Using Polymatroid Structures to Provide Fairness in Multiuser Systems

Mohammad A. Maddah-Ali, Amin Mobasher, and Amir K. Khandani
Coding & Signal Transmission Laboratory (www.cst.uwaterloo.ca),
Dept. of Elec. and Comp. Eng., University of Waterloo,
Waterloo, Ontario, Canada, N2L 3G1,
e-mail: {mohammad, amin, khandani}@cst.uwaterloo.ca

***Abstract*— For a wide class of multi-user systems, a subset of capacity region which includes the corner points and the sum-capacity facet has a special structure known as polymatroid. Any interior point of the sum-capacity facet can be achieved by time-sharing among corner points or by an alternative method known as *rate-splitting*. The main purpose of this paper is to find a point on the sum-capacity facet which satisfies a notion of fairness among active users. In one case, the corner point for which the minimum rate of the active users is maximized (max-min corner point) is computed for signaling. In another case, the polymatroid properties are exploited to locate a rate-vector on the sum-capacity facet which is optimally fair in the sense that the minimum rate among all users is maximized (max-min rate). It is shown that the problems of deriving the time-sharing coefficients or rate-spitting scheme can be solved by decomposing the problem to some lower-dimensional subproblems. In addition, a fast algorithm to compute the time-sharing coefficients to attain a general point on the sum-capacity facet is proposed.**

I. Introduction

In the multi-user scenarios, multiple transmitters/receivers share a common communication medium, and therefore there is an inherent competition in accessing the channel. Information theoretic results for such systems imply that in order to achieve a high spectral efficiency, the users with stronger channel should have a higher portion of the resources. Apparently, the drawback is losing the fairness among the users. Providing fairness among users, while achieving high-spectral efficiency, emerges as a challenging problem.

A lot of research works have addressed this problem and suggested different criteria to design a fair system. One of the first criteria is known as *max-min* measure. In this method, the main effort is to maximize the minimum rate of the users.

By relaxing the strict condition on fairness, the spectral efficiency can be increases. As compromising solution between fairness and throughput, the proportional fairness is proposed in [1]. In [2], a criterion based on Nash Bargaining solution in the context of Game Theory is proposed. This method generalizes the proportional fairness and increases the efficiency of the system.

All of the aforementioned methods deal with a general multi-user system. However, for a wide class of multi-user systems, the capacity region has special structure that we can exploit to provide fairness. Particularly, in some multiuser systems, the boundary of the capacity region includes a facet on which the sum-rate is maximum (Sum-capacity facet). In such systems, one can benefit from the available degrees of freedom, and determine the fairest rate vector on the sum-capacity facet.

As a special case, we consider a class of multi-user systems, in which the whole or a subset of the capacity region which includes the corner points and the sum-capacity facet forms a structure known as polymatroid. For this class of multi-user systems, the sum-capacity facet has $a!$ corner points, where a is the number of users with non-zero power (active users). The sum-capacity facet is the convex hull of these corner points. This means that the interior points of the sum-capacity facet can be attained by time-sharing among such corner points.

In [3], the optimal dynamic power allocation strategy for time-varying single-antenna multiple-access channel is established. To this end, the polymatroid properties of the capacity region for fixed multiple-access channel with fixed input distribution have been exploited. In [4], the polymatroid properties have been used to find a fair power allocation strategy. This problem is formulated by representing a point on the face of the contra-polymatroid (see [3], [5]) as a convex combination of its extreme points.

This article aims at finding a point on the sum-capacity facet which satisfies a notion of fairness among active users by exploiting the properties of polymatroids. In order to provide fairness, the minimum rate among all users is maximized (max-min rate). In the case that the rate of some users can not increase anymore (attain the max-min value), the algorithm recursively maximizes the minimum rate among the rest of the users. Since this rate-vector is in the face of polymatroid, it can be achieved by time sharing among the corner points. It is shown that the problem of deriving the time-sharing coefficients to attain this point can be decomposed to some lower-dimensional subproblems. An alternative approach to attain an interior point for multiple access channel is *rate splitting* [6], [7]. This method is based on splitting the power of all users except one user into two parts, and treating each spilt user as two virtual users. By splitting the powers appropriately and successive decoding of virtual users in a suitable order, any point on the sum-capacity facet can be attained [6], [7]. Similar to the time-sharing procedure, we show that the problem of rate-splitting can be decomposed to some lower dimensional subproblems.

There are cases that the complexity of achieving interior points is not feasible. This motivates us to compute the corner point for which the minimum rate of the active users is maximized (max-min corner point). A simple greedy algorithm is introduced to find the max-min corner point.

The rest of the paper is organized as follows. In Section II, the structure of the polymatroid is presented. In addition, the relationship between the capacity region and the polymatroid structure is described. Section III discusses the case in which the optimal fair corner point is computed. In Section IV, the optimal fair rate-vector on the sum-capacity facet is computed by exploiting polymatroid structures. In addition, it is shown that the problem of deriving the time-sharing coefficients and rate-splitting can be solved by decomposing the problem into some lower-dimensional subproblems.

Notation: All boldface letters indicate vectors (lower case) or matrices (upper case). $\det(\mathbf{H})$ denotes determinant and $\mathbf{H}^\dagger$ denotes transpose conjugate of the matrix $\mathbf{H}$. $\mathbf{M} \succeq 0$ represents that the matrix $\mathbf{M}$ is positive semi-definite. $\mathbf{1}_n$ represents an n dimensional vector with all entries equal to one. E is a set of integers $E = \{1, \cdots, |E|\}$, where $|E|$ denotes the cardinality of the set E. The set function $f : 2^E \longrightarrow \mathcal{R}_+$ is a mapping from a subset of E (there are a total of $2^{|E|}$ subsets) to the positive real numbers. A permutation of the set E is denoted by π and $\pi(i)$, $1 \leq i \leq |E|$, represents the element of the set E located in the i^{th} position after the permutation. For an a-dimensional vector $\mathbf{x} = \{x_1, x_2, \ldots, x_a\} \in \mathcal{R}^a$ and $S \subset E$, $\mathbf{x}(S)$ denotes $\sum_{i \in S} x_i$. Also, for a set of positive semi-definite matrices $\mathbf{D}_i$, $\mathbf{D}(S)$ represents $\sum_{i \in S} \mathbf{D}_i$.

II. Preliminaries

Definition [8, Ch. 18]: Let $E = \{1, 2, \ldots, a\}$ and $f : 2^E \longrightarrow \mathcal{R}_+$ be a set function. The polyhedron

$$\mathcal{B}(f, E) = \{(x_1, \ldots, x_a) : \mathbf{x}(S) \leq f(S), \forall S \subset E, \forall x_i \geq 0\} \tag{1}$$

is a polymatroid, if the set function f satisfies

$$(normalized) \qquad f(\emptyset) = 0 \tag{2}$$

$$(increasing) \qquad f(S) \leq f(T) \text{ if } S \subset T \tag{3}$$

$$(submodular) \qquad f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \tag{4}$$

Any function f that satisfies the above properties is termed as *rank function*. Note that (1) imposes $2^{|E|}$ constraints on any given vector $(x_1, \ldots, x_a) \in \mathcal{B}(f, E)$.

Corresponding to each permutation π of the set E, the polymatroid $\mathcal{B}(f, E)$ has a corner point $\mathbf{v}(\pi) \in \mathcal{R}_+^a$ which is equal to:

$$v_{\pi(i)}(\pi) = \begin{cases} f(\{\pi(i)\}) & i = 1 \\ f(\{\pi(1), \ldots, \pi(i)\}) & \\ \quad -f(\{\pi(1), \ldots, \pi(i-1)\}) & i = 2, \ldots, a \end{cases} \tag{5}$$

Consequently, the polymatroid $\mathcal{B}(f, E)$ has $a!$ corner points corresponding to different permutations of the set E. All the corner points are on the hyperplane $\mathbf{x}(E) = f(E)$. In addition, any point in the polymatroid on the facet $\mathbf{x}(E) = f(E)$ is in the convex hull of these corner points. The hyperplane $\mathbf{x}(E) = f(E)$ is called as dominant face, or simply face of the polymatroid. In this paper, we use the term *sum-capacity facet* to denote the face of the polymatroid.

For a wide class of multi-user systems, the whole or a subset of the capacity region forms a polymatroid structure. As the first example, consider a multi-access system with a users, where the distribution of inputs are independent and equal to $p(x_1), \ldots, p(x_M)$. Then, the capacity region of such a system is characterized by [9], [10]

$$\left\{\mathbf{r} \in \mathcal{R}_+^r | \mathbf{r}(S) \leq I\left(y; \{x_i, i \in S\} | \{x_i, i \in S^c\}\right), \forall S \subset E\right\}, \tag{6}$$

where y is the received signal, $\mathbf{r}$ represents rate vector, I denotes the mutual information, and S^c is equal to $E - S$. It has been shown that the above polyhedron forms a polymatroid [3].

As the second example, consider the capacity region of a Multiple-Antenna Broadcast System. A subset of the capacity region which includes the corner points and sum-capacity facet forms a polymatroid. The reader is refereed to [11] for more details.

III. The Fairest Corner Point

As mentioned, in some cases, the complexity of computing and implementing an appropriate time-sharing or rate-splitting algorithm is not feasible. This motivates us to compute the corner point for which the minimum rate of the active users is maximized (max-min corner point). In the following, we present a simple greedy algorithm to find the max-min corner point of a general polymatroid $\mathcal{B}(f, E)$.

Algorithm I

1) Set $\alpha = a$, $S = \varnothing$.
2) Set $\pi^*(\alpha)$ as

$$\pi^*(\alpha) = \arg \min_{z \in E, z \notin \mathcal{S}} f\left(E - S - \{z\}\right) \tag{7}$$

3) If $\alpha > 1$, then $S \longleftarrow S \cup \{\pi^*(\alpha)\}$, $\alpha \longleftarrow \alpha - 1$, and go to Step 2; otherwise stop.

The following theorem proves the optimality of the above algorithm.

Theorem 1 *Let the vector* $\mathbf{v}(\pi^*)$ *be the corner point of the polymatroid* $\mathcal{B}(f, E)$ *corresponding to the permutation* $\pi^* = (\pi^*(1), \ldots, \pi^*(a))$. *For any other permutation* $\pi = (\pi(1), \ldots, \pi(a))$,

$$\min_i v_{\pi^*(i)}(\pi^*) \geq \min_i v_{\pi(i)}(\pi). \tag{8}$$

Proof: see [11]. ■

Remark: In the case of multiple access channel, the above algorithm implies an interesting result. It suggests that to attain the fairest corner point with successive decoding, at each step, one should decode the strongest user (the user with the highest rate, while the signals of the remaining users are considered as interference). Note that in MAC, the corner point corresponding to the specific permutation π is obtained by the successive decoding in the reverse order of the permutation.

It is worth mentioning that by using a similar algorithm, one can find the corner point for which the maximum rate is minimum. The algorithm is as follows:

Algorithm II

1) Set $\alpha = 1$, $S = \varnothing$.
2) Set $\pi^*(\alpha)$ as

$$\pi^*(\alpha) = \arg \max_{z \in E, z \notin \mathcal{S}} f(S + \{z\}) \tag{9}$$

3) If $\alpha < a$, then $S \longleftarrow S \cup \{\pi^*(\alpha)\}$, $\alpha \longleftarrow \alpha + 1$, and go to Step 2; otherwise stop.

The optimality of the above algorithm can be proven by a similar method as used to prove Theorem 1.

IV. Optimal Rate-Vector on the Sum-Capacity Facet

A. Max-Min Operation over a Polymatroid

In the following, the polymatroid properties are exploited to locate an optimal fair point on the sum-capacity facet. For an optimal fair point, the minimum rate among all the users should be maximized (max-min rate). For a sum-capacity of r_{SC}, a fair rate allocation would ideally achieve an equal rate of $\frac{r_{\text{SC}}}{a}$ for the a active users. Although this rate vector is feasible for some special cases (see Fig. 1), it is not attainable in the general case (see Fig. 2). The maximum possible value for the minimum entry of a vector $\mathbf{x}$ where $\mathbf{x} \in \mathcal{B}(f, E)$ can be computed using the following lemma.

Lemma 1 *In the polymatroid $\mathcal{B}(f, E)$, define*

$$\begin{aligned} \delta \quad &= \quad \max \min_{i \in E} x_i. \\ &s.t. \quad (x_1, \ldots, x_a) \in \mathcal{B}(f, E). \end{aligned} \tag{10}$$

Then,

$$\delta = \min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}. \tag{11}$$

Proof: see [11]. ■

In minimization (11), if the minimizer is not the set E, then δ (the optimal max-min value) is less than $\frac{r_{\text{SC}}}{a}$ ($r_{\text{SC}} = f(E)$ is the sum-capacity), and therefore the ideal fairness is not feasible. For example in the polymatroid, depicted in Fig 2, the minimizing set in (11) is the set $\{3\}$, and therefore $\delta = f(\{3\})$.

In the following, a recursive algorithm is proposed to locate a rate vector $\mathbf{x}^*$ on the sum-capacity facet which not only attains the optimal max-min value δ, but also provides fairness among the users which have the rates higher than δ. The proposed algorithm partitions the set of active users into $t+1$ disjoint subsets, $S^{(0)}, ..., S^{(t)}$, such that in the i'th subset, the rate of all the users is equal to $m^{(i)}, i = 0, \cdots, t$, where $\delta = m^{(0)} < m^{(1)} < \cdots < m^{(t)}$. Starting from $m^{(0)}$, the algorithm maximizes $m^{(i)}$, $i = 1, \cdots, t$, given that $m^{(j)}$'s, $j = 0, \cdots, i-1$, are already at their maximum possible value.

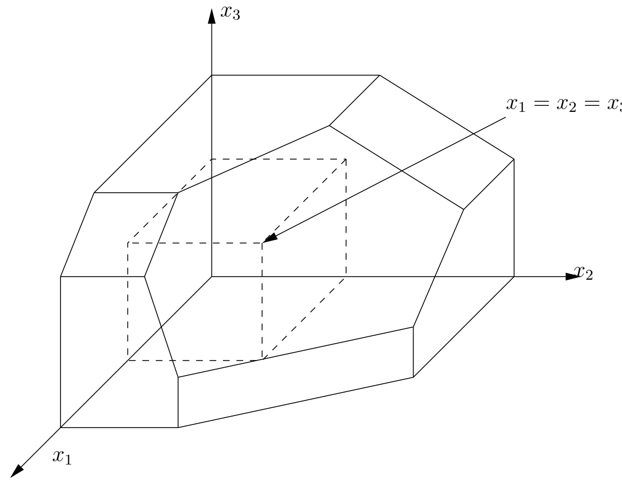

Fig. 1. All Equal Rate-Vector Is on the Sum-Capacity Facet

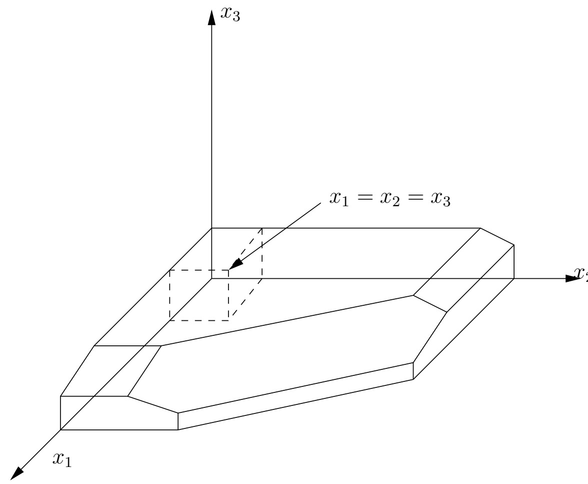

Fig. 2. All Equal Rate-Vector Is NOT on the Sum-Capacity Facet

To simplify this procedure, we establish a chain of nested polymatroids, $\mathcal{B}(f^\alpha, E^\alpha)$, $\alpha = 0, \ldots, t$, where

$$\mathcal{B}(f^t, E^t) \subset \mathcal{B}(f^{t-1}, E^{t-1}) \subset \ldots \subset \mathcal{B}(f^0, E^0) = \mathcal{B}(f, E). \tag{12}$$

In the algorithm, we use the result of the following lemma.

Lemma 2 *Let $E = \{1, \ldots, a\}$ and $A \subset E$, $A \neq E$. If the set function $f : 2^E \longrightarrow \mathcal{R}_+$ is a rank function, then $h : 2^{E-A} \longrightarrow \mathcal{R}_+$, defined as*

$$h(S) = f(S \cup A) - f(A), \quad S \subset E - A, \tag{13}$$

is a rank function.

Proof: By direct verification. ■

Using the following algorithm, one can compute the rate vector $\mathbf{x}^*$.

Algorithm III

1) Initialize the iteration index $\alpha = 0$, $E^{(0)} = E$, and $f^{(0)} = f$.

2) Find $m^{(\alpha)}$, where

$$m^{(\alpha)} = \min_{S\subset E^{(\alpha)}, S\neq\emptyset} \frac{f^{(\alpha)}(S)}{|S|}. \tag{14}$$

Set $S^{(\alpha)}$ equal to the optimizing subset.

3) For all $i \in S^{(\alpha)}$, set $x_i^* = m^{(\alpha)}$.

4) Define the polymatroid $\mathcal{B}(f^{(\alpha+1)}, E^{(\alpha+1)})$, where

$$E^{(\alpha+1)} = E^{(\alpha)} - S^{(\alpha)}, \tag{15}$$

and $\forall S \subset E^{(\alpha+1)}$,

$$f^{(\alpha+1)}(S) = f^{(\alpha)}(S \cup S^{(\alpha)}) - f^{(\alpha)}(S^{(\alpha)}). \tag{16}$$

5) If $E^{(\alpha+1)} \neq \emptyset$, set $\alpha \longleftarrow \alpha + 1$ and move to step two, otherwise stop.

This algorithm computes the optimization sets $S^{(\alpha)}$, $\alpha = 0, \cdots, t$ and their corresponding $m^{(\alpha)}$, where $E = \bigcup_{j=0}^{t} S^{(j)}$ and $x_i^* \in \{m^{(0)}, \cdots, m^{(t)}\}, i = 1, \cdots, a$.

In the following, we prove some properties of the vector $\mathbf{x}^*$.

Theorem 2 *Assume that the algorithm III is applied over the polymatroid $\mathcal{B}(f, E)$, then*

(I) $\mathbf{x}^* \in \mathcal{B}(f, E)$ *and is located on the sum-capacity facet* $\mathbf{x}(E) = f(E)$.

(II) *The minimum entry of the vector $\mathbf{x}^*$ attains the optimum value determined by Lemma 1 and*

$$\delta = m^{(0)} < m^{(1)} < \cdots < m^{(t)} \tag{17}$$

Proof: see [11]. ∎

The remaining issue in Algorithm II is how to compute $\min_{S\subset E, S\neq\emptyset} \frac{f(S)}{|S|}$. These types of problems are known as geometric minimization. In order to find the minimizer, the smallest value of β is desirable such that there is a set S with $f(S) = \beta|S|$. For the special case of single antenna Gaussian multi-access channel, computing such β is very simple. For the general case, β can be computed by Dinkelbach's discrete Newton method as follows [12].

The algorithm is initialized by setting β equal to $f(E)/|E|$, which is an upper bound for optimum β. Then, a minimizer Y of $f(S) - \beta|S|$ is calculated as will be explained later. Since $f(E) - \beta|E| = 0$, then $f(Y) - \beta|Y| \leq 0$. If $f(Y) - \beta|Y| = 0$, the current β is optimum. If $f(Y) - \beta|Y| < 0$, then we update $\beta = f(Y)/|Y|$, which provides an improved upper bound. By repeating this operation, the optimal value of β will be eventually calculated [13]. It is shown that the number of β visited by the algorithm is at most $|E|$ [12].

Using this approach, the minimization problem $\min_{S\subset E, S\neq\emptyset} \frac{f(S)}{|S|}$ is changed to $\min_{S\subset E, S\neq\emptyset} f(S) - \beta|S|$. By direct verification of (4), it is easy to see that $f(S) - \beta|S|$ is a submodular function. There have been a lot of research on submodular minimization problems [12], [14], [15]. In [14], [15], the first combinatorial polynomial-time algorithms for solving submodular minimization problems are developed. These algorithms design a combinatorial strongly polynomial algorithm for testing membership in polymatroid polyhedra.

B. Decomposition of the Time-Sharing Problem

In the following, we take advantage of the special properties of $\mathbf{x}^*$ and polymatroids to break down the time-sharing problem to some lower dimensional subproblems. In the previous subsection, a chain of nested polymatroids $\mathcal{B}(f^{\alpha}, E^{\alpha})$, $\alpha = 0, \ldots, t$, is introduced, where $\mathcal{B}(f^{(\alpha-1)}, E^{(\alpha-1)}) \subset \mathcal{B}(f^{(\alpha)}, E^{(\alpha)})$ for $\alpha = 1, \ldots, t$. Since $S^{(j)} \subset E^{(j)}$ for $j = 0, \ldots, t$ and regarding the definition of polymatroid, $\mathcal{B}(f^{(j)}, S^{(j)})$, $j = 1, \ldots, t$, is a polymatroid, which is defined on the dimensions $S^{(j)}$). According to the proof of the Theorem 2 [11], the vector $m^{(j)}\mathbf{1}_{|S^{(j)}|} \in \mathcal{B}(f^{(j)}, S^{(j)})$ is on the hyperplane $\mathbf{x}(S^{(j)}) = f(S^{(j)})$. Let $\{\pi_{\gamma_j}^{(j)}, \gamma_j = 1, \ldots, |S^{(j)}|!\}$ be the set of all permutations of the set $S^{(j)}$, and $\mathbf{u}^{(j)}(\pi_{\gamma_j}^{(j)})$ be the corner point corresponding to the permutation $\pi_{\gamma}^{(j)}$ in the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$. Then, there exists the coefficients $0 \leq \lambda_{\gamma}^{(j)} \leq 1$, $\gamma = 1, \ldots, |S^{(j)}|!$, such that

$$m^{(j)}\mathbf{1}_{|S^{(j)}|} = \sum_{\gamma_j=1}^{|S^{(j)}|!} \lambda_{\gamma_j}^{(j)} \mathbf{u}^{(j)}\big(\pi_{\gamma_j}^{(j)}\big), \tag{18}$$

where

$$\sum_{\gamma_j=1}^{|S^{(j)}|!} \lambda_{\gamma_j}^{(j)} = 1. \tag{19}$$

Note that $E = \bigcup_{j=0}^{t} S^{(j)}$. Consider a permutation $\pi_{\gamma_j}^{(j)}$ as one of the total $|S^{(j)}|!$ permutations of $S^{(j)}$, for $j = 0, \cdots, t$, then the permutation π formed by concatenating these permutations, i.e. $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$, is a permutation on the set E.

Theorem 3 *Consider the permutation $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$ of the set E.*

(I) *The corner point corresponding to this permutation in the polymatroid $\mathcal{B}(f, E)$ is $\mathcal{B}(f^{(j)}, S^{(j)})$*

$$v_i(\pi) = u_i^{(j)}(\pi_{\gamma_j}^{(j)}), \quad \text{for} \quad i \in S^{(j)}, \tag{20}$$

where $\mathbf{u}^{(j)}(\pi_{\gamma_j}^{(j)})$ is the corner point of the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$ corresponding to the permutation $\pi_{\gamma_j}^{(j)}$, and $u_i^{(j)}(\pi_{\gamma_j}^{(j)})$ denotes the value of $\mathbf{u}^{(j)}(\pi_{\gamma_j}^{(j)})$ over the dimension i, $i \in S^{(j)}$.

(II) *The vector $\mathbf{x}^*$ is in the convex hull of the set of corner points corresponding to the following set of permutations*

$$\left\{\left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right), 1 \leq \gamma_t \leq |S^{(t)}|!, \ldots, 1 \leq \gamma_0 \leq |S^{(0)}|!\right\}, \tag{21}$$

where the coefficient of the corner point corresponding to the permutation $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$ is equal to $\lambda_{\gamma_t}^{(t)} \ldots \lambda_{\gamma_0}^{(0)}$, i.e.

$$\mathbf{x}^* = \sum_{\gamma_t=1}^{|S^{(t)}|!} \cdots \sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda_{\gamma_t}^{(t)} \ldots \lambda_{\gamma_0}^{(0)} \mathbf{v}\Big(\left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)\Big). \tag{22}$$

Proof: see [11]. ■

Regarding the above statements, the problem of finding the time-sharing coefficients is decomposed to some lower dimensional subproblems. In this part, we develop an algorithm which finds the coefficients of the time-sharing directly over the corner points of polymatroid $\mathcal{B}(f, E)$ to attain $\mathbf{x}^*$ (or any other vector on the convex hull of the corner points).

Algorithm IV

1) Initialize $\alpha = 1$, $\mathbf{u}_1 = \mathbf{v}(\pi^*)$ (the fairest corner point obtained by algorithm I).
2) Solve the linear optimization problem

$$\begin{aligned} &\max \tau \\ s.t. \quad &\textstyle\sum_{i=1}^{\alpha} \mu_i \mathbf{u}_i - \mathbf{x} \geq \tau \\ &0 \leq \mu_i \leq 1 \end{aligned} \tag{23}$$

   Let μ_i^α, $i = 1, \ldots, \alpha$ be the optimizing coefficients.
3) If $\mathbf{x} = \sum_{i=1}^{\alpha} \mu_i^\alpha \mathbf{u}_i$, Stop.
4) $\alpha \longleftarrow \alpha + 1$. Set $\mathbf{e} = \mathbf{x} - \sum_{i=1}^{\alpha} \mu_i^\alpha \mathbf{u}_i$ and determine the permutation π for which $\mathbf{e}_{\pi(1)} \geq \mathbf{e}_{\pi(2)} \geq \ldots \geq \mathbf{e}_{\pi(|E|)}$. Set $\mathbf{u}_\alpha = \mathbf{v}(\pi)$ and move to step 2.

The idea behind the algorithm is as follows. In each step, the time-sharing among some corner points is performed. If all entries of the resulting vector are equal, the answer is obtained. If the entries are not equal, then a permutation π is determined such that $\mathbf{e}_{\pi(1)} \geq \mathbf{e}_{\pi(2)} \geq \ldots \geq \mathbf{e}_{\pi(|E|)}$. We can compensate the error vector $\mathbf{e}$ by including an appropriate corner point in the set of corner points participating in time-sharing. Clearly, the best one to be included is the one which has the highest possible rate for user $\pi(1)$ and lowest possible rate for user $\pi(|E|)$. Apparently, this corner point is $\mathbf{v}(\pi)$, computed by the above algorithm.

Note that the Algorithm IV can be applied over the sub-polymatroid to solve the lower dimensional problems or directly applied over the original polymatroid. If a and $|S^j|$ are relatively small numbers, the decomposition method has less complexity, but for large a applying the Algorithm IV over the original problem is less complex.

C. Decomposition of Rate-Splitting Approach

As mentioned, an alternative approach to achieve any rate vector on the sum-capacity facet of MAC is *rate splitting* [6], [7]. This method is based on splitting the power of all users except one into two parts. The users with split power is treated as two virtual users. Thus, there are at most $2a - 1$ virtual users. It is proven that by splitting the powers appropriately and successive decoding of virtual users in a suitable order, any point on the sum-capacity facet can be attained.

Similar to the time-sharing part, we prove that to attain the rate vector $\mathbf{x}^*$, the rate-splitting procedure can be decomposed into some lower dimensional subproblems. Consider a multi-access channel, where the capacity region is represented by polymatroid $\mathcal{B}(f, E)$ and the vector $\mathbf{x}^*$, derived in Algorithm III, is on its face. Assume that the users in the set $S^{(j)}$ is decoded before the set of users in $\{S^{(j-1)}, S^{(j-2)}, \ldots, S^{(0)}\}$ and after the users in the set $\{S^{(t), \ldots, S^{(j+2)}, S^{(j+1)}}\}$. It can be shown that the rate of the users in the set $S^{(j)}$ is characterized by the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$, where the rate vector $m^{(j)}\mathbf{1}_{|S^{(j)}|}$ is on its face [11]. Regarding the results presented in [6], [7], we can attain this rate vector by properly splitting the powers of all user, except for one, in the set $S^{(j)}$ to form $2|S^{(j)}| - 1$ virtual users and by choosing the proper order of the decoding of the virtual users. Consequently, using the following algorithm, we achieve the rate-vector $\mathbf{x}^*$ in the original polymatroid.

Algorithm V

1) Apply rate-splitting approach to attain the rate vector $m^{(j)}\mathbf{1}_{|S^{(j)}|}$ on the face of the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$, for $j = 0, \ldots, t$. Therefore, at most $2|S^{(j)}| - 1$ virtual users are specified with a specific order of decoding.
2) Starting from $j = t$, decode the virtual users in the set $S^{(j)}$ in the order found in Step 1. Set $j \leftarrow j - 1$. Follow the procedure until $j < 0$.

Acknowledgement

The authors would like to thank Mr. Mohammad H. Baligh and Mr. Shahab Oveis Gharan for helpful discussions.

References

[1] F.P. Kelly, "Charging and rate control for elastic traffi c," *European Transactions on Telecommunications*, vol. 8, pp. 33–37, 1997.

[2] Z. Han, Z. Ji, and K.J.R. Liu, "Fair multiuser channel allocation for OFDMA networks using Nash bargaining solutions and coalitions," *IEEE Transactions on Communications*, vol. 53, pp. 1366–1376, Aug 2005.

[3] D.N.C. Tse and S.V. Hanly, "Multiaccess fading channels. I. Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2796–2815, Nov. 1998.

[4] Y. Shi and E. J. Friedman, "Algorithms for implementing fair wireless power allocations," in *the 9th Canadian Workshop on Information Theory*, Montreal, Quebec, Canada, June 2005, pp. 171–174.

[5] S.V. Hanly and D.N.C Tse, "Multiaccess fading channels. II. delay-limited capacities," *IEEE Transactions on Information Theory*, vol. 44, pp. 2816–2831, Nov. 1998.

[6] B. Rimoldi and R. Urbanke, "A rate-splitting approach to the Gaussian multiple-access channel," *IEEE Transactions on Information Theory*, vol. 42, pp. 364–375, March 1996.

[7] A.J. Grant, B. Rimoldi, R.L. Urbanke, and P.A. Whiting, "Rate-splitting multiple access for discrete memoryless channels," *IEEE Transactions on Information Theory*, vol. 47, pp. 873–890, March 2001.

[8] D. J. A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

[9] R. Ahlswede, "Multiway communication channels," in *Proc. 2nd. Int. Symp. Information Theory*, Arminian S.S.R. Prague, 1971, pp. 23–52.

[10] H. Liao, *Multiple access channels*, Ph.D. thesis, Dep. Elec. Eng., Univ. of Hawaii., 1972.

[11] M. A. Maddah-Ali, A. Mobasher, and A. K. Khandani, "On the Fairest Corner Point of the MIMO-Broadcast Capacity Region," Tech. Rep. UW-E&CE 2005-15, University of Waterloo, 2005.

[12] L. Fleischer and S. Iwata, "A push-relabel framework for submodular function minimization and applications to parametric optimization," *Discrete Applied Mathematics*, vol. 131, no. 2, pp. 311–322, 2003.

[13] C. Scott, *Geometric programming*.

[14] A. Schrijver, "A combinatorial algorithm for minimizing submodular functions in strongly polynomial time," *Journal of Combinatorial Theory, B80*, pp. 346–355, 2000.

[15] S. Iwata, L. Fleischer, and S. Fujishige, "A combinatorial strongly polynomial time algorithm for minimizing submodular functions," *Journal of ACM*, vol. 48, no. 4, pp. 761–777, 2001.