Decoding of a Cartesian Product Set with a Constraint on an Additive Cost; Fixed-Rate Entropy-Coded Vector Quantization

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Consider a discrete set of points A composed of $K = |A|$ elements. A non-negative cost $c(a)$ is associated with each element $a \in A$. The n-fold cartesian product of A is dented as $\{A\}^n$. The cost of an n-fold element $\mathbf{a} = (a_0, \ldots, a_{n-1}) \in \{A\}^n$ is equal to: $c(\mathbf{a}) = \sum_i c(a_i)$. We select a subset of the n-fold elements, $S_n \in \{A\}^n$, with a cost less than or equal to a given value $c_{\max}$. We refer to A as the *constituent subset.*

Consider another set of n-tuples X_n denoted as the *input set.* A non-negative distance is defined between each $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in X_n$ and each $\mathbf{s} = (s_0, \ldots, s_{n-1}) \in S_n$. The distance between x_i and s_i is denoted as $d(x_i, s_i)$. The distance between $\mathbf{x}$ and $\mathbf{s}$ is equal to: $d(\mathbf{x}, \mathbf{s}) = \sum_i d(x_i, s_i)$. Decoding of an element $\mathbf{x} \in X_n$ is to find the element $\mathbf{s} \in S_n$ which has the minimum distance to $\mathbf{x}$.

A major application of this decoding problem is in the fixed-rate entropy-coded vector quantization where A is the set of reconstruction vectors of a vector quantizer and cost is equivalent to self-information.

In this work, the decoding problem is formulated in terms of a linear (zero-one) program. Using some special features of the problem, we present methods to substantially reduce the complexity of the corresponding simplex search. This results in a substantial reduction in complexity with respect to the schemes of [1], [2], [3], [4]. To formulate the decoding problem as a linear program, the elements of the ith constituent subset are identified by the use of a K-D binary vector $\{\delta_i(j), j = 0, \ldots, K-1\}$ where $\delta_i(j) = 0, 1$ and $\sum_j \delta_i(j) = 1$, $i = 0, \ldots, n-1$. The vector corresponding to the jth element is composed of a single one in the jth position and zeros elsewhere. The cost associated with the jth element of A is denoted as $c(j)$. For an n-tuple input $\mathbf{x}$, the distance of the ith component of $\mathbf{x}$ to the jth element of A is denoted as $d_i(j)$. Using these notations, the optimization problem is formulated as:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=0}^{n-1} \sum_{j=0}^{K-1} \delta_i(j) d_i(j) \\ \text{subject to:} \quad & \sum_{i=0}^{n-1} \sum_{j=0}^{K-1} \delta_i(j) c(j) \le c_{\max}, \\ & \delta_i(j) = 0, 1, \quad \forall i, j \quad \& \quad \sum_{j=0}^{K-1} \delta_i(j) = 1, \quad \forall i. \end{aligned} \tag{1}$$

Each of the equalities $\sum_j \delta_i(j) = 1$, $i = 0, \ldots, n-1$, is called an *indicator constraint.*

Theorem: There are at least one and at most two basic variables corresponding to each indicator constraint.

To solve the problem in (1), we relax the zero-one constraint and then apply the simplex search. Using the previous theorem, we conclude that in the final solution at most two basic variables are different from unity. If there is only one non-unity basic variable, the vector obtained by concatenating the nearest points of different constituent subsets satisfy the cost constraint. If there are two non-unity basic variables, we set one of them to zero and the other one to unity. The selection is achieved such that the cost constraint is not violated.

The problem has some special features which are used to reduce the complexity of the corresponding simplex search. These are: (i) All the constituent subsets are the same. This property allows us to reduce the complexity of the multiply-add operations involved in pivoting to one add per dimension and one multiply per set. (ii) The set of the indicator constraints are non-overlapping and there is only one constraint involving all the variables. These properties allow us to solve the problem using a reduced basis of size 2×2, as compared to $(n+2) \times (n+2)$, where the basis matrix is upper triangular with a unity element at the upper left corner. It is very easy to compute the inverse of this matrix.

References

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