$$

4.6.4 Distributive law

$$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t) \tag{213}$$

Proof:

$$F_1(\omega)[F_2(\omega) + F_3(\omega)] = F_1(\omega)F_2(\omega) + F_1(\omega)F_3(\omega) \tag{214}$$

4.6.5 Associative law

$$f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t) \tag{215}$$

Proof:

$$F_1(\omega)[F_2(\omega)F_3(\omega)] = [F_1(\omega)F_2(\omega)]F_3(\omega) \tag{216}$$

4.7 Convolution involving singularity functions

$$u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(t-\tau)d\tau \tag{217}$$

Let $x = t - \tau$, then,

$$u(t) * h(t) = \int_{-\infty}^{t} h(x)dx \tag{218}$$

$$f(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau-t_0)d\tau = f(t-t_0) \tag{219}$$

$$\delta(t-t_0) * \delta(t-t_1) = \int_{-\infty}^{\infty} \delta(\tau-t_0)\delta(t-\tau-t_1)d\tau = \delta(t-t_0-t_1) \tag{220}$$