Optimum Non-integer Rate Allocation Using Integer Programming*

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Abstract: Rate allocation is the problem of distributing a given quota of bits among a number of different quantizers such that the total distortion of all the quantizers is minimized. Conventionally, the problem of rate allocation is handled by limiting the number of levels allocated to each quantizer to be an integer power of two. Traditional methods are also based on assuming that the quantizer distortion curves have certain convexity property. In this work, we discuss a new method for the optimum rate allocation using an integer programming formulation, which unlike the conventional methods, does not make any assumption on either the integrality of the allocated bit rate, or on the convexity of the distortion curves. We apply this method to a fixed rate and also to a variable rate coding schemes. Numerical results are presented for the DCT coding of still images showing a noticeable improvement in the performance with no increase in the complexity.

1 Introduction

The basic promise of an image transform coding system is that the two-dimensional transform of an image has an energy distribution more suitable for coding than the spatial domain representation. In other words, as a result of the pixel-to-pixel correlation of image the energy in the transform domain tends to be clustered into a relatively small number of transform samples and therefore bandwidth reduction is achieved. In this case, in transmitting the transform samples instead of the image, low-magnitude transform samples can be discarded in an analog transmission system, or grossly quantized in a digital transmission system, without introducing serious image degradation.

The efficiency of the various 2-D transforms depends upon their energy compaction characteristic. The optimum transform which minimizes the distortion in the reconstructed signal turns out to be a non-causal, linear filter, known as the Karhunen-Loeve (KL) transform. In practice, the KL transform is substituted by a suboptimal, but fast unitary transform. Among the existing transforms, Discrete-Cosine-Transform (DCT) has been defined as an international standard (JPEG) because of its closeness to the optimal transform and easy computation.

2 Rate allocation

Rate allocation is a major concern in any coding scheme where a given total rate must be efficiently distributed among a number of different quantizers. The performance of such a set of quantizers is characterized by their quantizer distortion curves, defined by the average quantization distortion as a function of the rate. The set of quantizer distortion curves is used by the allocation algorithm to determine the best strategy which minimizes the overall distortion.

The conventional methods to the rate allocation are based on one of the following two main approaches [1]: (i) Treating the allocated bits (and also the quantizer distortion curves) as being continuous and using a Lagrangian optimization method to compute the bit allocation. (ii) Allocating the available bits in an incremental manner by first assigning zero bit to all the quantizers and

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then distributing the available bits one by one between the quantizers such that the decrease in the overall distortion due to the allocation of each bit is maximized. It can be shown that this second method results in the optimum bit allocation if the quantizer distortion curve poses certain convexity property [1, 2]. However, a serious drawback of this method is that it restricts the number of levels allocated to each quantizer to be an integer power of two (integer bit rate). In the following, we present an integer programming formulation of the rate allocation problem which does not suffer from any of these assumptions and shortcomings. This method is also applied in the case that the quantizer partitions are labeled by a set of Huffman codes (with the objective of minimizing the distortion subject to having a fixed average bit rate).

To formally define the problem of bit allocation, consider a quantization system which is used to quantize a set of K random variables, say $X^0, X^1, \dots, X^{K-1}$, each with zero mean and with variance $E[(X^i)^2]$, $i = 0, 1, \dots, K-1$. Assume that X^i is quantized with a scalar quantizer composed of N_i levels where N_i is a non-negative integer. Define the normalized quantizer profile $W_i(N_i)$ as the mean square error incurred in quantizing X^i with N_i levels. The gain factor G_i accompanies $W_i(N_i)$ to count for the variance of the input signal. In most cases, the overall performance of the quantization system is determined by the sum of the distortions associated with different quantizers. The level allocation problem is to find an allocation vector $\vec{\mathbf{N}} = (N_0, N_1, \dots, N_{K-1})$, which minimizes,

$$D(\vec{\mathbf{N}}) = \sum_{i=0}^{K-1} G_i W_i(N_i), \tag{1}$$

subject to,

$$\sum_{i=0}^{K-1} \log_2(N_i) \leq B, \quad 1 \leq N_i \leq u_i, \quad i = 0, \dots, K-1, \tag{2}$$

where B is the fixed quota of available bits, and u_i's are the upper limits for the admissible number of levels. In all cases, we have selected the u_i's at a high enough value such that they do not impose any constraint on the optimum solution.

3 Integer Programming Formulation

Integer programming (IP) deals with constrained optimization problems in which the variables are restricted to be integer. A linear integer programming problem can be written as,

$$\text{minimize } \{cx : Ax \leq b, x \in Z_+^n\},$$

where Z_+^n is the set of non-negative integral n-tuples, and the n-vector x is the set of unknowns (c is an n-vector, A is an $m \times n$ matrix, and b is an m-vector).

To formulate the rate allocation problem, we define the binary variables $\delta_i(j)$ as,

$$\delta_i(j) = \begin{cases} 1 & \text{if } j \text{ levels are allocated} \\ & \text{to the } i\text{th quantizer,} \\ 0 & \text{otherwise.} \end{cases}$$

Using these notations, we obtain the following IP formulation for the optimization problem given in Eq. (1) and (2):

$$\text{minimize } D = \sum_{i=0}^{K-1} \sum_{j=1}^{u_i} \delta_i(j) G_i W_i(j), \tag{3}$$

subject to:

$$\sum_{j=1}^{u_i} \delta_i(j) = 1, \quad \forall i \in \{0, .., K-1\}, \tag{4}$$

$$\sum_{i=0}^{K-1} \sum_{j=1}^{u_i} \delta_i(j) \log_2(j) \leq B, \tag{5}$$

$$\delta_i(j) \in \{0, 1\}. \tag{6}$$

Note that Eq. 4 ensures that each quantizer has only one value for the number of the allocated levels.

The major complicated operation in dealing with a non-integer bit rate is the problem of addressing (or labeling) of the quantizer partitions[1].

[1] In the conventional case of integer bit rate, addressing is simply achieved by concatenating the labels of the corresponding scalar quantizers. This method however is not applicable to the case of the non-integer bit allocation.

In this case, one can use a method based on an expansion of integers on a proper number system (with unequal values for different basis) to mix the labels of the corresponding scalar quantizers.

4 Variable Rate Scheme

In the case of using a variable rate code to index the corresponding quantizers, relationship (5) is modified to,

$$\sum_{i=0}^{K-1} \sum_{j=1}^{u_i} \delta_i(j) L_i(j) \leq B, \quad (7)$$

where $L_i(j)$ is the average length of the corresponding Huffman code for the case that j levels are allocated to the i'th quantizer. Using Huffman codes also solve the addressing problem which was a rather complicated operation in the case of using a fixed rate coding scheme (in conjunction with non-integer rate allocation).

5 Numerical Results

We have applied the proposed rate allocation methods for the quantization of the DCT coefficients of still images where the corresponding scalar quantizers are designed using the iterative method explained in [3]. In all cases, the corresponding integer optimization problem is solved using an application software called the General Algebraic Modeling System (GAMS) version 2.25. It should be mentioned that, in general, the complexity of solving a linear integer optimization problem is substantially higher than the complexity of solving the underlying linear program. The important point is that the linear solution of the optimization problem involved in this work satisfies the corresponding integrality constraints in the majority of the cases (no counter-example observed). It can be shown that in the event of the rare cases that this property does not hold, one can easily compute an integer solution (which is possibly slightly sub-optimum) using a simple round-off of the corresponding linear solution. This property is in the favor of the proposed method for the applications that the bit allocation problem should be solved dynamically (with a small computational complexity).

Rate	Case (a)	Case (b)	Case (c)
0.5 bits/pixel	20.0 dB	19.9 dB	20.2 dB
1.0 bits/pixel	22.8 db	22.7 db	23.1 db
1.5 bits/pixel	24.7 dB	24.6 dB	25.2 dB
2.0 bits/pixel	26.3 dB	26.2 dB	27.0 dB

Table 1: Signal-to-Noise ratio in dB for different rate allocation methods (for Lenna image).

Table 1 contains some examples of the results obtained. The numbers should be interpreted as follows: Case (a) is obtained by allocating $B = 0.5, 1, 1.5, 2$ bits/pixel using the formulation given in Eq. (3) through (6). Case (b) is obtained by allocating $B = 0.5, 1, 1.5, 2$ bits/pixel using the formulation given in Eq. (3) through (6) where the rates allocated are restricted to be integer. Note that Case (b) corresponds to the traditional bit allocation problem (which is solved here optimally using the proposed integer programming formulation). Finally, Case (c) is obtained by using Eq. (7) as the constraint on the rate, where the value of B is set equal to the average length of a set of Huffman codes designed to label the quantizers obtained as the outcome of Case (b).

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