Constellation Shaping as a Geometrical Approach to Solving a Constrained Optimization Problem

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This work addresses the following two problems in conjunction with a shaped constellation: (i) Constellation shaping as a means of spectral shaping, (ii) Optimum addressing[1] to minimize the effect of the channel errors on the end-to-end source distortion. These are expressed in terms of two constrained optimization procedures.

In spectral shaping, we minimize the average energy of a constellation for a fixed total rate, fixed number of points per dimension, and subject to some constraints on its power spectrum. Following spectral constraints are considered in detail: (i) A fraction of the total average energy equal to F_p is located in the frequency band $[0, f_c]$, and/or (ii) The spectrum has certain spectral nulls. To solve the problem, the cost of a point is defined as its energy plus some Lagrange multiplier(s) times the spectral constraint(s). The final constellation is selected as a subset of the points of the least cost.

For a wide class of spectral constraints, the cost of a high dimensional point is obtained by adding the costs of its lower dimensional components. This property allows us to apply most of the shaping techniques known for addressing or performance computation (mainly the methods of decomposition) in this new context.

We assume that the sine matrix is used as the constellation basis. In the case of spectral nulls, the basis is selected as the output eigenvectors of a linear system with the same set of spectral nulls.

We present: (i) A set of appropriate factors to measure the performance and the complexity of such a constellation[2]. (ii) Analytical methods for the computation of the corresponding shaping performance in a finite, infinite-dimensional space together with the related tradeoff curves. (iii) Efficient methods to facilitate the addressing of such a constellation. (iv) Spectral shaping using an arbitrary frequency window to give different weights to the energy in different frequency bands.

In the addressing problem, we find the optimum mapping between the source and the channel symbols which minimizes the end-to-end average distortion subject to a constraint on the average energy. The corresponding optimization problem is formulated as:

$$\text{Minimize} \quad \sum_{i=0}^{T-1}\sum_{j=0}^{T-1}\sum_{k=0}^{T-1}\sum_{l=0}^{T-1} P_s(i)P_c(l|j)D_s(i,k)x_{ij}x_{kl}$$

$$\text{Subject to:} \quad \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} P_s(i)E_c(j)x_{ij} \le \overline{E}$$

$$x_{ij} \in \{0,1\}$$

$$\sum_{i=0}^{T-1} x_{ij} = 1, \ \forall j \quad \& \quad \sum_{j=0}^{T-1} x_{ij} = 1, \ \forall i$$

where T denotes cardinality of the source, channel; $P_s(i)$'s denote the source symbols probabilities; $D_s(i,k)$'s denote the source distortion measure; $P_c(l|j)$'s denote the channel transition probabilities, and $\overline{E}$ denotes the maximum available average energy.

This can be easily written in terms of a QAP (Quadratic Assignment Problem) which is a standard problem in discrete optimization. QAP has the following general structure:

$$\text{Minimize} \quad \text{trace}\,(\mathbf{CX}^t + \mathbf{AXB}^t\mathbf{X}^t)$$

$$\left(= \sum_{i=0}^{T-1}\sum_{j=0}^{T-1} c_{ij}x_{ij} + \sum_{i=0}^{T-1}\sum_{j=0}^{T-1}\sum_{k=0}^{T-1}\sum_{l=0}^{T-1} a_{ik}b_{jl}x_{ij}x_{kl} \right)$$

Subject to: $\mathbf{X}$ is a permutation matrix

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{X}$ are matrices of elements a_{ij}, b_{ij}, c_{ij} and x_{ij}, respectively.

Shaping allows us to achieve the addressing between the N-fold source symbols, and the N-fold channel symbols. This ability to extend the system is a special feature which distinguishes our problem from a standard QAP.

[1] Addressing is the mapping between the source and the channel symbols.

[2] These are in main part generalization of the factors given in [1].

References

[1] G. D. Forney, Jr. and L. F. Wei, "Multidimensional constellations–Part I: Introduction, figures of merit, and generalized cross constellations," *IEEE J. Select. Areas Commun.*, vol. SAC-7, pp. 877–892, Aug. 1989.