On the Fairest Corner Point of the MIMO-BC Capacity Region*

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Abstract

A number of recent works have addressed the problem of characterizing the sum-capacity of the multiple-input multiple-output broadcast channels (MIMO-BC). However, the issue of the fairness remains an open problem. This article aims at finding a point on the sum-capacity hyperplane which satisfies a notion of fairness among active users. To facilitate the derivation of the results, we focus on a subset of capacity region which includes the corner points and their convex hull. It is shown that this region is a *Polymatroid*. The properties of polymatroids are exploited to locate a rate-vector on the sum-capacity hyperplane which is optimally fair. This means the minimum rate among all users is maximized (max-min rate). In the case that more than one user attain the max-min value, the algorithm recursively maximizes the minimum rate among the rest of the users. It is shown that the problem of deriving the time-sharing coefficients to attain this point can be decomposed to some lower-dimensional problems.

In some cases, the complexity of computing and implementing an appropriate time-sharing strategy is not feasible. This motivates us to compute the corner point for which the minimum rate of the active users is maximized (max-min corner point). A simple greedy algorithm is introduced to find the max-min corner point.

1 Introduction

Recently, there have been extensive efforts to develop spectral-efficient transmission schemes for wireless communications. By considering limits on the available bandwidth, multiple input multiple output (MIMO) systems are considered as the most promising approach to provide reliable and high data rate communication. More recently, the work on MIMO systems has been extended to MIMO multiuser channels [1–4]. In [1, 2, 5], the sum-capacity of the MIMO broadcast channel (MIMO-BC) is characterized as a convex optimization problem. In addition, it is shown that if the available power in the optimum solution is allocated to r users (active users), the capacity region has $r!$ corner points where the sum-capacity hyperplane is formed as the convex hull of these corner points.

This article aims at finding a point on the sum-capacity hyperplane which satisfies a notion of fairness among active users. To facilitate the derivation of the results, we

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focus on a subset of capacity region which includes the corner points and their convex hull. It is shown that this region is a *Polymatroid*. The properties of polymatroids are exploited to locate a rate-vector on the sum-capacity hyperplane which is optimally fair. This means the minimum rate among all users is maximized (max-min rate). In the case that more than one user attain the max-min value, the algorithm recursively maximizes the minimum rate among the rest of the users. It is shown that the problem of deriving the time-sharing coefficients to attain this point can be decomposed to some lower-dimensional problems.

In some cases, the complexity of computing and implementing an appropriate time-sharing strategy is not feasible. This motivates us to compute the corner point for which the minimum rate of the active users is maximized (max-min corner point). A simple greedy algorithm is introduced to find the max-min corner point.

In this paper, all the results are proven for a general polymatroid. Consequently, they can be applied for any other structure which satisfies the polymatroid conditions. For example, the results can be used for the capacity region of a single-input single-output multi-access system which is proven to have a polymatroid structure [6].

The rest of the paper is organized as follows. In Section 2, the system model and formulation of the sum-capacity are presented. In addition, the structure of polymatroid and its relation with the capacity region are discussed. In Section 3, the mentioned results are derived for a general polymatroid.

Notation: All boldface letters indicate vectors (lower case) or matrices (upper case). $\det(\mathbf{H})$ denotes determinant and $\mathbf{H}^\dagger$ denotes transpose conjugate of the matrix $\mathbf{H}$. $\mathbf{M} \succeq 0$ represents that the matrix $\mathbf{M}$ is positive semi-definite. $\mathbf{1}_n$ represents an n dimensional vector with all entries equal to one. E is a set of integers $E = \{1, \cdots, |E|\}$ where $|E|$ denotes the cardinality of the set E. The set function $f : 2^E \longrightarrow \mathcal{R}_+$ is a mapping from a subset of E (there are a total of $2^{|E|}$ subsets) to the positive real numbers. A permutation of the set E is denoted by π and $\pi(i)$, $1 \leq i \leq |E|$, represents the element of the set E located in the i^{th} position after the permutation. For a r-dimensional vector $\mathbf{x} \in \mathcal{R}^r$ and $S \subset E$, $\mathbf{x}(S)$ denotes $\sum_{i \in S} x_i$. Also, for a set of positive semi-definite matrices $\mathbf{D}_i$, $i = 1, \ldots, r$, $\mathbf{D}(S)$ represents $\sum_{i \in S} \mathbf{D}_i$.

2 Preliminaries

2.1 MIMO Broadcast

Consider a MIMO Broadcast channel (MIMO-BC) with M transmit antennas and K users, where the k^{th} user is equipped with N_k receive antennas. In a flat fading environment, the baseband model of this system is given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{s} + \mathbf{w}_k, \quad 1 \leq k \leq K, \tag{1}$$

where $\mathbf{H}_k \in \mathcal{C}^{N_k \times M}$ denotes the channel matrix from the base station to user k, $\mathbf{s} \in \mathcal{C}^{M \times 1}$ represents the transmitted vector, and $\mathbf{y}_k \in \mathcal{C}^{N_k \times 1}$ signifies the received vector by user k. The vector $\mathbf{w}_k \in \mathcal{C}^{N_k \times 1}$ is a white Gaussian noise with zero-mean and unit-variance.

Consider an order of the users $(\pi(1), \pi(2), \ldots, \pi(K))$. By assuming that user $\pi(i)$ knows the codewords selected for the users $\pi(j)$, $j = 1, \ldots, i-1$, the interference of the users $\pi(j)$, $j = 1, \ldots, i-1$ over user $\pi(i)$ can be effectively canceled based on dirty paper coding theorem [7]. Therefore, the rate of user $\pi(i)$ is equal to

$$R_{\pi(i)} = \log \frac{\det\left(\mathbf{I}_{N_k,N_k} + \mathbf{H}_{\pi(i)}\left(\sum_{j\geq i}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}^{\dagger}_{\pi(i)}\right)}{\det\left(\mathbf{I}_{N_k,N_k} + \mathbf{H}_{\pi(i)}\left(\sum_{j>i}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}^{\dagger}_{\pi(i)}\right)}, \quad i = 1, \ldots, K, \tag{2}$$

where $\mathbf{Q}_{\pi(j)}$ is the covariance of the signal vector to user $\pi(j)$. The dirty-paper capacity region is defined as the convex hull of the union of such rate vectors over all permutations $(\pi(1), \pi(2), \ldots, \pi(K))$ and over all positive semi-definite covariance matrices $\mathbf{Q}_i$, $i = 1, \ldots, K$ such that $\mathrm{Tr}\left(\sum_{i=k}^{K}\mathbf{Q}_k\right) \leq P_T$, where P_T denotes the total transmit power.

In [1, 2, 5], a duality between the MIMO-BC and the MIMO multi-access channel (MAC) is established. In the dual MIMO-MAC, the channel between user k and the base station is $\mathbf{H}_k^{\dagger}$ and the covariance of the power allocated to user k is $\mathbf{P}_k$. The relationship between $\mathbf{P}_k$ and $\mathbf{Q}_k$, $k = 1, \ldots, K$, has been derived [1]. The duality is used to characterize the sum-capacity of the MIMO-BC as follows

$$\begin{aligned} R_{\text{Sum-Capacity}} = \quad & \max\nolimits_{\mathbf{P}_1,\ldots,\mathbf{P}_K} \log\det(\mathbf{I}_{M,M} + \sum_{k=1}^{K}\mathbf{H}_k^{\dagger}\mathbf{P}_k\mathbf{H}_k). \\ s.t. \quad & \textstyle\sum_{k=1}^{K}\mathrm{Tr}(\mathbf{P}_k) \leq P_T, \\ & \mathbf{P}_k \succeq 0 \end{aligned} \tag{3}$$

The above optimization problem determines the power allocated to each user in dual MIMO-MAC, and consequently, the power of each user in MIMO-BC. Note that only a subset of users are active and the power allocated to the rest is zero. Equation (3) determines a hyperplane on the boundary of the capacity region of MIMO-BC, the so-called sum-capacity hyperplane. If the cardinality of the set of active users is r, i.e. $E = \{1, \cdots, r\}$, the sum-capacity hyperplane has $r!$ corner points corresponding to different permutations of the active users. Note that the rates of the non-active users remain zero regardless of the permutation. The corner point corresponding to a permutation can be computed using (2). Assuming the active users are users $i = 1, \ldots, r$, we define

$$\mathbf{D}_i = \mathbf{H}_i^{\dagger}\mathbf{P}_i^{*}\mathbf{H}_i \quad i = 1, \ldots, r, \tag{4}$$

where $\mathbf{P}_i^{*}$, $i = 1, \ldots, r$, correspond to optimizing matrices in (3). It is shown that the corner point in (2) can be reformulated as [1]

$$R_{\pi(i)} = \log \frac{\det\left(\mathbf{I}_{M,M} + \sum_{j\leq i}\mathbf{D}_{\pi(j)}\right)}{\det\left(\mathbf{I}_{M,M} + \sum_{j<i}\mathbf{D}_{\pi(j)}\right)}, \quad i = 1, \ldots, r, \tag{5}$$

which is the corner point of the dual multi-access channel. To achieve the rate vector (5) in dual MIMO-MAC, the decoding-cancelation method is used. By starting from $i = K$, the data of the user $\pi(i)$ is decoded, while the signals of the users $\pi(1), \ldots, \pi(i-1)$ are considered as interference. After decoding, the signal of the user $\pi(i)$ is canceled from the received signal. The next user to be decoded is user $\pi(i-1)$. Note that to achieve the same corner point in the broadcast and the multi-access channels, the order of encoding in the broadcast channel and the order of decoding in the multi-access channel is reversed. In other words, the set of users encoded *before* user $\pi(i)$ in the MIMO-BC is the same of the set of users decoded *after* user $\pi(i)$ in the MIMO-MAC.

2.2 Polymatroid Structure

Definition [8, Ch. 18]: Let $E = \{1, 2, \dots, r\}$ and $f : 2^E \longrightarrow \mathcal{R}_+$ a set function. The polyhedron

$$\mathcal{B}(f, E) = \{(x_1, x_2, \dots, x_r) : \mathbf{x}(S) \leq f(S) \quad \forall S \subset E, \quad \forall x_i \geq 0\} \tag{6}$$

is a polymatroid, if the set function f satisfies

$$(normalized) \qquad f(\emptyset) = 0 \tag{7}$$

$$(increasing) \qquad f(S) \leq f(T) \text{ if } S \subset T \tag{8}$$

$$(submodular) \qquad f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \tag{9}$$

Any function f that satisfies the above properties is termed as *rank function*. Corresponding to each permutation π of the set E, the polymatroid $\mathcal{B}(f, E)$ has a corner point $\mathbf{v}(\pi) \in \mathcal{R}_+^r$ which is equal to:

$$v_{\pi(i)} = \begin{cases} f(\{\pi(i)\}) & i = 1 \\ f(\{\pi(1), \dots, \pi(i)\}) - f(\{\pi(1), \dots, \pi(i-1)\}) & i = 2, \dots, r \end{cases} \tag{10}$$

Consequently, the polymatroid $\mathcal{B}(f, E)$ has $r!$ corner points corresponding to different permutations of the set E. Any point in the polymatroid on the hyperplane $\mathbf{x}(E) = f(E)$ is in the convex hull of the corner points.

Here, we introduce a special polymatroid and establish its relationship with the capacity region of the MIMO-BC channel. For a set of positive semi-definite matrices $\mathbf{D}_i$, define the set function g as

$$g(S) = \log\det\left(\mathbf{I} + \mathbf{D}(S)\right) \quad \text{for} \quad S \subset E. \tag{11}$$

Lemma 1 *Given $g(S)$ defined in (11), the polyhedron $\mathcal{B}(g, E)$ defined as follows is a polymatroid.*

$$\mathcal{B}(g, E) = \{(x_1, x_2, \dots, x_r) \in \mathcal{R}_+^r : \mathbf{x}(S) \leq g(S) \quad \forall S \subset E\}. \tag{12}$$

Proof: Clearly, $g(\emptyset) = 0$. Assume $\mathbf{B} \succeq 0$ and $\mathbf{C} \succeq 0$ are two Hermitian matrices. If $\mathbf{B} - \mathbf{C} \succeq 0$, then $\det(\mathbf{B}) \geq \det(\mathbf{C})$ [4, Proposition I.2]. Furthermore, if $\mathbf{\Delta} \succeq 0$, then

$$\frac{\det(\mathbf{\Delta} + \mathbf{B} + \mathbf{C})}{\det(\mathbf{\Delta} + \mathbf{B})} \leq \frac{\det(\mathbf{B} + \mathbf{C})}{\det(\mathbf{B})}. \tag{13}$$

Using above properties, it is straight-forward to prove (8) and (9) for the set function $g(.)$. ■

In the set function $g(S)$, define $\mathbf{D}_i$ as in (4). It is easy to verify that the polymatroid $\mathcal{B}(g, E)$ is a subset of the capacity region of the MIMO-BC. The hyperplane $\mathbf{x}(E) = g(E)$ and its corner points are the same as the sum-capacity hyperplane and its corner points. Due to this property, we focus on the polymatroid $\mathcal{B}(g, E)$ (see Fig. 1).

3 Max-Min Operation over a Polymatroid

The following lemma determines the maximum possible value for the minimum entry of a vector $\mathbf{x}$ where $\mathbf{x} \in \mathcal{B}(f, E)$.

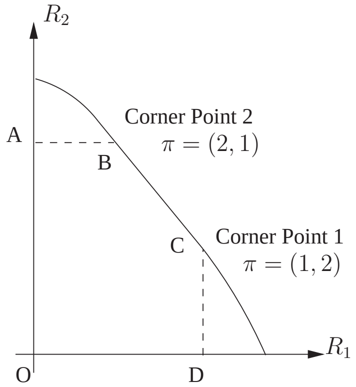

Figure 1: Capacity region of the MIMO-BC and its corner Points. The region OABCD is a polymatroid

Lemma 2 *In the polymatroid $\mathcal{B}(f, E)$, define*

$$\delta = \max \quad \min_{i \in E} x_i. \tag{14}$$
$$s.t. \quad (x_1, \ldots, x_r) \in \mathcal{B}(f, E). \tag{15}$$

Then,

$$\delta = \min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}. \tag{16}$$

Proof: Consider $\mathbf{x} \in \mathcal{B}(f, E)$, and let $\sigma = \min_i x_i$. Therefore,

$$\forall S \subset E, \ \sigma|S| \leq \mathbf{x}(S). \tag{17}$$

Noting $\forall S \subset E, \ \mathbf{x}(S) \leq f(S)$ and using the above inequality, we have

$$\forall S \subset E, \ \sigma|S| \leq f(S). \tag{18}$$

Consequently, $\sigma \leq \min_{S \subset E, \ S \neq \emptyset} \frac{f(S)}{|S|}$. Therefore, $\min_{S \subset E, \ S \neq \emptyset} \frac{f(S)}{|S|}$ provides an upper bound on $\min_i x_i$. By selecting $\mathbf{x} = \delta \mathbf{1}_r \in \mathcal{B}(f, E)$, where $\delta = \min_{S \subset E, \ S \neq \emptyset} \frac{f(S)}{|S|}$, the upper bound is achieved, resulting in (16). ■

In the following, we present an algorithm which computes a point $\mathbf{x}^* \in \mathcal{B}(f, E)$ on the hyperplane $\mathbf{x}(E) = f(E)$ which attains δ defined in Lemma 2. The main idea behind the algorithm is that in each step we increase the rate of all users until some users are saturated (the rate of these users cannot further increase within the polymatroid region). We fix such users at this value and increase the rate of the rest. This recursive procedure is repeated until the rates of all users are determined.

Algorithm I

1. Initialize the iteration index $\alpha = 0$, $E^{(0)} = E$, and $f^{(0)} = f$.

2. Find $m^{(\alpha)}$, where
$$m^{(\alpha)} = \min_{S \subset E^{(\alpha)}, S \neq \emptyset} \frac{f^{(\alpha)}(S)}{|S|}. \tag{19}$$
Set $S^{(\alpha)}$ equal to the optimizing subset.

3. For all $i \in S^{(\alpha)}$, set the i^{th} entry of the vector $\mathbf{x}^*$ equal to $m^{(\alpha)}$ (i.e. $x_i^* = m^{(\alpha)}$).

4. Set
$$E^{(\alpha+1)} = E^{(\alpha)} - S^{(\alpha)}, \tag{20}$$
and define $f^{(\alpha+1)}(S)$ as,
$$f^{(\alpha+1)}(S) = f^{(\alpha)}(S \cup S^{(\alpha)}) - f^{(\alpha)}(S^{(\alpha)}) \quad \text{for} \quad S \subset E^{(\alpha+1)}. \tag{21}$$

5. Set $\alpha \longleftarrow \alpha + 1$. If $E^{(\alpha+1)} \neq \emptyset$, move to step two, otherwise stop.

This algorithm computes the optimization sets $S^{(\alpha)}$, $\alpha = 0, \cdots, t$ and their corresponding $m^{(\alpha)}$, where $E = \bigcup_{j=0}^{t} S^{(j)}$ and $x_i^* \in \{m^{(0)}, \cdots, m^{(t)}\}, i = 1, \cdots, r$. In the following, we prove some properties of the vector $\mathbf{x}^*$. The following lemma is used in the proof.

Lemma 3 *Let $E = \{1, \ldots, r\}$ and $A \subset E$, $A \neq E$. If the set function $f^{(0)} : 2^E \longrightarrow \mathcal{R}_+$ is a rank function, then $f^{(1)} : 2^{E-A} \longrightarrow \mathcal{R}_+$ defined as*
$$f^{(1)}(S) = f^{(0)}(S \cup A) - f^{(0)}(A), \quad S \subset E - A \tag{22}$$
is a rank function.

Proof: By direct verification. ■

Theorem 1 *Assume that the above algorithm is applied over the polymatroid $\mathcal{B}(f, E)$, then*

(I) $\mathbf{x}^* \in \mathcal{B}(f, E)$ and is located on the hyperplane $\mathbf{x}(E) = f(E)$.

(II) The minimum entry of the vector $\mathbf{x}^*$ attains the optimum value determined by Lemma 2.

Proof:

Part (I): We show that $\mathbf{x}^* \in \mathcal{B}(f, E)$. According to the algorithm, we have $m^{(0)} = \min_{S \subset E, S \neq \emptyset} \frac{f(S)}{|S|}$, where $S^{(0)}$ is the minimizing set. In addition, $x_i^* = m^{(0)}$ for all $i \in S^{(0)}$. It is straight-forward to check that the allocated values for $x_i^*, i \in S^{(0)}$ satisfy the condition of the polymatroid $\mathcal{B}(f, E)$. By substituting the selected values for $x_i, i \in S^{(0)}$ in the constraints of the polymatroid $\mathcal{B}(f, E)$, the constraints over the coordinate i, $i \in E - S^{(0)}$ are updated. From the definition of the polymatroid, we have a set of constraints on $\mathbf{x}(S)$, $S \subset E - S^{(0)}$, which has the following format
$$\forall A \subset S^{(0)}, \mathbf{x}(S \cup A) \leq f^{(0)}(S \cup A). \tag{23}$$
Since $S \cap A = \emptyset$, then $\mathbf{x}(S \cup A) = \mathbf{x}(S) + \mathbf{x}(A)$. Consequently, from (23), we have,
$$\forall A \subset S^{(0)}, \mathbf{x}(S) \leq f^{(0)}(S \cup A) - \mathbf{x}(A). \tag{24}$$
The above set of inequalities results in,
$$\mathbf{x}(S) \leq \min_{A \subset S^{(0)}} \{f^{(0)}(S \cup A) - \mathbf{x}(A)\}, \quad \forall\, S \subset E - S^{(0)}. \tag{25}$$
We claim that $\min_{A \subset S^{(0)}} \{f^{(0)}(S \cup A) - \mathbf{x}(A)\}$ is equal to $f^{(0)}(S \cup S^0) - f^{(0)}(S^{(0)})$. The proof is as follows,
$$\forall A \in S^{(0)}, \qquad f^{(0)}(S \cup A) - \mathbf{x}(A) \tag{26}$$
$$\geq f^{(0)}(S \cup A) - f^{(0)}(A) \tag{27}$$
$$\geq f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)}) \tag{28}$$

The first inequality relies on the fact that $\forall A,\ \mathbf{x}(A) \le f^0(A)$. The second inequality is proven by using (9) and the fact that $A \subset S^{(0)}$ and $S \cap S^{(0)} = \emptyset$.

Regarding the above statements, for the non-allocated entries of $\mathbf{x}$, we have the following set of constraints,

$$\forall S \subset E - S^{(0)},\ \mathbf{x}(S) \le f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)}) \tag{29}$$

According to the algorithm, set $E^{(1)} = E^{(0)} - S^{(0)}$, $f^{(1)}(S) = f^{(0)}(S \cup S^{(0)}) - f^{(0)}(S^{(0)})$, $\forall S \subset E^{(1)}$. Using Lemma 3, the set of constraints (29), and the definition of polymatroid, $\mathcal{B}(f^{(1)}, E^{(1)})$ is a polymatroid, which is a subset of $\mathcal{B}(f, E)$. Now, we use the same procedure that is applied for $\mathcal{B}(f^{(0)}, E^{(0)})$ over $\mathcal{B}(f^{(1)}, E^{(1)})$, and continue recursively. Therefore, in each step, the rates of a subset of coordinates are determined, satisfying the constraints of the polymatroid $\mathcal{B}(f, E)$. Therefore, $\mathbf{x}^* \in \mathcal{B}(f, E)$. Direct verification proves that $\mathbf{x}^*(E) = f(E)$.

Part (II): We must show that the smallest entries of $\mathbf{x}^*$ is equal to $\min_{S\subset E} \frac{f(S)}{|S|}$. According to the algorithm, for all $i \in E$, we have $x_i^* \in \{m^{(0)}, ..., m^{(t)}\}$. Furthermore, $m^{(0)} = \min_{S\subset E} \frac{f(S)}{|S|}$.

From the algorithm, we have

$$m^{(j)} = \frac{f^{(j)}(S^{(j)})}{|S^{(j)}|} = \min_{S\subset E^{(j)}} \frac{f^{(j)}(S)}{|S|} \le \frac{f^{(j)}(S^{(j)} \cup S^{(j+1)})}{|S^{(j)} \cup S^{(j+1)}|} = \frac{f^{(j)}(S^{(j+1)} \cup S^{(j)})}{|S^{(j+1)}| + |S^{(j)}|}. \tag{30}$$

Therefore,

$$m^{(j)} \le \tfrac{f^{(j)}(S^{(j+1)} \cup S^{(j)})}{|S^{(j+1)}| + |S^{(j)}|} \quad \Longrightarrow \tag{31}$$

$$m^{(j)} \le \tfrac{f^{(j)}(S^{(j+1)} \cup S^{(j)}) - m^{(j)}|S^{(j)}|}{|S^{(j+1)}|} \quad \Longrightarrow \tag{32}$$

$$m^{(j)} \le \tfrac{f^{(j)}(S^{(j+1)} \cup S^{(j)}) - f^{(j)}(S^{(j)})}{|S^{(j+1)}|} \quad = m^{(j+1)} \tag{33}$$

Consequently, $\forall j, m^{(j)} \ge m^{(0)}$ and the proof is complete. ■

Since $\mathbf{x}^* \in \mathcal{B}(f, E)$ is on the hyperplane $\mathbf{x}(E) = f(E)$, $\mathbf{x}^*$ belongs to the convex hull of the corner points of the polymatroid. This means that $\mathbf{x}^*$ can be represented as a linear superposition of the corner points, where the corresponding coefficients are between zero and one with summation equal to one. In the following, we take advantage of the special structures of $\mathbf{x}^*$ and polymatroids to break down the above problem to some lower dimensional sub-problems.

Using Lemma 3, set of constraints (29), and considering the definition of polymatroid, it is easy to verify that $\mathcal{B}(f^{(j)}, S^{(j)})$ is a polymatroid and the vector $m^{(j)}\mathbf{1}_{|S^{(j)}|} \in \mathcal{B}(f^{(j)}, S^{(j)})$ is on the hyperplane $\mathbf{x}(S^{(j)}) = f(S^{(j)})$. Let $\{\pi_\gamma^{(j)}, \gamma = 1, \ldots, |S^{(j)}|!\}$ be the set of all permutations of the set $S^{(j)}$, and $\mathbf{u}(\pi_\gamma^{(j)})$ be the corner point corresponding to the permutation $\pi_\gamma^{(j)}$ in the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$. Then, there exists the coefficients $0 \le \lambda_\gamma^{(j)} \le 1$, $\gamma = 1, \ldots, |S^{(j)}|!$, such that

$$m^{(j)}\mathbf{1}_{|S^{(j)}|} = \sum_{\gamma=1}^{|S^{(j)}|!} \lambda_\gamma^{(j)} \mathbf{u}\big(\pi_\gamma^{(j)}\big), \tag{34}$$

where

$$\sum_{\gamma=1}^{|S^{(j)}|!} \lambda_\gamma^{(j)} = 1. \tag{35}$$

Note that $E = \bigcup_{j=0}^{t} S^{(j)}$. Consider a permutation $\pi_{\gamma_j}^{(j)}$ as one of the total $|S^{(j)}|!$ permutations of $S^{(j)}$, for $j = 0, \cdots, t$, then the permutation π formed by concatenating these permutations, i.e. $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$, is a permutation for the set E.

Theorem 2 *(I) Consider the permutation $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$ of the set E. The corner point corresponding to this permutation in the polymatroid $\mathcal{B}(f, E)$ is*

$$v_i(\pi) = u_i(\pi_{\gamma}^{(j)}), \quad \text{for} \quad i \in S^{(j)}. \tag{36}$$

(II) The vector $\mathbf{x}^*$ is in the convex hull of the set of corner points corresponding to the following set of permutation

$$\left\{ \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right), 1 \le \gamma_t \le |S^{(t)}|!, \dots, 1 \le \gamma_0 \le |S^{(0)}|! \right\} \tag{37}$$

where the coefficient of the corner point corresponding to the permutation $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$ is equal to $\lambda_{\gamma_t}^{(t)} \dots \lambda_{\gamma_0}^{(0)}$, i.e.

$$\mathbf{x}^* = \sum_{\gamma_t=1}^{|S^{(t)}|!} \dots \sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda_{\gamma_t}^{(t)} \dots \lambda_{\gamma_0}^{(0)} \mathbf{v}\left(\left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right) \right). \tag{38}$$

Proof: **Part (I)** From recursive equation (21), we can show that

$$\text{For } S \in E - \bigcup_{i=0}^{j-1} S^{(i)}, \quad f^{(j)}(S) = f\left(S \cup \left\{ \bigcup_{i=0}^{j-1} S^{(i)} \right\} \right) - f\left(\left\{ \bigcup_{i=0}^{j-1} S^{(i)} \right\} \right). \tag{39}$$

Consider the permutation $\pi = \left(\pi_{\gamma_t}^{(t)}, \cdots, \pi_{\gamma_0}^{(0)}\right)$. Set $\xi = \sum_{i=1}^{j} |S^{(i)}|$. Using (10) and (39), for $\xi < \kappa \le \xi + |S^{(j+1)}|$, $v_{\pi(\kappa)}(\pi)$ is equal to

$$v_{\pi(\kappa)}(\pi) = f(\{\pi(1), \dots, \pi(\kappa)\}) - f(\{\pi(1), \dots, \pi(\kappa - 1)\}) \tag{40}$$

$$= f\left(\left\{ \bigcup_{i=0}^{j-1} S^{(i)}, \pi(\xi) \dots, \pi(\kappa) \right\} \right) - f\left(\left\{ \bigcup_{i=0}^{j-1} S^{(i)}, \pi(\xi), \dots, \pi(\kappa - 1) \right\} \right) \tag{41}$$

$$= f^{(j)}(\{\pi(\xi) \dots, \pi(\kappa)\}) - f^{(j)}(\{\pi(\xi) \dots, \pi(\kappa - 1)\}). \tag{42}$$

According to definition, $f^{(j)}(\{\pi(\xi) \dots, \pi(\kappa)\}) - f^{(j)}(\{\pi(\xi) \dots, \pi(\kappa - 1)\})$ is the value of $x_{\pi(\kappa)}$ in the corresponding corner point of the polymatroid $\mathcal{B}(f^{(j)}, S^{(j)})$.

Part (II) Since $\sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda_{\gamma_0}^{(0)} = 1$ and by using (34) and part (i) of the theorem, it is easy to verify that in the resulting vector of the following summation

$$\sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda_{\gamma_0}^{(0)} \mathbf{v}\left(\pi_{\gamma_t}^{(t)}, \dots, \pi_{\gamma_0}^{(0)}\right) \tag{43}$$

the entry i, $i \in S^{(0)}$ is equal to $m^{(0)}$. Similarly, in the resulting vector of the following summation,

$$\sum_{\gamma_1=1}^{|S^{(1)}|!} \lambda_{\gamma_1}^{(1)} \sum_{\gamma_0=1}^{|S^{(0)}|!} \lambda_{\gamma_0}^{(0)} \mathbf{v}\left(\pi_{\gamma_t}^{(t)}, \dots, \pi_{\gamma_0}^{(0)}\right), \tag{44}$$

the entry i, $i \in S^{(1)}$ is equal to $m^{(1)}$, while the entry i, $i \in S^{(0)}$ remains $m^{(0)}$. By continuing this procedure, the part (ii) of the algorithm is proven. ■

In the following we present an algorithm to find a max-min corner point.
Algorithm II

1. Set $\alpha = r$, $S = \varnothing$.
2. Set $\pi^*(\alpha)$ as
$$\pi^*(\alpha) = \arg \min_{z \in E, z \notin S} f\left(E - S - \{z\}\right) \tag{45}$$
3. If $\alpha > 1$, then $\alpha \longleftarrow \alpha - 1$, $S \longleftarrow S \cup \{\pi^*(\alpha)\}$, and go to Step 2; otherwise stop.

The following theorem proves the optimality of the above algorithm.

Theorem 3 *Let the vector $\mathbf{v}(\pi^*)$ be the corner point of the polymatroid $\mathcal{B}(f, E)$ corresponding to the permutation $\pi^* = (\pi^*(1), \ldots, \pi^*(r))$. For any other permutation $\pi = (\pi(1), \ldots, \pi(r))$,*
$$\min_i v_{\pi^*(i)}(\pi^*) \geq \min_i v_{\pi(i)}(\pi). \tag{46}$$

Proof: Assume that in the permutation π^*, the user θ which is located in position l in the permutation π^* (i.e. $\theta = \pi^*(l)$), has the minimum rate,
$$v_{\pi^*(l)}(\pi^*) = \min_i v_{\pi^*(i)}(\pi^*) \tag{47}$$

Let us define two sets:

- The set of users located before $\pi^*(l)$ in π^*: $\Phi = \{\pi^*(1), \ldots, \pi^*(l-1)\}$.
- The set of users located after $\pi^*(l)$ in π^*: $\Psi = \{\pi^*(l+1), \ldots, \pi^*(r)\}$.

Using (10), we have
$$v_\theta(\pi^*) = f(\Phi \cup \{\theta\}) - f(\Phi). \tag{48}$$
In the following, we consider different scenarios which generate new permutations and prove that in all cases, (46) is valid.

Case 1. *Permutation in Φ and Ψ:* By considering (48), it is apparent that any permutation of the users in Φ and Ψ does not change the rate of the user $\pi^*(l)$.

Case 2. *Moving a set of users from Ψ to the set Φ:* Assume a set Υ of users, $\Upsilon \subset \Psi$, is moved from Ψ to the set Φ to generate a new permutation π. The rate of the user θ in the new permutation is equal to:
$$v_\theta(\pi) = f(\Phi \cup \Upsilon \cup \{\theta\}) - f(\Phi \cup \Upsilon). \tag{49}$$

From (9), we can show that
$$f(\Phi \cup \{\theta\}) + f(\Phi \cup \Upsilon) \geq f(\Phi \cup \Upsilon \cup \{\theta\}) + f(\Phi) \tag{50}$$

Using (48), (49), and (50), we conclude that $v_\theta(\pi) \leq v_\theta(\pi^*)$, and therefore, $\min_i v_{\pi(i)}(\pi) \leq \min_i v_{\pi^*(i)}(\pi^*)$.

Case 3. *Moving one or more users from the set Φ to the set Ψ (with or without moving some users from the set Ψ to the set Φ):* Assume that one or more users move from Φ to Ψ (with or without moving some users from the set Ψ to the set Φ) to generate the new permutation π. Consider the user ν which is the last user in permutation π among the moved users from Φ to Ψ (user $\pi(1)$ is the first user and user $\pi(r)$ is the last user in the permutation π). Let Ω be the set of users located before the user ν in the permutation π. It is clear that,

$$\{\theta\} \cup \Phi - \{\nu\} \subset \Omega. \tag{51}$$

On the other hand, this user is in the set Φ in permutation π^*. It means that in Step 2 of the algorithm, this user has been compared with other users in the set $\Phi \cup \{\pi^*(l)\}$ to be located in the position l, but the user $\pi^*(l)$ has been chosen for the position, i.e. $f(\Phi \cup \{\theta\} - \{\theta\}) \leq f(\Phi \cup \{\theta\} - \{\nu\})$, therefore,

$$f(\Phi) \leq f(\Phi \cup \{\theta\} - \{\nu\}). \tag{52}$$

Using (10), we have,

$$v_\nu(\pi) = f(\Omega \cup \{\nu\}) - f(\Omega). \tag{53}$$

Using (9) and (51), we can show that

$$f(\Omega \cup \{\nu\}) - f(\Omega) \leq f(\Phi \cup \{\theta\}) - f(\{\theta\} \cup \Phi - \{\nu\}). \tag{54}$$

Using (48), (52), (53), and (54), we conclude that $v_\nu(\pi) \leq v_\theta(\pi^*)$, and therefore, we have $\min_i v_{\pi(i)}(\pi) \leq \min_i v_{\pi^*(i)}(\pi^*)$. Note that the permutation of users located before (or after) the user ν in the permutation π does not increase $v_\nu(\pi)$. ■

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