Power Allocation and Asymptotic Achievable Sum-Rates in Single-Hop Wireless Networks

Masoud Ebrahimi, Mohammad A. Maddah-Ali, and Amir K. Khandani
Coding and Signal Transmission Laboratory
Electrical and Computer Engineering Department
University of Waterloo
Waterloo, ON, Canada
Emails: {masoud, mohammad, khandani}@cst.uwaterloo.ca

***Abstract*— A network of n communication links operating over a shared wireless channel is considered. Power management is crucial to such interference-limited networks to improve the aggregate throughput. We consider sum-rate maximization of the network by optimum power allocation when conventional linear receivers (without interference cancellation) are utilized. It is shown that in the case of $n = 2$ links, the optimum power allocation strategy is such that either both links use their maximum power or one of them uses its maximum power and the other keeps silent. An asymptotic analysis for large n is carried out to show that in a Rayleigh fading channel the average sum-rate scales at least as $\log(n)$. This is obtained by deriving an *on-off power allocation strategy*. The same scaling law is obtained in the work of Gowaikar et al., where the number of links, their end-points (source-destination pairs), and the relay nodes are optimally chosen all by a central controller. However, our proposed strategy can be implemented in a decentralized fashion for any number of links, arbitrary transmitter-receiver pairs, and without any relay nodes. It is shown that the proposed power allocation scheme is optimum among all on-off power allocation strategies in the sense that no other strategies can achieve an average sum-rate of higher order.**

I. INTRODUCTION

In a wireless network, a number of source nodes transmit data to their designated destination nodes through a shared wireless channel. Capacity of such networks has received considerable attention in the literature [1]–[3]. Based on the network structure, throughput optimization can be executed in different ways, e.g. by power control [4], bandwidth allocation [5], [6], transmission scheduling [7], routing [1], [3], [8], base station selection [9], etc. Among these various challenging problems, power control has a prominent role in the ongoing research in this area.

In this work, we investigate the sum-rate of single-hop wireless networks achievable by means of power allocation. In our model, interference is treated as Gaussian noise. Thus, one can de£ne the *signal to interference-plus-noise ratio* ($SINR$) and obtain the achievable rate from the Shannon capacity formula. The network under consideration consists of n transmitter-receiver pairs, referred to as *links* or *users*. This model includes single-hop wireless networks, cellular networks, and code division multiple access (CDMA) systems as its special cases.

The problem of sum-rate maximization has been frequently appeared in the literature [10]–[13]. This problem translates to the problem of maximizing a product of linear fractional functions[1], which is a non-convex problem. Although there are algorithms to £nd the global optimum of such problems [14], their complexity precludes them from being implemented practically. Thus, one should think of £nding suboptimum methods which are simple and yet their performance is not far from the optimum. One approach is to utilize numerical optimization methods [15] to solve the problem (see e.g. [12]). Since the problem is non-convex, these methods may converge to local optimum solutions. Another approach is to adopt an approximation of the objective function such that the problem can be converted to a convex program. Speci£cally, one common technique that has been utilized in [10]–[12] is the assumption of large *SINR*; with this assumption, the 1 in the Shannon capacity formula is neglected and the rate of each link becomes proportional to the logarithm of the corresponding $SINR$ (i.e. $\log(1 + SINR) \approx \log(SINR)$). As a result, the problem is easily converted to a convex program [16]. Unfortunately, in the interference channel, the assumption of large $SINR$ is not valid. The reason is that due to the presence of interference the solution of the optimization problem is not guaranteed to satisfy this condition even when the noise power is quite low. In addition to being suboptimal, the above methods do not provide a suitable framework for analyzing the achievable sum-rates.

In this work, we show that in the case of 2 links, the optimum power allocation is such that either both transmitters transmit with maximum power or one them remains off and the other transmits with maximum power. Stimulated by this result and the result of [13], we consider a power allocation scheme where the powers of all links are selected from 0 and a maximum value.

Followed by the pioneering work of Gupta and Kumar [1], considerable attention has been paid to £nd out how the throughput of wireless networks scales with n, the number of nodes (see [3], [17] and the references therein). The random behaviour of the channel is considered in [3], where it is shown that the throughput of the network heavily depends on the channel distribution. In particular, in a Rayleigh fading channel, it is shown that the throughput scales at least as

[1]See Section III for illustrations.

$\log(n)$ for large n.

A common attribute of the works in [1], [3], and [17] is that the direction of the information ¤ow is chosen optimally to maximize the system throughput. However, in a realistic network, the source nodes and their corresponding destinations are predetermined. In this work, we show that in our wireless network model, which has preset links and no relay nodes, power allocation can achieve a throughput equal to $\log(n)$ for Rayleigh fading. This result is the same as what is obtained in [3], while unlike [3], the source-destination pairs are £xed and no relays are used. An optimum power allocation strategy is suggested that can be implemented in a decentralized fashion.

The rest of the paper is organized as follows. In Section II, the system description is provided. The formulation of the sum-rate maximization is presented in Section III. We derive a lower bound on the average sum-rate in Section IV, and prove its optimality among all on-off strategies in Section V. Finally, we conclude the paper in Section VI.

Notation: Bold face lower case (upper case) letters denote vectors (matrices); $\mathbf{0}_n$ and $\mathbf{1}_n$ stand for the all-zero and all-one column vectors of length n, respectively; $\mathbb{N}_n$ represents the set of natural numbers less than or equal to n; $\boldsymbol{x}_{-i}$ is a vector obtained by eliminating the ith element of $\boldsymbol{x}$; $\boldsymbol{x} \leq \boldsymbol{y}$ or $\boldsymbol{x} < \boldsymbol{y}$ denote element-wise inequality; $\log$ is the natural logarithm function; for any functions $f(n)$ and $h(n)$, $h(n) = O(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| < \infty$, $h(n) = \Omega(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| > 0$, $h(n) = o(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| = 0$, $h(n) = \omega(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| = \infty$, and $h(n) = \Theta(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| = c$, where $0 < c < \infty$.

II. System Description

We consider a wireless communication network with n pairs of transmitters and receivers. Each transmitter aims to send data to its corresponding receiver. We denote the vector of transmit powers by $\boldsymbol{p} = (p_1, \cdots, p_n)$, where p_i is the transmit power of link i. Due to hardware constraints the transmit powers of the transmitters can not exceed some certain values. This power constraint is represented as

$$\mathbf{0}_n \leq \boldsymbol{p} \leq \boldsymbol{a}, \tag{1}$$

where $\boldsymbol{a} = (a_1, \cdots, a_n)$ is the vector of maximum allowed transmit powers.

The channel is represented by coef£cients $G_{ji} = |\alpha_{ji}|^2$, where α_{ji} is the channel gain between receiver i and transmitter j. This means the received power from transmitter j at the receiver i equals $G_{ji}p_j$. The channel gains, in general, depend on small scale and large scale fadings, path attenuation, processing gain of the CDMA system, etc. For the sake of convenience, we collect all channel coef£cients in the channel matrix $\boldsymbol{G} = \{G_{ji}\}_{n\times n}$.

We consider an additive white Gaussian noise (AWGN) with variance σ_i^2 at the receiver i. The receivers are conventional, linear receivers, i.e., without multiuser detection. Since the transmissions occur simultaneously within the same environment, the signal from each transmitter acts as interference for other links. Assuming Gaussian signal transmission from all links, the distribution of the interference will be Gaussian as well. Thus, we can de£ne the *SINR* of the receiver i as

$$\gamma_i(\boldsymbol{p}) = \frac{G_{ii}p_i}{\sigma_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{n} G_{ji}p_j}. \tag{2}$$

Throughout the paper, we occasionally use γ_i instead of $\gamma_i(\boldsymbol{p})$. The *SINR* determines different QoS measures such as the maximum possible data rate, or the error probability of link i.

In this paper, we are interested in rates at which the transmitters can send data to their corresponding receivers without any error. According to the Shannon capacity formula [18], the maximum rate of link i is equal to

$$r_i = \log(1 + \gamma_i) \quad nats/channel\ use. \tag{3}$$

The network rate vector is de£ned as $\boldsymbol{r} = (r_1, \cdots, r_n)$. In a network, we desire to have all rates as large as possible. However, due to the interplay between the rates of different links (see (2) and (3)), it is not possible to maximize all the rates simultaneously. Instead, one may consider maximizing a utility function of the network which is increasing in all rates. A common utility function is the sum-rate of the network.

III. Problem Formulation and Preliminary Results

The problem of sum-rate maximization is formulated as follows:

$$\begin{aligned} \max_{\boldsymbol{p}} \quad & \sum_{i=1}^{n} \log(1 + \gamma_i(\boldsymbol{p})), \\ \text{s.t.} \quad & \mathbf{0}_n \leq \boldsymbol{p} \leq \boldsymbol{a}, \end{aligned} \tag{4}$$

which is a non-convex optimization problem. Thus, the algorithms developed for convex problems may converge to local optimum points.

Proposition 1: In the optimum solution $\boldsymbol{p}^*$ of (4), the power of at least one link takes its maximum allowed value.

Proof: If $p_i^* < a_i$ for all $i \in \mathbb{N}_n$, we can scale $\boldsymbol{p}^*$ by a coef£cient greater than one and still stay in the feasible region. Speci£cally, de£ne $\hat{\boldsymbol{p}} = \alpha^*\boldsymbol{p}^*$, where $\alpha^* > 1$ is de£ned as

$$\alpha^* = \min_{i\in\mathcal{J}} \frac{a_i}{p_i^*}, \tag{5}$$

and $\mathcal{J} \subseteq \mathbb{N}_n$ is the set of indices $i \in \mathbb{N}_n$ for which $p_i^* > 0$. From the de£nition of γ_i in (2), it is easy to show that $\gamma_i(\alpha\boldsymbol{p})$ is increasing in α. Thus, we have

$$\gamma_i(\hat{\boldsymbol{p}}) = \gamma_i(\alpha^*\boldsymbol{p}^*) > \gamma_i(\boldsymbol{p}^*), \qquad \forall i \in \mathcal{J}, \tag{6}$$

which implies that the sum-rate obtained by $\hat{\boldsymbol{p}}$ is larger than that of $\boldsymbol{p}^*$. This is in contradiction to the optimality of $\boldsymbol{p}^*$. Thus, we should have $p_i^* = a_i$ for at least one $i \in \mathcal{J}$. ■

The results in [13] indicate that in the special case when $G_{ji} = G_j$ for all $j \in \mathbb{N}_n$ the power of all links take the value of zero or the maximum allowed value except for at most one link. This result is not valid for a general distribution of channel coef£cients. However, as discussed below there are other special cases where an analogous result holds.

Special Case 1 ($\sigma_i^2 \to \infty$ or $\sigma_i^2 \to 0$): It is obvious that when the noise power is very large it dominates the effect of interference. Thus, to maximize the sum-rate all transmitters should transmit with maximum power. On the other hand, if the noise power is very small, the system can enjoy an extremely large sum-rate by having the link with the largest direct coef£cient transmit at its maximum power and keeping the other links silent.

Special case 2 ($n = 2$): When there are only two links sharing a wireless channel, we have the following interesting result; it indicates that there are only 3 candidates for the optimum power vector.

Proposition 2: The optimum solution of (4) for $n = 2$ is obtained when both transmitters transmit with maximum power or one of them is silent and the other one transmits with maximum power.

Proof: See the Appendix. ■

Obviously, if in the optimum solution only one link is active, it should be the link with the largest direct channel coef£cient.

Special case 3 (low SINR regime): If we know that the *SINR* of all links is small, we can use the approximation $\log(1+x) \approx x$ to write (4) as follows

$$\begin{aligned} \max_{\boldsymbol{p}} \quad & T(\boldsymbol{p}) = \sum_{i=1}^{n} \gamma_i(\boldsymbol{p}), \\ \text{s.t.} \quad & \mathbf{0}_n \leq \boldsymbol{p} \leq \boldsymbol{a}. \end{aligned} \tag{7}$$

Although this problem is again non-convex, the following result can be concluded that allows for obtaining the optimum solution by enumerating the vertices of the power domain.

Proposition 3: In the optimum solution $\boldsymbol{p}^*$ of (7), all transmit powers satisfy one of the power constraints by equality, i.e., $p_i^* \in \{0, a_i\}$ for all $i \in \mathbb{N}_n$.

Proof: If $\boldsymbol{p}_{-i}^* = \mathbf{0}_{n-1}$, clearly $p_i^* = a_i$ maximizes the sum-rate and the proof is complete. If $\boldsymbol{p}_{-i}^* \neq \mathbf{0}_{n-1}$, by substituting the values of $\gamma_i(\boldsymbol{p})$ from (2) in the objective function of (7) and computing the second order partial derivative with respect to p_i we obtain

$$\frac{\partial^2 T(\boldsymbol{p})}{\partial p_i^2} = 2 \sum_{j \neq i} G_{ij}^2 \frac{\gamma_j(\boldsymbol{p})}{d_j^2(\boldsymbol{p})}, \tag{8}$$

which is positive for all $\boldsymbol{p}_{-i} \neq \mathbf{0}_{n-1}$. Thus, $T(\boldsymbol{p})$ is convex with respect to p_i. As a result, the maximizing value of p_i lies on one end of the interval $[0, a_i]$. ■

According to Proposition 3, if we somehow know that the optimum solution of (4) satis£es the low *SINR* condition, then the optimum value of (7) is an approximation for the optimum value of (4) provided that the optimum solution of (7) satis£es the low *SINR* condition.

IV. A Lower Bound on Sum-Rate

It is interesting to know how the throughput of a wireless network scales with the number of nodes, when this number is large. In this section, we present a simple heuristic power allocation scheme, which yields to a lower bound on the average sum-rate of the wireless network described before. This scheme is based on the *on-off power allocation strategy*.

De£nition 1: A power allocation strategy is called an *on-off power allocation strategy* or brie¤y an *on-off strategy* if the power of link i is selected from the two element set $\{0, a_i\}$.

To facilitate the analysis, in the rest of the paper we assume that the channel between each transmitter and each receiver is Rayleigh fading. Also, all channels are pairwise independent. Thus, the entries of $\boldsymbol{G}$ are independent exponentially distributed random variables with mean 1 and variance 1, i.e., the pdf function of each entry is $f_X(x) = e^{-x}u(x)$. Moreover, we assume that all links have power constraints equal to 1, i.e., $\boldsymbol{a} = \mathbf{1}_n$. Furthermore, the noise powers at all receivers are limited and the same, i.e., $\sigma_i^2 = \sigma^2 < \infty$. The next theorem states the main result of this section.

Theorem 1: In a wireless network with Rayleigh channels and n links, with probability 1 the sum-rate grows with n at least as $\log(n) + O(\log\log n)$.

Proof: We provide a power allocation strategy and show that with probability 1 its corresponding sum-rate is larger than $\log(n) + O(\log\log n)$. Consider a threshold t and assume that link i is activated and transmit with full power if $G_{ii} > t$; otherwise, it is kept off. Note that the performance of this on-off strategy depends on the value of the threshold t; if t is very large, the quality of the selected links will be very good, but the number of such links is small and as a result the achieved sum-rate will be small; on the other hand, if t is very small, many links are chosen, but it causes a large interference and again the sum-rate will be small. Thus, it is crucial to choose a proper value for t.

As the channel coef£cients are exponentially distributed, the probability of a link being chosen is $q = e^{-t}$. Thus, the number of active links k is a binomial random variable with parameters n and q. However, according to the central limit theorem [19], if $\zeta = o(\sqrt{nq})$, then, we have

$$\Pr\{nq - \zeta\sqrt{nq} < k < nq + \zeta\sqrt{nq}\} \to 1, \tag{9}$$

as $n \to \infty$. Since $\zeta\sqrt{nq} = o(nq)$, we can assume $k = nq = ne^{-t}$.

Without loss of generality, we assume that the active links are indexed by $1, 2, \cdots, k$. The corresponding sum-rate is equal to

$$\begin{aligned} R_{tot} &= \sum_{i=1}^{k} \log\left(1 + \frac{G_{ii}}{\sigma^2 + \sum_{\substack{j=1 \\ j\neq i}}^{k} G_{ji}}\right) \\ &\geq \sum_{i=1}^{k} \log\left(1 + \frac{t}{\sigma^2 + \sum_{\substack{j=1 \\ j\neq i}}^{k} G_{ji}}\right), \end{aligned} \tag{10}$$

where the inequality is because $G_{ii} > t$ for the selected links. According to the law of large numbers, we can rewrite (10) as

$$R_{tot} \geq k\mathbb{E}\left[\log\left(1+\frac{t}{\sigma^2+\sum_{\substack{j=1\\j\neq i}}^{k} G_{ji}}\right)\right]. \quad (11)$$

Using the convexity of $\log$ with respect to G_{ji}, we obtain

$$\begin{aligned} R_{tot} &\geq k\log\left(1+\frac{t}{\sigma^2+\sum_{\substack{j=1\\j\neq i}}^{k}\mathbb{E}\left\{G_{ji}\right\}}\right) \\ &= k\log\left(1+\frac{t}{k}\right), \end{aligned} \quad (12)$$

where the equality relies on $\mathbb{E}\left\{G_{ji}\right\}=1$. Consequently, by substituting $k=ne^{-t}$ in (12) we have

$$R_{tot} \geq ne^{-t}\log\left(1+\frac{t}{ne^{-t}}\right). \quad (13)$$

The last step is to £nd the optimum value of t, such that the lower bound (13) becomes as tight as possible. By setting the derivative of this lower bound equal to zero, we obtain:

$$\log\left(1+\frac{te^t}{n}\right)=\frac{(1+t)e^t}{n+te^t}. \quad (14)$$

We can verify that the solution of the above equation is obtained as:

$$t_{opt}=\log n-2\log\log n+\log 2+O\left(\frac{\log\log n}{\log n}\right). \quad (15)$$

By substituting this value in the lower bound (13), we obtain

$$R_{tot} \geq \log n-2\log\log n+O(1), \quad (16)$$

which gives a lower bound on the sum-rate that can be achieved with probability 1. ■

As an immediate result of Theorem 1, we have the following corollary.

Corollary 1: In a Rayleigh fading channel, the average sum-rate $\overline{R}_{tot}$ of a network with n links is lower bounded as $\overline{R}_{tot} \geq \log n + O(\log\log n)$.

From the proof of Theorem 1, we have the following corollary on the number of active links in the suggested on-off strategy.

Corollary 2: In a Rayleigh fading channel, with probability 1 the number of active links of a network with n links scales as $O(\log^2 n)$.

We can deduce from Corollary 2 that the rate per link scales as $O\left(\frac{1}{\log n}\right)$.

It is worth mentioning that in the suggested on-off strategy, no coordination is required between the links. All one transmitter needs to know is wether its direct channel coef£cient is above the threshold t_{opt}. Based on this information, it decides wether to transmit with full power or remain silent.

V. Optimum Link Activation Strategy

Stimulated by Propositions 2 and 3 and the results of [13], we limit the power allocation problem to the on-off power allocation strategies. Finding the optimum subset of users who should transmit with maximum power requires solving a complicated integer program. However, it is possible to £nd suboptimum algorithms that perform close to the optimum. Indeed, in this section we show that an average sum-rate larger than the lower bound provided in the previous section is not achievable. Thus, the strategy described in the proof of Theorem IV is optimum in the sense of the order of the average sum-rate.

Lemma 1: Let m^* denote the number of active links in the sum-rate maximizing strategy. Then, with probability 1,

$$m^* \to \infty \qquad \text{as} \qquad n \to \infty. \quad (17)$$

Proof: To prove this lemma, it is enough to show that for any given integer m, $\Pr\left\{m^* \leq m\right\} \to 0$ as $n \to \infty$. We have

$$\begin{aligned} \Pr\left\{m^* \leq m\right\} &= \sum_{k=1}^{m}\Pr\left\{m^*=k\right\} \\ &\leq \sum_{k=1}^{m}\binom{n}{k}q_k \end{aligned} \quad (18)$$

where q_k is the probability that a given k-tuple set of active links achieves the maximum sum-rate. The inequality is because of the union bound. Let's denote the pdf and the cdf of the sum-rate achieved by a k-tuple set of active links by $f_k(x)$ and $F_k(x)$, respectively. It is clear that q_k is upper bounded by the probability of the event that the sum-rate of the k-tuple set of active links is larger than the rate of any of the remaining $n-k$ links when it is the only active link, i.e.,

$$q_k < \int_0^\infty F_1^{n-k}(x)f_k(x)dx \quad (19)$$

From (18) and (19), we obtain

$$\begin{aligned} \Pr\left\{m^* \leq m\right\} &< \sum_{k=1}^{m}\binom{n}{k}\int_0^\infty F_1^{n-k}(x)f_k(x)dx \\ &\approx \sum_{k=1}^{m}\int_0^\infty\left(\frac{e}{kF_1(x)}\right)^k n^k F_1^n(x)f_k(x)dx \end{aligned} \quad (20)$$

The equality is obtained by using Stirling's approximation for factorial. As it is seen, the integrand includes a product of a term polynomial in n and a term exponential in n. Since $F_1(x) < 1$, the integrand goes to 0 as $n \to \infty$. Thus, the summation goes to zero and consequently,

$$\Pr\left\{m^* \leq m\right\} \to 0 \quad \text{as} \quad n \to \infty \quad (21)$$

This means with probability 1, m^* grows with n. ■

The above result is valid for any channel conditions. The following Lemma states how fast the optimum number of active links should grow in a Rayleigh fading channel.

Lemma 2: If the channel is Rayleigh, then with probability 1

$$m^* = \Omega\left(\frac{\log n}{\log\log n}\right). \tag{22}$$

Proof: Assume by contradiction that $\Pr\left\{m^* = \Omega\left(\frac{\log n}{\log\log n}\right)\right\} = 1-\epsilon$, for some constant $\epsilon > 0$. Hence, $m^* = o\left(\frac{\log n}{\log\log n}\right)$ happens with probability ϵ. If this happens, the sum-rate is upper bounded by the sum-rate of a network for which the m^* links with largest direct coef£cients have no interference on each other. For such an imaginary network, the optimum strategy is to assign maximum power to the best m^* links and keep the other links off. Thus, we have

$$\begin{aligned} R_{tot} &\leq m^* \mathbb{E}\left[\log\left(1+\frac{G_{ii}}{\sigma^2}\right)\right] \\ &\leq o\left(\frac{\log n}{\log\log n}\right)\log\left(1+\frac{\mathbb{E}[G_{ii}]}{\sigma^2}\right), \end{aligned} \tag{23}$$

where the second inequality is obtained by the concavity of the $\log$ function. Recalling that the expected value of the largest of n i.i.d. exponential random variables is of order $O(\log n)$ [20], we can write $\mathbb{E}[G_{ii}] \leq O(\log n)$. Thus, (23) can be written as:

$$\begin{aligned} R_{tot} &\leq o\left(\frac{\log n}{\log\log n}\right)\log\left(1+\frac{O(\log n)}{\sigma^2}\right) && (24) \\ &= o(\log n). && (25) \end{aligned}$$

This means that with probability ϵ the sum-rate if of order less than $o(\log n)$, which is in contradiction to the result of Theorem 1. ∎

The previous lemma implies that in order to achieve the maximum sum-rate the growth rate of the number of active links should be larger than some certain orders. However, due to the effect of interference, the number of active links should not grow very fast. In the next lemma, we address this effect. Indeed, we show that when m, the number of active links, growth larger than some certain orders, then the interference seen by most of the links is of order $\Omega(m)$.

Lemma 3: If $m = \Omega\left(\frac{\log n}{\log\log n}\right)$, the probability that there exists an m-tuple set of active links, with the property that the number of links with interference of order $o(m)$ be larger than αm (for some constant $0 < \alpha < 1$), approaches zero as n goes to in£nity.

Proof: Consider an arbitrary set of m active links. Let Y denote the interference seen by one of these links, which has chi-squared distribution with $2(m-1)$ degrees of freedom. The cdf of Y is upper bounded as

$$\begin{aligned} F_Y(y) &= e^{-y}\sum_{k=m}^{\infty}\frac{y^k}{k!} \\ &< e^{-y}\sum_{k=m}^{\infty}\left(\frac{ye}{k}\right)^k \\ &< e^{-y}\sum_{k=m}^{\infty}\left(\frac{ye}{m}\right)^k \\ &= \frac{e^{-y}\left(\frac{ye}{m}\right)^m}{1-\frac{ye}{m}} \end{aligned} \tag{26}$$

where the £rst inequality is obtained by using the Stirling's approximation for the factorial function. By considering $y = o(m)$, (26) is converted to

$$F_Y(y) < e^{-y}\left(\frac{ye}{m}\right)^m. \tag{27}$$

Let π represent the probability of having at least αm links with interference of order $o(m)$. It turns out that

$$\begin{aligned} \pi &= \sum_{i=\alpha m}^{m}\binom{m}{i}F_Y^i(y)\left(1-F_Y(y)\right)^{m-i} \\ &< 2^m F_Y^{\alpha m}(y), \end{aligned} \tag{28}$$

where the inequality is obtained by replacing $F_Y^i(y)$ and $\left(1-F_Y(y)\right)^{m-i}$ by the larger values $F_Y^{\alpha m}(y)$ and 1, respectively. Let ρ denote the probability of having at least one m-tuple of active links in which at least αm of links experience interference of order $o(m)$. By using the union bound, we obtain

$$\begin{aligned} \rho &\leq \binom{n}{m}\pi \\ &< \left(\frac{ne}{m}\right)^m 2^m e^{-\alpha m y}\left(\frac{ye}{m}\right)^{\alpha m^2} \\ &= \exp\left[m(\log n + 1 - \log m) + m\log 2 \right. \\ &\qquad \left. -\alpha m y + \alpha m^2(\log y + 1 - \log m)\right], \end{aligned} \tag{29}$$

where the second inequality is obtained by using (27) and (28). To complete the proof it suf£ces to show that the exponent of the above expression approaches minus in£nity as n goes to in£nity. Taking into account the fact that $y = o(m)$ and neglecting the non-dominant terms, the condition for having negative exponent is obtained as

$$m\log n - \alpha m^2\log m < 0. \tag{30}$$

This is equivalent to having

$$m = \Omega\left(\frac{\log n}{\log\log n}\right), \tag{31}$$

which is exactly the hypothesis of the lemma. ∎

Theorem 2: In a Rayleigh fading channel, the power allocation policy of activating the links with direct coef£cients larger than $\log n$ is optimum among all on-off strategies.

Proof: According to Lemma 2 with probability 1, we have $m^* = \Omega\left(\frac{\log n}{\log\log n}\right)$. However, according to Lemma 3, for this value of m^*, the interference in most of the links is $\Omega(m^*)$. Thus, if a strategy creates an interference of order $\Theta(m^*)$, it is optimum (because otherwise the interference is of order $\omega(m)$ and yields a sum-rate of lower order). Moreover, if a group of such strategies exists, since the interferences are from the same order, the best one is the one that chooses the links with largest direct channel coef£cients. The decentralized power allocation scheme described in previous section possesses both two aforementioned properties. Hence, it is optimum. ∎

VI. Conclusion

In this paper, the problem of sum-rate maximization by means of power allocation was investigated. This problem is non-convex and only suboptimum solutions have been reported for it in previous works. We proved that in the case of $n = 2$ links the optimum solution is one of the corner points of the power domain. By limiting the power allocation problem to a on-off strategies, we showed that the average sum-rate scales as $O(\log n)$. Moreover, it was proved that this sum-rate can be achieved by a simple decentralized scheme.

Appendix

Assume for simplicity that channel coef£cients and noise powers are scaled such that the maximum allowed power of both links and also the direct channel coef£cients G_{ii} are equal to one. According to Proposition 1, in the optimum solution of (4) the power of at least one link should be equal to one; without loss of generality assume $p_2 = 1$. It suf£ces to show that the maximum of the function

$$f(p_1) = \log\left(1 + \frac{p_1}{\sigma_1^2 + G_{21}}\right) + \log\left(1 + \frac{1}{\sigma_2^2 + G_{12}p_1}\right) \tag{32}$$

is obtained either at $p_1 = 0$ or $p_1 = 1$. By computing the derivative of $f(p_1)$ and simplifying it we obtain

$$f'(p_1) = \frac{Ap_1^2 + Bp_1 + C}{d(p_1)}, \tag{33}$$

where $A = G_{12}^2$, $B = 2\sigma_2^2 G_{12}$, $C = \sigma_2^2(\sigma_2^2 + 1) - G_{12}(\sigma_1^2 + G_{21})$, and $d(p_1)$ is a polynomial in p_1 with all coef£cients non-negative. Thus, the sign of $f'(p_1)$ is determined by its numerator. Note that $A, B \geq 0$. If $C \geq 0$, the numerator (and thus $f'(p_1)$) is always non-negative for $p_1 \geq 0$. Thus, $f(p_1)$ is increasing in p_1 and achieves its maximum at $p_1 = 1$. If $C < 0$, the numerator has exactly one positive (p_1') and one negative (p_1'') roots. Thus, $f(p_1)$ has a minimum at p_1' and attains its maximum at 0 or 1.

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