A New Space-Time Code Based on Permutation Matrices

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Abstract — **A full diversity block space-time code over two transmit antennas and two symbol periods is introduced. The proposed method results in substantial performance improvement over the widely used scheme of Alamouti [1] and at the same time allows for a simple maximum likelihood decoding algorithm.**

I. Introduction

Block orthogonal space-time coding is a method to design full diversity space-time codes with orthogonality property [1][2] (orthogonality property makes the decoding simple). It has been shown that using an orthogonal structure sacrifices the achievable coding gain and is also restrictive in the sense that it does not allow to achieve full channel capacity for more than one receive antenna [3]. A very well known example of orthogonal code is so-called *Alamouti Code* [1] which takes advantage of two transmit antennas over two symbol periods. Knowing the sub-optimality of the Alamouti Code and its practical significance, there have recently been some attempts to improve this structure. The best known result in this category is the work of [4] which uses an algebraic structure to construct non-orthogonal codes with large distance. However, finding codes with good distance properties which at the same time allow for a simple Maximum Likelihood (ML) decoding algorithm remains an open problem.

In this paper, a new full diversity full rate space-time code over two transmit antennas and two symbol periods is introduced. This structure is based on maximization of coding advantage through decomposition of the code to two permutation matrices. This procedure results in a full diversity code with high coding advantage. On the other hand, a new simple decoding method with low complexity is presented. Finally, simulations show significant improvement in Symbol Error Rate (SER) compared with Alamouti code, and similar performance with Damen code [4]. Note that the decoding method of the Damen code is based on sphere decoding which is normally a complex operation.

II. Preliminaries

In slow flat fading environment, MIMO channels with M transmit and N receive antennas over T symbol periods is modelled by

$$Y = \sqrt{\frac{\rho}{M}} HX + V \qquad (1)$$

where $Y \in \mathcal{C}^{N\times T}$ denotes the received matrix and $X \in \mathcal{C}^{M\times T}$ denotes the transmitted matrix, $H \in \mathcal{C}^{N\times M}$ denotes the channel matrix, and $V \in \mathcal{C}^{N\times T}$ denotes additive, spatially and temporally i.i.d white noise with complex gaussian distribution. Matrix X and noise N are normalized such that ρ is SNR at each receive antennas. Matrix X is selected from a finite set, so-called *codebook*. Considering two matrices, X^i and X^j in codebook, we define B_{ij} and A_{ij} as $B_{ij} = X^i - X^j$ and $A_{ij} = B_{ij}B^*_{ij}$. The following criteria are the basic criteria for space-time code designing [5]:

- *The Rank Criterion*: Diversity advantage is determined with minimum rank of the matrices A_{ij} (or B_{ij}) over all pairs of codewords in the codebook. In order to achieve full diversity, the matrices A_{ij} (or B_{ij}) have to be full rank for all pairs in codebook.
- *The Determinant Criterion*: In order to improve coding gain, we must maximize the minimum values of multiplication of nonzero eigenvalues of A_{ij} for all pairs of codewords in the codebook. Thus, if the code is full diversity, we must maximize the minimum of the $\det(A_{ij})$ for all pairs of X^i and X^j in the codebook.

III. Code Structure

In this section, we introduce a new full diversity block space time code. The main objective of this design is space-time code with high coding advantage and simple decoding method. In the following, we concentrate on using an 8-PSK constellation, however, all the results can be easily generalized to a PSK constellation with a different number of points.

We suggest a space-time code with following codebook structure

$$C = \{C_{m,n} = A^m + DA^n \qquad m,n = 0,1,\ldots,7\} \qquad (2)$$

where

$$A = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_3 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & \theta_3 \\ 1 & 0 \end{bmatrix} \qquad (3)$$

and

$$\theta_n = \exp(j\frac{2\pi}{8}n) \qquad (4)$$

We can consider this structure as a result of coding advantage optimization. On the other hand, it is possible to consider it as the result of attempts to design a code with simple decoding. In this section, we will explain the first point of view. At first, let us define the problem.

The final objective of this design is introduction of a 64-size codebook C with maximum coding advantage. The codebook includes 64 , 2×2 matrices as space-time codewords, each of them represents two three-bit symbols m and n. In other words, we are looking for a codebook C

$$C = \{C_{m,n} \in \mathcal{C}^{2\times 2} \qquad m,n = 0,1,\ldots,7\} \tag{5}$$

such that coding advantage Δ is maximized, where Δ is equal

$$\Delta = \min_{m,n,m',n'} \left(\det\{C_{m,n} - C_{m',n'})(C_{m,n} - C_{m',n'})^*\}\right)^{1/2} \tag{6}$$

for $(m,n) \neq (m',n')$. It can be readily checked that finding optimal codebook C by exhaustive search is not possible, due to complexity issue. On the other hand, it is very likely that the worst case of pairwise error is obtained when (m,n) and (m',n') defer in only one element. Thus we can narrow down optimization to the case that $m = m'$ or $n = n'$. Also for simplicity, we assume that where $n = n'$, $C_{m,n} - C_{m',n'}$ depends only on m and m' and does not depend on $n = n'$. The same assumption is valid for n and n', where $m = m'$. Considering the special structure of permutation matrices , one of the best approach to construct that code is decomposition $C_{m,n}$ to sum of two permutation matrixes, U_m and W_n, such that each of them depends just on m and n, respectively. Thus, we can write

$$C_{m,n} = U_m + W_n \tag{7}$$

where

$$U_m = \begin{bmatrix} u_m^1 & 0 \\ 0 & u_m^2 \end{bmatrix}, W_n = \begin{bmatrix} 0 & w_n^1 \\ w_n^2 & 0 \end{bmatrix} \tag{8}$$

As a result, we can see that if $n = n'$, then

$$C_{m,n} - C_{m',n'} = U_m - U_{m'} \tag{9}$$

Similarly, if $m = m'$ then

$$C_{m,n} - C_{m',n'} = W_n - W_{n'} \tag{10}$$

Now, we can narrow down our optimization to set U

$$U = \{U_0, U_1, \cdots, U_7\} \tag{11}$$

such as δ is maximized, where δ is defined as

$$\delta = \min_{m,m'} \det\{(U_m - U_{m'})(U_m - U_{m'})^*\} \tag{12}$$

where $m \neq m'$. In fact, when $n = n'$, Δ is reduced to δ. The result of the optimization is 14 individual sets as follows

$$U = \left\{E \times \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_3 \end{bmatrix}^m, m = 0,1,\ldots,7\right\} \tag{13}$$

and

$$U = \left\{E \times \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_5 \end{bmatrix}^m, m = 0,1,\ldots,7\right\} \tag{14}$$

where E is equal to

$$E = \begin{bmatrix} 1 & 0 \\ 0 & \theta_k \end{bmatrix} \tag{15}$$

The function θ_n is defined in (3). By choosing $k = 0,1,\ldots,7$, fourteen individual sets will be obtained. It is easy to see that the optimization for $W = \{W_0, W_1, \cdots, W_7\}$ has the same results with some small modifications. The result of the optimization for W is

$$W = \left\{D \times \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_3 \end{bmatrix}^n, n = 0,1,\ldots,7\right\} \tag{16}$$

and

$$W = \left\{D \times \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_5 \end{bmatrix}^n, n = 0,1,\ldots,7\right\} \tag{17}$$

where D is equal to

$$D = \begin{bmatrix} 0 & \theta_k \\ 1 & 0 \end{bmatrix} \tag{18}$$

So, there is fourteen individual choices for W. One of the best choice for U and W with highest coding advantage is the choice mentioned in (2).

Generalization of this code to other 2^b-PSK is very simple. Consider a code book with the following structure

$$C = \{C_{m,n} = A^m + DA^n \in \mathcal{C}^{2\times 2} \quad m,n = 0,1,\ldots,2^b - 1\} \tag{19}$$

where

$$A = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_k \end{bmatrix} \qquad D = \begin{bmatrix} 0 & \theta_r \\ 1 & 0 \end{bmatrix} \tag{20}$$

and

$$\theta_n = \exp(j\frac{2\pi}{2^b}n) \tag{21}$$

In table 1, the best values for k and r for 2^b-PSK, $2 \leq b \leq 5$, are listed.

	k	r	Coding Adv.
4PSK	1,3	1,3	2
8PSK	3	1,3,5,7	1.0824
16PSK	7	2,6,10,14	0.4483
32PSK	7,23	3,5,11,19,21,27,29	0.1175

Table 1: Code structure for 2^b-PSK

IV. Decoding

To formulate the ML decoding, we have

$$P(Y|H, C_{m,n}) = \frac{1}{\pi^{NT}} \exp(-d^2(m,n)) \tag{22}$$

where H is the channel transfer matrix, Y is the matrix corresponding to the received signal, and

$$d^2(m,n) = tr\left[(Y - \sqrt{\frac{\rho}{M}}HC_{m,n})(Y - \sqrt{\frac{\rho}{M}}HC_{m,n})^*\right] \tag{23}$$

The goal of ML decoding is to find m and n to maximize $P(Y|H, C_{m,n})$ or minimize $d^2(m,n)$. The straightforward approach for ML decoding is to calculate different values of $d^2(m,n)$ for all possible values of m and n, and find the minimum value of $d^2(m,n)$ using an exhaustive search. It is clear that the complexity of such an exhaustive search is very high.

Let us define K, $f(n)$, $g(m)$ and $h(m-n)$ as

$$K = tr\left\{YY^* + 2\frac{\rho}{M}HH^*\right\} \tag{24}$$

$$f(m) = -\sqrt{\frac{\rho}{M}}tr\{HA^mY^* + YA^{-m}H^*\} \tag{25}$$

$$g(n) = -\sqrt{\frac{\rho}{M}}tr\{HDA^nY^* + YA^{-n}D^*H^*\} \tag{26}$$

$$h(m-n) = \frac{\rho}{M}tr\{HA^{m-n}D^*H^* + HDA^{n-m}H^*\}. \tag{27}$$

Using these notations, it is easy to show that,

$$d^2(m,n) = K + f(m) + g(n) + h(m-n) \tag{28}$$

To prove (28), we use these facts that

$$A^* = A^{-1} \qquad \text{and} \qquad D^* = D^{-1} \tag{29}$$

This is another advantage of selection of permutation matrices. Considering that fact that $A^8 = A^0 = I_{2\times 2}$, it is easy to prove that $h(m-n) = h(m-n \mod 8)$. Thus there are 8 different values for each of $f(m)$, $g(n)$, and $h(m-n)$ where $0 \le m \le 7$ and $0 \le n \le 7$.

Ignoring the constant part K, for minimizing $f(m) + g(n) + h(m-n)$, we can use Viterbi Algorithm over the trellis structure shown in Fig. 1. In this figure, $k = [(m-n) \mod 8]$.

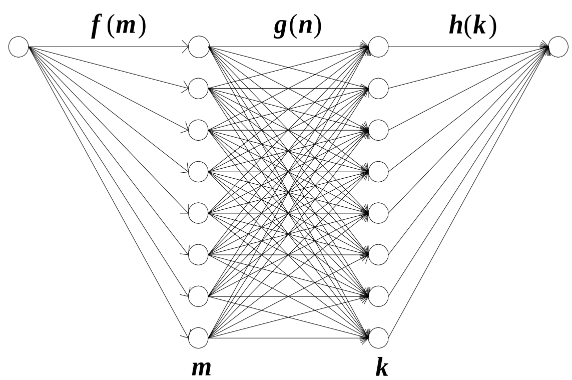

Figure 1: Trellis constructed based on code structure

Another effective ML decoding method with less complexity is as follows: Let us sort $f(m)$ and $g(m)$ in the increasing order and specify the corresponding arguments as $m_0, m_1 \ldots m_7$ and $n_0, n_1 \ldots n_7$, i.e.

$$f(m_0) \le f(m_1) \le \ldots \le f(m_7) \tag{30}$$

and,

$$g(n_0) \le g(n_1) \le \ldots \le g(n_7) \tag{31}$$

We define two sets, named *potential set* and *final set*. Each set has 8 entries corresponding to different values of $[(m-n) \mod 8]$. The k^{th} entries of *final set*, $0 \le k \le 7$, is the best pair of (m,n) in terms of the minimization of $f(m) + g(n)$ such that $k = [(m-n) \mod 8]$. The k^{th} entry of *potential set*, $0 \le k \le 7$, is the best pair of (m,n) in terms of the minimization of $f(m) + g(n)$ such that $k = [(m-n) \mod 8]$ until then, and for those values of k that are not yet in the *final set*. The *final set* will be gradually filled using the following algorithm:

For $0 \le \alpha \le 14$, starting from $\alpha = 0$,

1. Find $S_\alpha = \{(i,j) | i+j = \alpha, 0 \le i \le 7, 0 \le j \le 7\}$.
2. Find $(i,j) \in S_\alpha$ that minimize $f(m_i) + g(n_j)$.
3. Set $k = [(m_i - n_j) \mod 8]$. Compare (m_i, n_j) with the pair in k^{th} row of *potential set*, if any, in terms of $f(m) + g(n)$ and put the best one in k^{th} row of *final set* (if k^{th} row of *final set* is not filled yet).
4. For other pairs of $(i,j) \in S_\alpha$, compute $k = [(m_i - n_j) \mod 8]$ and compare $f(m_i) + g(n_j)$ with the related value in the k^{th} row of the *potential set*. Put the better of these two values in the k^{th} row of the *potential set*.
5. If the *final set* is not filled yet, set $\alpha \leftarrow \alpha + 1$ and go to step 1.
6. Compute $f(m) + g(n) + h(m-n)$ for pairs in the *final set* and select the pair that minimize $f(m) + g(n) + h(m-n)$.

The core of the prove of the algorithm is as follows: If $(i,j) \in S_\alpha$ minimize $f(m_i) + g(n_j)$ and $k = [(m_i - n_j) \mod 8]$, then there is no $(i',j') \in S_{\alpha'}$, $(\alpha < \alpha')$ such that $k = [(m_{i'} - n_{j'}) \mod 8]$ and $f(m_i) + g(n_j) > f(m_{i'}) + g(n_{j'})$.

Simulations show that *final set* will be filled in early steps, most of the time in $\alpha = 7$, so the complexity of this algorithm is much smaller than the complexity of Viterbi algorithm.

V. Simulation and Comparison

As was described in section II, we use *rank* and *determinant* criteria as the basic tools to design space-time codes. These two criteria are based on minimization of the worst case of pairwise error probability. Although these criteria are very helpful for designing codes, we need simulation to completely evaluate the code performance. In general, Symbol Error Rate (SER) curves are used to evaluate code performance.

In this section, a complete evaluation of code performance by means of simulation measuring of SER is presented. In addition, we compare the performance of this code with two well-known codes, so-called Alamouti code [1] and Damen code [4]. This comparison is presented in the case of three and four bit Per Channel Use (PCU).

In the case of three bit PCU, we use 8-PSK modulation. Damen throughput is restricted to even numbers which means that we can not compare proposed scheme with this code for three bit PCU. Table 2 shows the coding advantages of the Alamouti Code and the proposed Code for three bit throughput. This table shows significant improvement of the coding advantage in the proposed code compared with the Alamiuti code.

We now look at the Symbol Error Rate (SER) curves of these two codes. For brevity, we just show the SER curves in the case of two receive antennas. Fig. 2 is the SER curves for Alamouti code and proposed scheme. This figure shows more than 2dB improvement in coding gain.

Structure	*Alamouti*	*Proposed*
Coding Adv.	0.5857	1.0824

Table 2: Coding advantage of the proposed and Alamouti codes in 3 bit PCU

For 4 bit PCU, 16-PSK constellation is used for modulation in the Alamouti code and the proposed code. Table 3 shows the coding advantage of Alamouti Code, Damen Code, and proposed Code. It is apparent that the coding advantage has significant improvement compared with the Alamouti code, but it is very closed to coding advantage of Damen code.

Structure	*Alamouti*	*Damen*	*Proposed*
Coding Adv.	0.1522	0.4738	0.4483

Table 3: Coding advantage of proposed, Alamouti and Damen codes in 4 PCU

Now, we compare the symbol error rate for these three codes. Fig. 3 shows the SER curves for two receive antennas. It is apparent that in this case the performance of the proposed code is similar to Damen code but is better than Alamouti code more than 5 dB.

VI. Conclusion

In this paper, a new full diversity full rate space-time block is introduced. At this point, this structure is presented for two transmit antennas over two symbol periods. This structure is the result of coding advantage optimization. The approach for optimization is decomposition of space-time matrix to two permutation matrices. This procedure results in a full diversity code with high coding advantage.

For decoding, a trellis constructed based on the code structure are developed which means that can use the Viterbi algorithm for decoding. A new method of decoding with very low complexity is also introduced .

Finally, simulations show significant improvement in coding gain of the proposed code where compared with the Alamouti code. On the other hand, the performance of the code is closed to that of the Damen code. Note that decoding method of Damen code is based on sphere decoding which is generally a complex operation.

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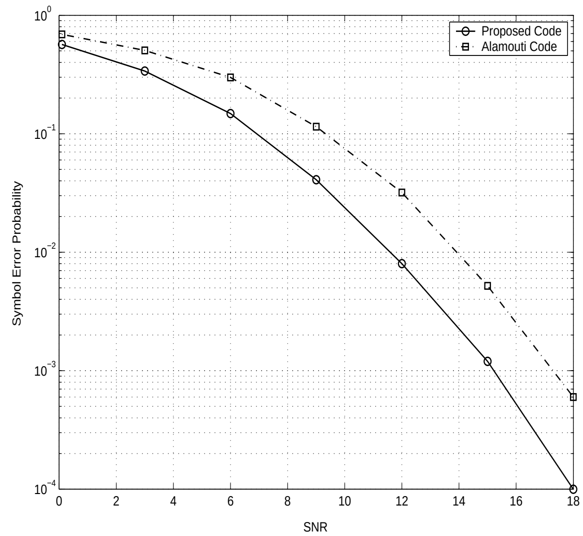

Figure 2: Symbol-Error-Rate for two transmit and two receive antennas-3 bit PCU

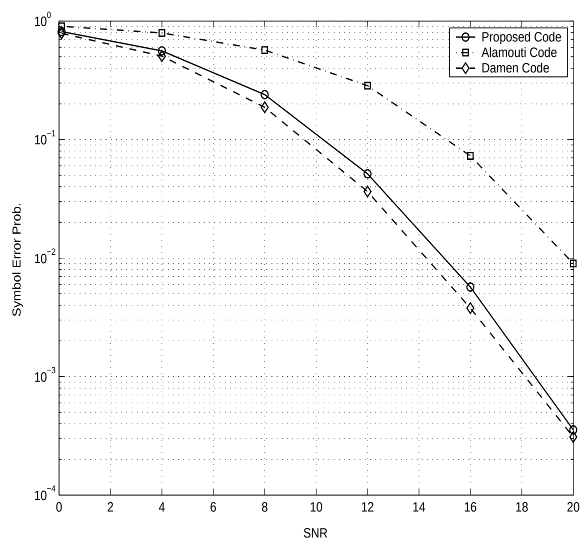

Figure 3: Symbol-Error-Rate for two transmit and two receive antennas-4 bit PCU