On Structure and Decoding of Product Codes

S. A. Miri[1]
Department of Mathematics
University of Toronto, Toronto
Ontario, Canada, M5S 3G3
samiri@math.utoronto.ca

A. K. Khandani
Dep. of Elect. & Comp. Eng.
University of Waterloo, Waterloo
Ontario, Canada, N2L 3G1
khandani@shannon.uwaterlo.ca

Abstract — **Product codes have been an effective coding method for communication channels where both random and burst error occur. In this paper, we present a new approach to the structure and Maximum Likelihood (ML) decoding of product codes using Tanner graphs. For product codes having a sub-code which is a product of simple parity codes and repetition codes, we show how to obtain a sub-code with an acyclic Tanner graph and the largest possible distance. We show that in all cases of interest, a n-dimensional product code has such a structure. Wagner rule decoding is used on this sub-code and its cosets to obtain an effective and efficient maximum-likelihood decoding of the given product code.**

I. Introduction

The product codes first proposed by Elias in the 1950's are multi-dimensional codes constructed by combining simpler component codes. Experience has shown that product codes generally have good random-error-correction and burst-error-correction capabilities. In [1], Tanner extended earlier works by Gallager on low-density parity-check codes to product codes using bipartite graphs, since known as Tanner graphs. It is well known that using this approach one can construct convergent decoding algorithms for codes with acyclic graphs. The question of which codes have acyclic Tanner graphs was answered categorically in [2]. In [3], it was shown that decomposition of a code into an acyclic sub-code and its cosets can provide an efficient method for the maximum-likelihood decoding of some of the best known linear block codes. In the present work, we concentrate on product codes for which the row and column codes are based on well known linear block codes such as Golay code and Reed-Muller codes. This assumption is justified by the fact that the minimum distance and dimension of a given product code is directly related to the distance and dimension of its component codes. For this reason, we are interested in product codes using good binary block codes as components. Extending the work in [3], we will provide a systematic way of obtaining an optimal sub-code with an acyclic, uniform Tanner graph with the largest possible distance such that the number of the corresponding cosets are minimized and decoding complexity is lowered.

II. Main

It is well known that the generator matrix for product of codes A and B is given by the Kronecker product of their generators, that is $G_A \otimes G_B$. It is also known that if two matrices G and G' differ by a permutation of row and columns, then their corresponding Tanner graphs are isomorphic. Allowing for row and column permutations, we will show that if C' and M' be sub-codes of codes C and M respectively, and the decomposition of corresponding generators be $G_C = G_{C'} + G_{C/C'}$ and $G_M = G_{M'} + G_{M/M'}$, then the product of C and M is equal to the union of the sub-code $C' \otimes M'$ and its cosets which can be easily calculated from appropriate products of C', M', C/C', and M/M'.

Consider an n-dimensional product of good codes. It was shown in [3] that each of these codes has an acyclic sub-code with a generator of the form $\mathcal{R}_m \otimes \mathcal{E}_n$ where $\mathcal{R}_m$, $\mathcal{E}_n$ are matrix generators of some repetition codes and simple-parity check codes of length m and n, respectively. Hence, an n-dimensional product code will have a sub-code of the form,

$$(\mathcal{R}_{j_1} \otimes \mathcal{E}_{i_1}) \otimes (\mathcal{R}_{j_2} \otimes \mathcal{E}_{i_2}) \otimes \cdots (\mathcal{R}_{j_n} \otimes \mathcal{E}_{i_n}).$$

Regrouping, and using the facts the Kronecker product is associative, and that the Kronecker product of repetition codes is simply another repetition code, this can be rewritten as

$$\mathcal{R}_L \otimes (((\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2}) \otimes \mathcal{E}_{i_3}) \cdots \mathcal{E}_{i_n}), \qquad \text{for some } \mathcal{R}_L.$$

We will show using the results of [2] that the product code given by the above equation always has cycles if it includes more than one parity check code. The aim is to show how to obtain an acyclic sub-code of the form $\mathcal{R} \otimes \mathcal{E}$ for these cases. We will first show how to find an optimal acyclic sub-code for the case of $\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2}$. We use this result to find an optimal acyclic sub-code for $\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2} \otimes \mathcal{E}_{i_3}$ in a recursive manner, since this product can be considered as $(\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2}) \otimes \mathcal{E}_{i_3}$ and it has a sub-code of the form $\mathcal{R} \otimes (\mathcal{E} \otimes \mathcal{E}_{i_3})$. Using this approach $n-1$ times, it follows that $(\mathcal{E}_{i_1} \otimes \mathcal{E}_{i_2} \otimes \cdots \mathcal{E}_{i_n})$, and consequently, the n-dimensional product code will have an acyclic sub-code of the form $\mathcal{R} \otimes \mathcal{E}$ of appropriate sizes. Finally, following earlier work in [3], the simple structure of $\mathcal{R} \otimes \mathcal{E}$ allows it to be easily decoded using the Wagner rule in conjunction with the trellis representation of the corresponding cosets.

References

[1] R. M. Tanner, "A Recursive Approach to Low Complexity Codes", *IEEE Trans. Inform. Theory*, Vol. IT-27, No. 5, pp. 533–547, September 1981.

[2] T. Etzion, A. trachtenberg, and A. Vardy, "Which Codes Have Cycle-Free Tanner Graphs?", *Proceedings of 1998 ISIT*, New York, NY, pp. 207.

[3] M. Esmaeili and A. K. Khandani, "Acyclic Tanner graphs and two-level decoding of linear block codes", *Proceedings of 1998 ISIT*, New York, NY, pp. 91.

[1]This work was supported in part by the Natural Sciences and Engineering Research Council of Canada and in part by Communications and Information Technology Ontario (CITO).