A New Method for Performance Evaluation of Turbo-Codes

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***Abstract*— In this paper[1], a new method for the performance evaluation of Turbo-Codes is presented. The method is based on estimating the Probability Density Function (pdf) of the bit Log Likelihood Ratio (LLR) by using an exponential model. It is widely known that the pdf of the bit LLR is close to the normal density, and the proposed approach takes advantage of this property to simplify the calculations. The moment matching method is combined with the maximum entropy principle to estimate the parameters of the new model. A simple method for computing the confidence intervals for the estimated parameters, as well as for the Bit Error Rate (BER) is presented. The corresponding results are adopted to compute the number of samples that are required for a given precision of the estimated values. It is demonstrated that this method requires significantly fewer samples than the conventional Monte-Carlo (MC) simulation.**

***Index Terms*— Bit Decoding, Log Likelihood Ratio, Maximum Entropy, Probability Density Function, Turbo-Code.**

I. Introduction

In the application of channel codes, one of the most important issues is to develop an efficient method for performance evaluation, since the Monte-Carlo (MC) simulation is extremely time consuming for low Bit Error Rate (BER) values. There have been numerous efforts devoted to the performance evaluation of Turbo-Codes. These approaches derive some bounds on the average performance of Turbo-Codes by assuming Maximum Likelihood (ML) decoding [1]–[3]. Some researchers [4]–[10] have employed the Importance Sampling (IS) method to improve the performance of the MC simulation by increasing the weight of the rare error events. In this method, instead of choosing the samples from the original distribution, the samples are selected from a modified distribution which concentrates the points where the rare error events occur. This modified distribution is obtained from the original distribution by the application of a biasing function. This ensures a variance reduction if the biasing function is appropriately selected. The Turbo-Product Codes (of a small block length) are simulated in [11] by partitioning the error regions and by using a biasing function for each sub region independently. This method becomes inefficient, as the complexity of the code increases. In the case of Turbo-Codes with a large block length, the search for the appropriate biasing functions may be lengthy, which renders this method even more complicated than the conventional MC simulation.

It is observed in [3], [12], [13] that the pdf of the bit Log Likelihood Ratio (LLR) is nearly Gaussian. There have been some efforts on estimating this pdf using Gaussian mixture model [14], [15]. Two soft decision MC simulation techniques based on probabilities or LLR values are introduced in [16], and later on compared in [17].

An exponential model with a polynomial in the exponent is proposed. The aforementioned model has the ability to efficiently capture the deviation of the desired pdf from Gaussian. The moments of the bit LLR is used to estimate the parameters for the proposed model, in this article.

II. Modeling the pdf of the Bit LLR

A common tool to express the bit probabilities in bit decoding algorithms is based on the so-called LLR. The LLR of the k^{th} bit position is defined by the following equation:

$$LLR(k) = \log \frac{P(c_k = 1|\mathbf{x})}{P(c_k = 0|\mathbf{x})}, \tag{1}$$

where c_k is the value of the k^{th} bit in the transmitted code-word, $\mathbf{x}$ is the received vector, and $\log$ represents the natural logarithm. Let us define the random variable $Y = LLR(k)$ with its pdf denoted as $f(y)$. It is proved in [18] that the pdf of the bit LLR is independent of the transmitted code-word, as long as the value of the bit position under consideration remains unchanged. By using this result and without the loss of generality, the case of sending the all-zero code-word is considered. It is proved in [19] that the pdf of the bit LLR has the following symmetry property:

$$f(y) = e^{-y} f(-y). \tag{2}$$

Taking the logarithm from both sides of (2) and utilizing the power series, it easily follows that the following model can be used for the pdf of the bit LLR:

$$f(y) \simeq \exp(-y/2 + \sum_{i=0}^{N} a_i y^{2i}). \tag{3}$$

The received bit is decoded to 0 (or 1), if the corresponding LLR is negative (or positive). Therefore, the following integral simplifies the remaining BER calculations:

$$P_e \simeq \int_0^{\infty} f(y) dy. \tag{4}$$

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In the next section, the maximum entropy principle is used to find the parameters of the proposed model by using the moments of the bit LLR.

III. Moment Matching Using the Maximum Entropy Principle

There are various methods for parameter estimation. Typically, the unknown parameters of a pdf can be found by adopting moment matching, entropy matching, or ML. In this paper, the moment matching method with the maximum entropy principle is used simply because it is mathematically tractable, and has been successfully implemented in a variety of applications [20]. An attractive feature of the class of the distributions with the maximum entropy is that a simple iterative maximization technique can be employed to compute their parameters. The maximum entropy principle was first introduced by Jaynes [20] in 1982. Since then, it has been widely used in various applications. In this method, the search, while satisfying the constraints on the moments, is limited to the pdf with the maximum entropy. For more recent discussions on this method, refer to [21], [22]. We follow an approach that is similar to the one introduced in [23]. The maximum entropy density can be found by maximizing the following with respect to $\hat{f}(y)$:

$$\text{Maximize } -\int_{-\infty}^{+\infty} \hat{f}(y)\log[\hat{f}(y)]dy, \tag{5}$$

$$\text{Subject to: } \hat{\mu}_i = \mu_i, \quad i = 1, 2, \ldots, N, \tag{6}$$

where μ_i denotes the true i^{th} moment, $\hat{\mu}_i$ stands for the estimated i^{th} moment, and N is the number of moments used in the parameter estimation. This maximization problem can be solved with the Lagrange multipliers by following the methods of the calculus of variations [24]. One can estimate the parameters a_k, $k = 1, \ldots, N$ using the first N moments of the bit LLR. In practice, the statistical estimates of the moments are used instead of the true moments.

IV. Confidence Interval Statements

An analytical approximate method to compute the confidence intervals for the estimated parameters of the model in terms of the covariance matrix of the estimated moments is derived. Subsequently, a relationship between the confidence interval on the BER and the confidence interval on the parameters is established. If the moment estimator satisfies a set of mild conditions, it follows that the estimated parameters are asymptotically normal with a derivable covariance matrix [25]. This allows for the confidence interval statements to be made concerning $\hat{f}(y)$. A method to compute the covariance matrix of the estimated parameters in terms of the covariance matrix of the moments is developed. With the covariance matrix of the parameters, the desired confidence intervals for the parameters can be easily computed. Let us define m_i as follows

$$m_i = \int_0^{\infty} y^{2i} \exp(-y/2 + \sum_{i=0}^{N} a_i y^{2i}) dy. \tag{7}$$

It can be shown that ΔP_e, the error in the BER estimation, is a linear combination of m_i's, which can be estimated during the procedure of the moment computation by considering the positive samples only. This analysis enables us to make confidence interval statements on the estimated BER in terms of the confidence intervals for the model parameters.

V. Numerical Results

A Turbo-Code of the length 100 and rate $1/2$ is employed to perform the simulations. In Table 1, variances of the BER estimations are computed for both methods. The number of samples and the variance to the mean ratio of the BER are denoted as $n(.)$ and $v(.)$, respectively. The variance of the MC method can be computed analytically, although this analysis is very complex for the proposed method and one need to estimate the variances with numerical methods. The variance of the proposed method can be computed by repeating the experiment for J times (generating J independent sets of moments), and computing the variance of the resulting sequence of the BER values. In the computations of Table 1, we set $J = 1000$ to obtain a reasonable approximation, and at the same time, render the analysis feasible in the sense of the required time.

$\frac{E_b}{N_0}$	BER	v(new)	n(new)	v(MC)	n(MC)	G
1	3.8E-2	6.8E-5	1E4	9.5E-5	1E4	1.4
2	4.9E-3	1.4E-5	1E4	9.9E-5	1E4	6.8
3	1.7E-4	4.9E-6	1E5	9.9E-6	1E6	20.2
4	3.5E-6	2.3E-8	1E6	1.0E-8	1E8	43.5

Table 1 : Comparison of the proposed method and the MC simulation, where the variances are computed as described in Section V and $\frac{E_b}{N_0}$ is in dB.

The relative gain G is used in Table 1 as a measure to compare the two methods. To incorporate both the variance reduction and the sample reduction advantage of the new method, and noting that $v(\text{MC})$ is inversely proportional to $n(\text{MC})$, define G as follows:

$$G = \frac{v(\text{MC})}{v(\text{new})} \cdot \frac{n(\text{MC})}{n(\text{new})}. \tag{8}$$

Simulation results are shown in Figure 1, where the same number of samples is used as indicated in Table 1. It is evident that increasing the number of moments (the order of approximation) that are involved from two to five significantly improves the approximation.

In addition the confidence intervals are computed by using the proposed method in Section IV for this example. This confidence interval is closely related to n, the number of samples used to compute it. In Table 2, this relation is presented for three different values of n at E_b/N_0=2dB. We compute $p(|\Delta P_e| < \theta)$ for the different values of n and θ. When we compare the proposed method with the MC simulation in Table 1 and Table 2, the number of samples required for the BER calculations indicate a significant reduction for our method. It can be seen that the proposed method is more accurate than the MC simulation even by using significantly fewer samples.

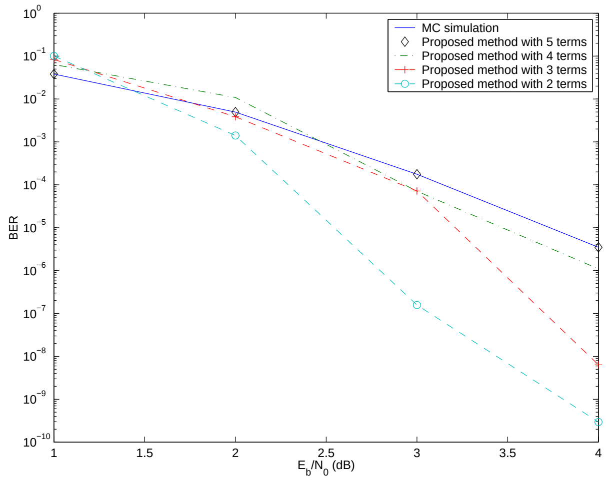

Fig. 1. BER curves for Turbo-Code of the length 100 and rate $1/2$ in comparison with the MC simulation.

n	θ	p_1	p_2
10^4	0.0058	0.95	0.67
10^5	0.0058	0.96	0.70
10^6	0.0058	0.97	0.96
10^4	0.0020	0.94	0.66
10^5	0.0020	0.95	0.70
10^6	0.0020	0.96	0.96
10^4	0.0005	0.93	0.33
10^5	0.0005	0.94	0.68
10^6	0.0005	0.95	0.95

Table 2 : Relation between n and confidence interval at E_b/N_0=2dB for the new method and the MC simulation. $p_1 = p(|\Delta P_e| < \theta)$ for new method and $p_2 = p(|\Delta P_e| < \theta)$ for MC.

VI. Concluding Remarks

In this paper, we have proposed a new method for the performance evaluation of Turbo-Codes. Although our focus is on Turbo-Codes, the application of the proposed method is not necessarily restricted to this class of channel codes. The problem of finding the BER in high signal to noise ratio regions can be solved with this method, since the MC simulation may not be feasible. We take advantage of the symmetry properties of the pdf of the bit LLR to propose a suitable model for this unknown density. The moment matching method is employed to find the density with the maximum entropy which satisfies the moment constraints. A simple method is introduced to make confidence interval statements both for the parameters of the model and the BER integral, which enables us to compute the BER values accurately. It is demonstrated that significantly fewer samples, compared to those required in the MC simulation, are necessary to compute the statistical moments that are accurate enough.

References

[1] T. M. Duman and M. Salehi, “New Performance Bounds for Turbo Codes,” *IEEE Transactions on Communications*, vol. 46, no. 6, pp. 717-723, June 1998.

[2] I. Sason and S. Shamai, “Improved Upper Bounds on the ML Decoding Error Probability of Parallel and Serial Concatenated Turbo Codes via their Ensemble Distance Spectrum,” *IEEE Transactions on Information Theory*, vol.46, no.1, pp. 24-47, January 2000.

[3] D. Divsalar, S. Dolinar and F. Pollara, “Iterative Turbo Decoder Analysis Based on Density Evolution,” *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 5, pp. 891-907, May 2001.

[4] N. C. Beaulieu, “An Investigation of Gaussian Tail and Rayleigh Tail Density Functions for Importance Sampling Digital Communication System Simulation,” *IEEE Transactions on Communications*, vol. 38, no. 9, pp. 1288-1292, September 1990.

[5] K. S. Shanmugam and P. Balaban, “A Modified Monte-Carlo Simulation Technique for the Evaluation of Error Rate in Digital Communication Systems,” *IEEE Transactions on Communications*, vol. 28, no. 11, pp. 1916-1924, November 1980.

[6] M. C. Jeruchim, “On the Application of Importance Sampling to the Simulation of Digital Satellite and Multi-hop Links,” *IEEE Transactions on Communications*, vol. 32, no. 10, pp. 1088-1092, October 1984.

[7] Q. Wang and V. K. Bhargava, “On the Application of Importance Sampling to BER Estimation in the Simulation of Digital Communication Systems,” *IEEE Transactions on Communications*, vol. 35, no. 11, pp. 1231-1233, November 1987.

[8] J. S. Sadowsky, “A New Method for Viterbi Decoder Simulation Using Importance Sampling,” *IEEE Transactions on Communications*, vol. 38, no. 9, pp. 1341-1351, September 1990.

[9] J. C. Chen, D. Lu, J. S. Sadowsky and K. Yao, “On Importance Sampling in Digital Communications. I. Fundamentals,” *IEEE Journal on Selected Areas in Communications*, vol. 11, no. 3, pp. 289-299, April 1993.

[10] J. A. Bucklew and R. Radeke, “On the Monte-Carlo Simulation of Digital Communication Systems in Gaussian Noise,” *IEEE Transactions on Communications*, vol. 51, no. 2, pp. 267-274, February 2003.

[11] M. Ferrari and S. Bellini, “Importance Sampling Simulation of Turbo-Product Codes,” *Proceedings of IEEE International Conference on Communications (ICC 2001)*, vol. 9, pp. 2773-2777, June 2001.

[12] C. Berrou, A. Glavieux and P. Thitimajshima, “Near Shannon Limit Error-Correcting Coding and Decoding: Turbo-Codes (1),” *Proceedings of IEEE International Conference on Communications*, Geneva, Switzerland, pp. 1064-1070, May 1993.

[13] H. El-Gamal and A. R. Hammons Jr., “Analyzing the Turbo Decoder Using the Gaussian Approximation,” *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 671-686, February 2001.

[14] S. Y. Chung, T. J. Richardson, R. L. Urbanke, “Analysis of Sum-Product Decoding of Low-Density-Parity-Check Codes Using Gaussian Approximation,” IEEE Transactions on Information Theory, vol. 47, no. 2, pp. 657-670, February 2001.

[15] N. Letzepis and A. Grant, “Bit error rate estimation for turbo decoding,” in Proc. IEEE Int. Symp. Inform. Theory (ISIT), pp. 437, Yokohama, Japan, June/July 2003.

[16] I. Land, P.A. Hoeher, and U. Sorger, ”Log-Likelihood Values and Monte Carlo Simulation Some Fundamental Results,” in Proc. Int. Symp. on Turbo Codes and Related Topics, Brest, France, Sept. 2000.

[17] I. Land and P.A. Hoeher, ”New results on Monte Carlo bit error simulation based on the a posteriori log-likelihood ratio,” in Proc. Int. Symp. on Turbo Codes and Related Topics, Brest, France, Sept. 2003.

[18] A. Abedi and A. K. Khandani, “Some Properties of Bit Decoding Algorithms for Binary Linear Block Codes,” *IEEE Transactions on Information Theory*, Revised, March 2004.

[19] T. J. Richardson, M. A. Shokrollahi and R. L. Urbanke, “Design of Capacity Approaching Irregular Low-Density-Parity-Check Codes,” *IEEE Transactions on Information Theory*, vol.47, no.2, pp. 619-637, February 2001.

[20] E. T. Jaynes, “On The Rationale of Maximum-Entropy Methods,” *Proceedings of the IEEE*, vol. 70, no. 9, pp. 939-952, September 1982 (Invited paper).

[21] M. Grendar Jr. and M. Grendar, “Maximum Entropy: Clearing up Mysteries,” Journal of Entropy, vol. 3, pp. 58-63, 2001.

[22] P.D. Grunwald and A. P. Dawid, “Game Theory, Maximum Generalized Entropy, Minimum Discrepancy, Robust Bayes and Pythagoras,” *Information Theory Workshop*, Bangalore, India, pp. 94-07, October 2002.

[23] M. Kavehrad and M. Joseph, “Maximum Entropy and the Method of Moments in Performance Evaluation of Digital Communications Systems,” *IEEE Transactions on Communications*, vol.34, no.12, pp. 1183-1189, December 1986.

[24] R. Weinstock, *Calculus of Variations with Applications to Physics and Engineering*, New York: Dover, 1974.

[25] S. Kullback, *Information Theory and Statistics*, New York: Wiley, 1959.