4 Fourier Transform

A periodic function is obtained by repeating a signal after some fixed period of time T (refer to Fig. 10). We know that such a periodic signal has a Fourier series representation. For a given time function $f(t)$, we use the notation $f_T(t)$ to show the periodic function obtained by repeating $f(t)$ for the time period T. This means that,

$$f_T(t) \equiv \sum_{n=-\infty}^{\infty} f(t - nT) \tag{109}$$

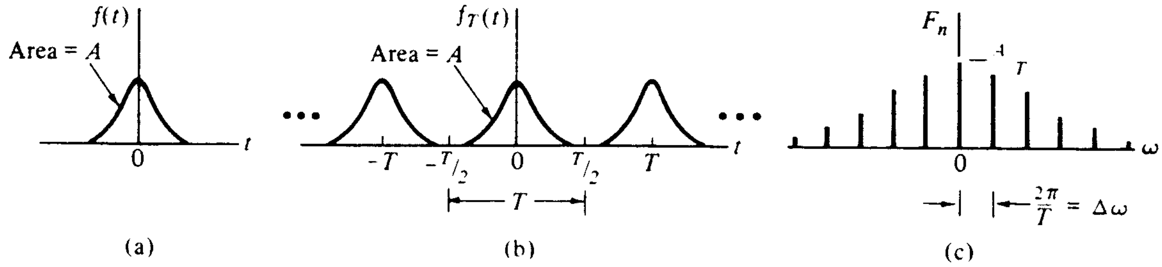

Figure 10: Generation of a periodic signal and the corresponding Fourier series coefficients.

We saw in the case of discrete Fourier series that as the period T is made larger, $\boxed{\text{(i)}}$ the fundamental frequency $(2\pi/T)$ becomes smaller, $\boxed{\text{(ii)}}$ the spacing between the frequency lines $(2\pi/T)$ decreases, $\boxed{\text{(iii)}}$ the amplitude of each frequency component decreases, $\boxed{\text{(vi)}}$ the general shape of the spectrum remains unchanged. We now consider the limiting case of $T \to \infty$.

$$\lim_{T\to\infty} f_T(t) = f(t) \tag{110}$$

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{111}$$

where

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_0 t}dt \qquad \omega_0 = 2\pi/T \tag{112}$$

Define:

$$\omega_n \equiv n\omega_0 \tag{113}$$

$$F(\omega_n) \equiv TF_n \tag{114}$$

This results in

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{T}F(\omega_n)e^{j\omega_n t} \tag{115}$$

$$F(\omega_n) = \int_{-T/2}^{T/2} f_T(t)e^{-j\omega_n t}dt \tag{116}$$

Spacing between adjacent frequency lines is equal to: $\Delta\omega = 2\pi/T$. Substituting in (115), we obtain,

$$f_T(t) = \sum_{n=-\infty}^{\infty} F(\omega_n)e^{j\omega_n t}\frac{\Delta\omega}{2\pi} \tag{117}$$

As T tends to infinity, $\Delta\omega = 2\pi/T$, which is the spacing between the frequency lines, tends to zero and the frequency spectrum becomes continuous. In this case, the relationships (116), (117) change into,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \equiv \mathcal{F}\{f(t)\} \tag{118}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \equiv \mathcal{F}^{-1}\{F(\omega)\} \tag{119}$$

We also have,

$$F_n = \frac{1}{T}F(n\omega_0) \tag{120}$$

The function $F(\omega)$ is known as the *spectral density* function of $f(t)$. One can think of the Fourier transform as the representation of a give signal in terms of an infinite sum of complex exponentials each weighted by $F(\omega)d\omega$. This means that we have infinite number of terms where the contribution of each term tends to zero. In contrast to this, a period waveform has all its amplitude components at discrete frequencies where each component has a definite contribution.

Note that relationship (120) expresses the Fourier series coefficients of a periodic signal in terms of the samples of the Fourier transform of one period of the same signal.

It is usual to express the Fourier transform in terms of f or $\omega = 2\pi f$. In changing from f to ω, or vice versa, the following relationship is useful,

$$\delta(f) = 2\pi\delta(\omega) \tag{121}$$

Example:

$$\text{rect}(t/\tau) \equiv \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{cases} \tag{122}$$

$$\begin{aligned} F(\omega) = & \int_{-\infty}^{\infty} \text{rect}(t/\tau)e^{-j\omega t}dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}dt \\ & (e^{-j\omega\tau/2} - e^{j\omega\tau/2})/(-j\omega) = \tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)} \end{aligned} \tag{123}$$

$$\mathcal{F}\{\text{rect}(t/\tau)\} = \tau\text{Sa}(\omega\tau/2) \tag{124}$$

4.1 Parseval's theorem

$$\begin{aligned} E = & \int_{-\infty}^{\infty} |f(t)|^2dt = \int_{-\infty}^{\infty} f(t)f^*(t)dt = \\ & \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right]^* dt = \\ & \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)e^{-j\omega t}d\omega\right] dt = \\ & \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)\left[\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right] d\omega = \\ & \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(\omega)F(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2d\omega \end{aligned} \tag{125}$$

This means that,

$$\int_{-\infty}^{\infty} |f(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2d\omega \tag{126}$$

The quantity $|F(\omega)|^2$ is called the *energy spectral density* (unit of joule per Hz) of $f(t)$. The quantity $|F(\omega)|^2d\omega$ is the energy in a small frequency band of $d\omega$ around ω.
Note that in the case of a periodic signal, the parseval relationship computes the average power of the signal and is of the form,

$$\frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2dt = \sum_{n=-\infty}^{\infty} |F_n|^2 \tag{127}$$

In this case, $|F_n|^2$ is the power in the frequency $n\omega_0 = n2\pi/T$.

4.2 Some Fourier transforms involving impulse functions

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1 \tag{128}$$

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-j\omega t}dt = e^{-j\omega t_0} \tag{129}$$

$$\mathcal{F}^{-1}\{\delta(\omega \mp \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega \mp \omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{\pm j\omega_0 t} \tag{130}$$

$$\mathcal{F}\{e^{\pm j\omega_0 t}\} = 2\pi\delta(\omega \mp \omega_0) \tag{131}$$

$$\omega_0 = 0 \quad \rightarrow \quad \mathcal{F}\{1\} = 2\pi\delta(\omega) \tag{132}$$

$$\mathcal{F}\{\cos\omega_0 t\} = \mathcal{F}\{\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\} = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \tag{133}$$

$$\mathcal{F}\{\sin\omega_0 t\} = \mathcal{F}\{\frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}\} = \frac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)] \tag{134}$$

$$\text{sgn}(t) = \frac{|t|}{t} = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \tag{135}$$

$$\mathcal{F}\{\text{sgn}(t)\} = \mathcal{F}\left\{\lim_{a\to 0}[e^{-a|t|}\text{sgn}(t)]\right\} = \tag{136}$$

$$\lim_{a\to 0}\left\{\int_{-\infty}^{\infty}[e^{-a|t|}\text{sgn}(t)]e^{-j\omega t}dt\right\} = \tag{137}$$

$$\lim_{a\to 0}\left\{-\int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt\right\} = \tag{138}$$

$$\lim_{a\to 0}\left\{-\frac{1}{a-j\omega} + \frac{1}{a+j\omega}\right\} = \tag{139}$$

$$\mathcal{F}\{\text{sgn}(t)\} = \lim_{a\to 0}\left\{\frac{-2j\omega}{a^2+\omega^2}\right\} = \frac{2}{j\omega} \tag{140}$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t) \tag{141}$$

$$\mathcal{F}\{u(t)\} = \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\mathrm{sgn}(t)\} \tag{142}$$

$$\mathcal{F}\{u(t)\} = \pi\delta(\omega) + 1/j\omega \tag{143}$$

The spectral density function of the unit step contains an impulse at $\omega = 0$ corresponding to the average value of 1/2 in the step function. It also has all the high frequency components of the sgn() function reduced by one-half.

Table 1 contains a list of some important Fourier transform pairs. Figure 11 shows some of these Fourier transform pairs.

4.3 Periodic signals

For a periodic signal $f_T(t)$, we have,

$$f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}, \qquad \omega_0 = 2\pi/T \tag{144}$$

$$\mathcal{F}\{f_T(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right\} \tag{145}$$

$$\mathcal{F}\{f_T(t)\} = \sum_{n=-\infty}^{\infty} F_n \mathcal{F}\{e^{jn\omega_0 t}\} \tag{146}$$

$$\mathcal{F}\{f_T(t)\} = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0) \tag{147}$$

<u>Periodic signal of unit impulses:</u>

$$\delta_T(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{148}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{149}$$

where

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \tag{150}$$

$$\delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} \tag{151}$$

	$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
1.	$e^{-at}u(t)$	$1/(a + j\omega)$
2.	$te^{-at}u(t)$	$1/(a + j\omega)^2$
3.	$e^{-a\|t\|}$	$2a/(a^2 + \omega^2)$
4.	$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}$
5.	$\mathrm{sgn}\,(t)$	$2/(j\omega)$
6.	$j/(\pi t)$	$\mathrm{sgn}\,(\omega)$
7.	$u(t)$	$\pi\delta(\omega) + 1/(j\omega)$
8.	$\delta(t)$	1
9.	1	$2\pi\delta(\omega)$
10.	$e^{\pm j\omega_0 t}$	$2\pi\delta(\omega \mp \omega_0)$
11.	$\cos\,\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
12.	$\sin\,\omega_0 t$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
13.	$\mathrm{rect}\,(t/\tau)$	$\tau\,\mathrm{Sa}\,(\omega\tau/2)$
14.	$\dfrac{W}{2\pi}\,\mathrm{Sa}\,(Wt/2)$	$\mathrm{rect}\,(\omega/W)$
15.	$\dfrac{W}{\pi}\,\mathrm{Sa}\,(Wt)$	$\mathrm{rect}\,(\omega/(2W))$
16.	$\Lambda(t/\tau)$	$\tau[\mathrm{Sa}\,(\omega\tau/2)]^2$
17.	$\dfrac{W}{2\pi}[\mathrm{Sa}\,(Wt/2)]^2$	$\Lambda(\omega/W)$
18.	$\cos\,(\pi t/\tau)\,\mathrm{rect}\,(t/\tau)$	$\dfrac{2\tau}{\pi}\,\dfrac{\cos\,(\omega\tau/2)}{1 - (\omega\tau/\pi)^2}$
19.	$\dfrac{2W}{\pi^2}\,\dfrac{\cos\,(Wt)}{1 - (2Wt/\pi)^2}$	$\cos\,[\pi\omega/(2W)]\,\mathrm{rect}\,[\omega/(2W)]$
20.	$\delta_T(t)$	$\omega_0\,\delta_{\omega_0}(\omega)$, where $\omega_0 = 2\pi/T$

Table 1: Some selected Fourier transform pairs.

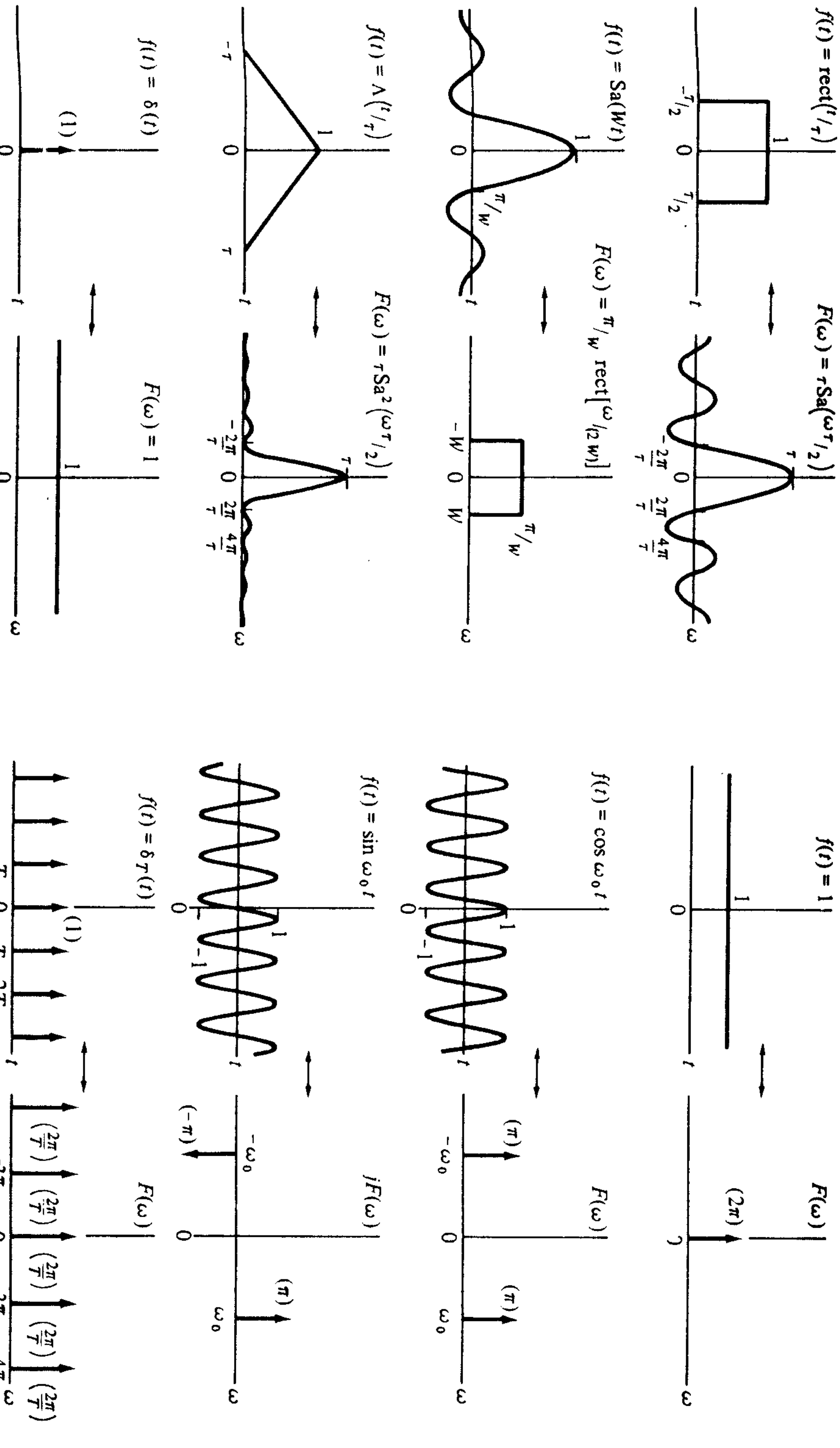

Figure 11: Some selected Fourier transform pairs.

$$\mathcal{F}\{\delta_T(t)\} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0) = \omega_0 \sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0) \tag{152}$$

$$\mathcal{F}\{\delta_T(t)\} = \omega_0\delta_{\omega_0}(\omega) \tag{153}$$

$\boxed{\text{Example:}}$ Consider the rectangular pulse,

$$\text{rect}(t/\tau) \equiv \begin{cases} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{cases} \tag{154}$$

We already saw that,

$$\mathcal{F}\{\text{rect}(t/\tau)\} = \tau\text{Sa}(\omega\tau/2) \tag{155}$$

Now consider a periodic signal obtained by repeating $\text{rect}(t/\tau)$ over a period of T. Using (120), the corresponding Fourier series coefficients are equal to,

$$F_n = \frac{1}{T}F(n\omega_0) = \frac{\tau}{T}\frac{\sin(n\omega_0\tau/2)}{(n\omega_0\tau/2)} \tag{156}$$

which is the same as the result obtained earlier.

The Fourier transform of this period rectangular signal is equal to,

$$\begin{aligned} \mathcal{F}\{\text{rect}_T(t)\} &= 2\pi\sum_{n=-\infty}^{\infty} F_n\delta(\omega - n\omega_0) \\ &= \frac{2\pi\tau}{T}\frac{\sin(n\omega_0\tau/2)}{(n\omega_0\tau/2)}\delta(\omega - n\omega_0) \end{aligned} \tag{157}$$

These are shown in Fig. 12.

4.4 Properties of Fourier Transform

- $\boxed{\text{Linearity (superposition)}}$

$$\mathcal{F}\{a_1f_1(t) + a_2f_2(t)\} = a_1\mathcal{F}\{f_1(t)\} + a_2\mathcal{F}\{f_2(t)\} \tag{158}$$

- $\boxed{\text{Complex conjugate}}$

$$\begin{aligned} \mathcal{F}\{f^*(t)\} &= \int_{-\infty}^{\infty} f^*(t)e^{-j\omega t}dt \\ &= \left[\int_{-\infty}^{\infty} f(t)e^{j\omega t}dt\right]^* \\ &= F^*(-\omega) \end{aligned} \tag{159}$$

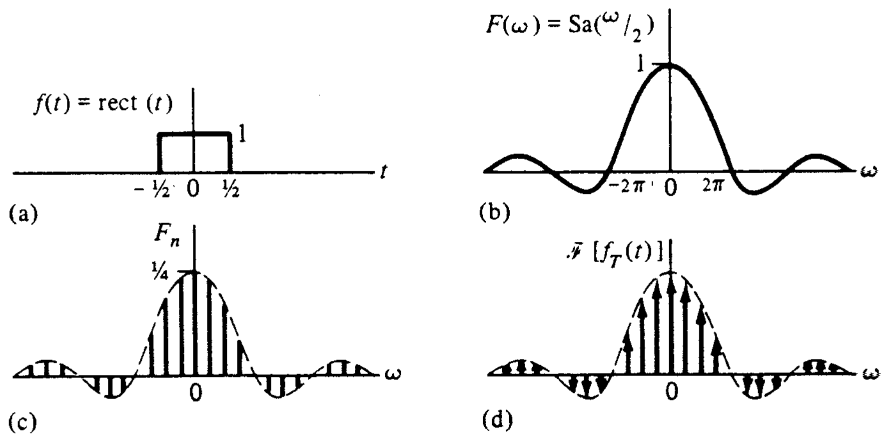

Figure 12: (a) The rectangular pulse, (b) its Fourier transform. (c) The Fourier series coefficients of the periodic rectangular signal, (d) The Fourier transform of the periodic rectangular signal.

Real $f(t)$, then $f^*(t) = f(t)$ and $F^*(-\omega) = F(\omega)$.

Using this result, it is easy to show that for real $f(t)$, the real/imaginary parts of $F(\omega)$ should have even/odd symmetry.

- $\boxed{\text{Symmetry}}$

Any signal can be written as the sum of an even part and an odd part, $f(t) = f_e(t) + f_o(t)$. We have,

$$
\begin{aligned}
\mathcal{F}\{f_e(t)\} &= F_e(\omega) \quad \text{and real} \\
\mathcal{F}\{f_o(t)\} &= F_o(\omega) \quad \text{and imaginary}
\end{aligned}
\tag{160}
$$

To prove, consider an even function $f_e(t)$. We have,

$$
\begin{aligned}
\mathcal{F}\{f_e(t)\} &= \int_{-\infty}^{\infty} f_e(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} f_e(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_e(t)\sin(\omega t)dt
\end{aligned}
\tag{161}
$$

The first integral involves an even function and is equal to two times the value of the integral over $[0, \infty]$. The second integral involves an odd function and is equal to zero. This results in,

$$
\mathcal{F}\{f_e(t)\} = 2\int_{0}^{\infty} f_e(t)\cos(\omega t)dt \tag{162}
$$

This is a $\boxed{\text{real, even}}$ function of ω. Similarly, for an odd function, we have,

$$
\begin{aligned}
\mathcal{F}\{f_o(t)\} &= \int_{-\infty}^{\infty} f_o(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} f_o(t)\cos(\omega t)dt - j\int_{-\infty}^{\infty} f_o(t)\sin(\omega t)dt
\end{aligned}
\tag{163}
$$

The first integral involves an odd function and is equal to zero. The second integral involves an even function and is equal to two times the value of the integral over $[0, \infty]$. This results in,

$$
\mathcal{F}\{f_o(t)\} = -2j\int_{0}^{\infty} f_o(t)\sin(\omega t)dt \tag{164}
$$

This is an $\boxed{\text{imaginary, odd}}$ function of ω. Using these results together, we conclude (160).