Unsymmetrical Boundary Shaping with Application to Spectral Shaping

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Abstract: We discuss the selection of the boundary of a signal constellation which has nonequal values of average energy along different dimensions. This nonequal energy allocation, in conjunction with a nondiagonal modulating matrix, is used to shape the corresponding power spectrum. The objective is to optimize the "rate versus energy tradeoff" subject to some constraints on the corresponding power spectrum. In a rectangular constellation, the nonequal energy allocation is achieved by using different number of points along different dimensions. It is shown that the optimization procedure reduces to maximizing the determinant of an autocorrelation matrix subject to some linear constraints on its elements. In a shaped constellation, the number of points along the one-D (one-dimensional) subspaces are the same and the nonequal energy allocation is achieved by selecting an appropriate boundary in a higher dimensional space.

1 Introduction

Consider a regular array of points. A signal constellation is a finite subset of these points bounded within a shaping region. Conventionally, the shaping region is selected to minimize the average energy of the constellation for a fixed number of points [1], [2]. In the present work, we impose some additional constraints on the corresponding power spectrum. To satisfy the spectral constraints, the modulating matrix is selected to be non-diagonal and nonequal values of average energy are allocated to different dimensions of this matrix.

The problem of the line coding is a well established subject. The major difference between our approach to this problem and most of the works reported in the literature is that in our case the memory of the code is limited to the elements within a block.

The body of the paper is as follows: In Section 1.1, the block diagram of the system is introduced. In Section 1.2, we explain our figures of merit in designing an unsymmetrical shaping region. In Section 2, we discuss how to maximize the rate of a rectangular constellation for a fixed total average energy and subject to some constraints on its power spectrum. In Section 3, we discuss how to minimize the average energy of a shaped constellation for a fixed total rate, fixed number of point per dimension, and subject to some constraints on its power spectrum. To solve this constrained optimization problem, the cost of a point is defined as its energy plus some Lagrange multiplier(s) times the spectral constraint(s). The final constellation is selected as a subset of the points of the least cost. For a wide class of the spectral constraints, the cost of a high dimensional point is obtained by adding the costs of its lower dimensional components. This is a useful property which allows us to apply most of the known shaping techniques in this new context.

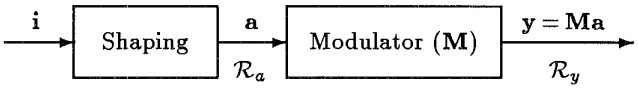

Figure 1: System block diagram.

1.1 System block diagram

Figure 1 shows the block diagram of the system under consideration. Each block is composed of N_m successive time periods. In each signaling interval, a data vector $\mathbf{i}$ is encoded. The shaping block maps the vector $\mathbf{i}$ to a point $\mathbf{a}$ in the baseband constellation $\boldsymbol{A}$. This is a finite set of the N-D points, $N \leq N_m$, selected from an array of points which is geometrically uniform and bounded within the shaping region $\mathcal{R}_a$. We assume that the points of $\boldsymbol{A}$ are used with equal probability. Using continuous approximation and normalizing the volume of the Voronoi region around each constellation point to unity, the rate of $\boldsymbol{A}$ is found as:

$$H(\boldsymbol{A}) = \log[V(\mathcal{R}_a)], \tag{1}$$

where $V(\mathcal{R}_a)$ is the volume of $\mathcal{R}_a$. The average energy (second moment) along the i'th dimension of $\mathcal{R}_a$ is denoted as λ_i. We assume that the λ_i's are strictly positive. The diagonal matrix $\boldsymbol{\Lambda}_a$ is defined as: $\boldsymbol{\Lambda}_a = \mathrm{diag}[\lambda_0, \ldots, \lambda_{N-1}]$.

The columns of the $N_m \times N$ (modulating) matrix $\mathbf{M}$ are the dimensions (line codes) of the constellation $\mathbf{y} \in \boldsymbol{Y}$. We can have up to $N_m - N$ nulls in the power spectrum of $\boldsymbol{Y}$. We have $\mathbf{M}^t\mathbf{M} = \mathbf{I}$ where $\mathbf{I}$ is the $N \times N$ identity matrix. This results in an isometry between the space containing $\boldsymbol{A}$ and the space containing $\boldsymbol{Y}$. This results in $V(\mathcal{R}_y) = V(\mathcal{R}_a)$, and consequently $H(\boldsymbol{Y}) = H(\boldsymbol{A})$, while the distance property and consequently the performance in noise of the two constellations are identical.

There exists a tradeoff between the rate $H(\boldsymbol{Y})$, and the total average energy $\sum_i \lambda_i$. The objective is to optimize this

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tradeoff subject to some constraints on the power spectrum of $\boldsymbol{Y}$. Our tools are the selection of the region $\mathcal{R}_a$ and the matrix $\mathbf{M}$.

1.2 Figures of merit

We assume that the projection of the constellation on its one-D subspaces (denoted as the one-D subconstellations) are finite portions of the one-D half integer grid[1]. A rectangular constellation is equal to the cartesian product of its one-D subconstellations while a shaped constellation is an appropriate subset of this product. The following definitions are based on the block diagram given in Section 1.1.

We consider two reference rectangular regions $\mathcal{C}_1$, $\mathcal{C}_2$, both with the same volume as $\mathcal{R}_a$. Region $\mathcal{C}_1$ is of dimensionality N and has the average energy $\gamma_s\lambda_i$ along its ith dimension. The factor γ_s, called the shaping gain of $\mathcal{R}_a$, reflects the reduction in the average of the shaping region $\mathcal{R}_a$ with respect to the reference region $\mathcal{C}_1$. The proportionality of the energies guarantees that the corresponding power spectrum are identical within the scale factor γ_s. Using continuous approximation, we obtain,

$$\gamma_s = \frac{1}{12}\left[\frac{V^2(\mathcal{R}_a)}{\prod_{i=0}^{N-1}\lambda_i}\right]^{1/N}. \tag{2}$$

Region $\mathcal{C}_2$ is of dimensionality N_m and has equal average energy along all its dimensions (cubic region). The equal allocation of energy results in a white spectrum. The increase in the total average energy of the reference region $\mathcal{C}_1$ with respect to the reference region $\mathcal{C}_2$ is measured by the factor P_l (performance loss). This factor reflects the price in energy that we pay for the shaping of the power spectrum. Using continuous approximation, we obtain,

$$P_l = \frac{N_m\left(\prod_{i=0}^{N-1}\lambda_i\right)^{1/N_m}}{(12\gamma_s)^{1-\frac{N}{N_m}}\sum_{i=0}^{N-1}\lambda_i}. \tag{3}$$

The overall changing in the average energy of the region $\mathcal{R}_a$ with respect to the reference region $\mathcal{C}_2$ is obtained by multiplying the shaping gain given in (2) with the performance loss given in (3).

A price to be paid for shaping is an increase in the factor Constellation-Expansion-Ratio [1]. We define the unsymmetrical Constellation-Expansion-Ratio, CER_u, as the ratio of the maximum number of points per dimension $C_{\max}$ to the minimum necessary number of points per dimension, i.e.,

$$\mathrm{CER}_u = \frac{C_{\max}}{[V(\mathcal{R}_a)]^{1/N}}. \tag{4}$$

This is a reasonable definition because the complexity of the modulator and demodulator is essentially determined by the maximum number of points per dimension.

[1]The one-D half integer grid is the set of points $[-\infty,\ldots,-3/2,-1/2,1/2,3/2,\ldots,+\infty]$.

For the reference region $\mathcal{C}_1$, using (4), we obtain,

$$\mathrm{CER}_r = \frac{\lambda_{\max}^{1/2}}{\left(\prod_{i=0}^{N-1}\lambda_i^{1/2}\right)^{1/N}}, \tag{5}$$

where $\lambda_{\max}$ is the maximum value of λ_i for $i=0,\ldots,N-1$.

The CER_r reflects the effect of the spectral shaping. To specify the effect of the boundary shaping, we define the multiplicative factor RCE (Ratio-of-Constellation-Expansions) as:

$$\mathrm{RCE} = \frac{\mathrm{CER}_u}{\mathrm{CER}_r} = \frac{C_{\max}}{\sqrt{12\gamma_s\lambda_{\max}}}. \tag{6}$$

In general, considering the saving in the energy due to the boundary shaping, the factor RCE can be less than unity.

2 Spectral shaping using a rectangular constellation

In all our discussions concerning a rectangular shaping region, we assume that the average energy per time interval is normalized to unity. This results in a total average energy of N_m units per block for the system given in Section 1.1. In this case, the optimization of the "rate versus energy tradeoff" is interpreted in terms of maximizing the rate for a fixed average energy.

Using continuous approximation over a rectangular region, we obtain,

$$H(\boldsymbol{Y}) = \frac{N}{2}\log(12) + \frac{1}{2}\sum_{\lambda_i(\mathbf{R}_y)\neq 0}\log\left[\lambda_i(\mathbf{R}_y)\right], \tag{7}$$

where $\lambda_i(\mathbf{R}_y)$ is the ith eigenvalue of $\mathbf{R}_y$. The objective in selecting the matrices $\mathbf{M}$ and $\mathbf{\Lambda}_a$ is to maximize the second term in (7) subject to some constraints on the power spectrum of $\mathbf{y}$. If all the eigenvalues are nonzero, we have $\sum_i \log\left[\lambda_i(\mathbf{R}_y)\right] = \log\left(|\mathbf{R}_y|\right)$ where $|\,.\,|$ denotes the determinant. In the following, we study some spectral constraints.

2.1 Spectral constraints

Due to the linear relationship between the spectrum and the elements of the autocorrelation matrix matrix $\mathbf{R}_y$, most of the spectral constraints can be formulated as linear constraints on the elements of $\mathbf{R}_y$. In the following, we study the F_p-constraint and the spectral null.

For a given cutoff frequency ω_c, define the power-ratio of a spectrum as the fraction of the total average energy in the frequency band $[0,\omega_c]$. The F_p-constraint is the constraint of having a power-ratio less than or equal to F_p.

A spectral null at zero frequency or at Nyquist frequency results in at least one zero eigenvalue for $\mathbf{R}_y$. In this case, we consider $\mathbf{y}$ as the output of a linear system $\mathbf{A}$ with the same spectral null and reformulate the problem at the system input, $\mathbf{x}$.

The final optimization problem is as follows:

$$\begin{cases} \text{Maximize} & \log(|\mathbf{R}_x|), \\ \text{Subject to:} & \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} B_l(i,j)R_x(i,j) \le e_l, \quad l \in [0, L-1], \\ & \mathbf{R}_x \text{ is positive–definite,} \end{cases} \tag{8}$$

where L denotes the total number of spectral constraints. The eigenvalues of $\mathbf{R}_x$ determine the allocation of the energy and the eigenvectors of $\mathbf{R}_x$, concatenated with the system corresponding to the spectral nulls, determine the basis.

It can be shown that the optimization problem in (8) is convex. As a result, the maximum point is unique and can be computed by using the Lagrange method. The Lagrange multipliers are denoted by ξ_l, $l \in A_c$ where A_c is the set of the active constraints. Calculating the derivatives and considering that the derivative of the determinant with respect to the (i,j)'th element is equal to the determinant of the corresponding adjoint matrix, we obtain,

$$\text{adj}\,[\mathbf{R}_x] = \sum_{l \in A_c} \xi_l \mathbf{B}_l, \tag{9}$$

where $\text{adj}\,[\mathbf{R}_x]$ is the adjoint matrix of $\mathbf{R}_x$ and $\mathbf{B}_l$ is the matrix of the elements $B_l(i,j)$ in (8). To calculate the Lagrange multipliers, we first calculate $\mathbf{R}_x$ using,

$$\mathbf{R}_x = |\text{adj}\,\mathbf{R}_x|^{\frac{1}{N-1}} \times (\text{adj}\,[\mathbf{R}_x])^{-1}, \tag{10}$$

and then apply the active constraints to the result. By iteratively satisfying the constraints, the multipliers are computed.

It is easy to show that for the spectral nulls and/or the F_p-constraint, the energy constraint is always active. For $F_p \in [F_{\min}, F_{\max}]$ (given $F_{\min}$ and $F_{\max}$), the F_p-constraint is active. For $F_p < F_{\min}$, the optimization problem has no answer. For $F_p > F_{\max}$, the F_p-constraint is not active and the power-ratio is equal to $F_{\max}$. The $F_{\max}$ can be calculated by relaxing the F_p-constraint and finding the power-ratio of the result. Without spectral null constraint, this results in a white spectrum and $F_{\max} = \omega_c/\pi$.

3 Spectral shaping using a non-rectangular constellation

In this section, we discuss the *optimum* boundary shaping using a fixed set of basis. The one-D subconstellations are identical and are bounded within the range $[-A, A]$. The optimization of the "rate versus energy tradeoff" is interpreted in terms of minimizing the energy for a fixed rate. The optimum constellation has the minimum average energy for a given number of points per dimension, given total rate and given set of spectral constraints. First, the asymptotic behavior in an infinite dimensional space is studied, and then we consider the finite dimensional case.

3.1 Optimum unsymmetrical shaping in infinite dimensionality

Assume that the space dimensionality is extended to infinity while the modulation is achieved on sub-blocks of an effective dimensionality N. Using the calculus of variations, the probability distribution which is zero outside region $[-A, A]$ and maximizes the rate for a given average energy is a truncated Gaussian distribution. For the ith dimension, $i = 0, \ldots, N-1$, we have,

$$P_i(X) = \begin{cases} C(\alpha_i)\exp\left(\dfrac{-\alpha_i X^2}{A^2}\right) & \text{for } X \in [-A, A], \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

where,

$$C(\alpha_i) = \left[2A\int_0^1 \exp(-\alpha_i x^2)dx\right]^{-1}. \tag{12}$$

The average energy and the rate along the ith dimension are computed as:

$$E_i = A^2 \frac{\displaystyle\int_0^1 x^2 \exp(-\alpha_i x^2)dx}{\displaystyle\int_0^1 \exp(-\alpha_i x^2)dx}, \tag{13}$$

$$H_i = \frac{\alpha_i \lambda_i}{A^2} - \log C(\alpha_i). \tag{14}$$

We use A as a parameter to adjust the tradeoff point. The α_i's, $i = 0, \ldots, N-1$, are selected such that $E_i = \lambda_i$ where E_i is given in (13) and the λ_i's are computed using the method explained earlier. For a given A, the maximum average energy is obtained for $\alpha = 0$ corresponding to a uniform distribution. This maximum value is equal to $A^2/3$. To be able to adjust the energies, we should have $A \ge \sqrt{3\lambda_{\max}}$.

The shaping performances can be computed by replacing,

$$V(\mathcal{R}_a) = \exp\left(\sum_{i=0}^{N-1} H_i\right), \quad \text{and} \quad C_{\max} = 2A, \tag{15}$$

in the relationships given in Section 1.2. The result of these computations are shown in Fig. 2.

3.2 Optimum unsymmetrical shaping in finite dimensionality

Consider a conventional shaping problem in a space of dimensionality N. We have a set of N, one-D subconstellations. The energy of each one-D point is considered as a cost associated with that point. In the cartesian product space, the cost associated with an N-D point is obtained by adding the costs of its components. The final constellation is selected as a subset of the N-D points of the least cost and a total cardinality T. Addressing is a one-to-one mapping between the set of the integer numbers $[0, T-1]$ and the set of the constellation points.

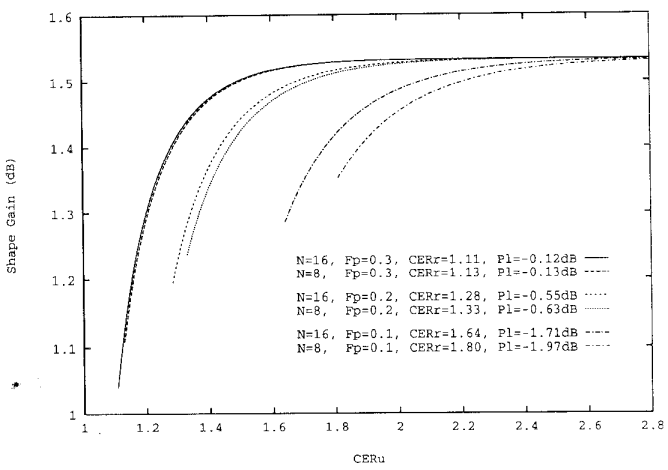

Figure 2: Examples of the optimum tradeoff curves in an infinite dimensional space, sine basis, F_p-constraint with $f_c = 0.2$, $\mathrm{CER}_u \geq \mathrm{CER}_r$. The constellation expansion due to the boundary shaping can be computed using $\mathrm{RCE} = \mathrm{CER}_u / \mathrm{CER}_r$.

In shaping, we are usually concerned with a set of a huge cardinality. This fact complicates the addressing and also the performance analysis of a shaped constellation. A major attribute of shaping is due to the additivity property of the cost in a cartesian product space. This property allows us to reduce the complexity of the problem by using the notion of *shell*.

A shell is defined as the collection of points of the same cost. In the process of shaping, we either keep a complete shell or discard all of its points. This means that the overall complexity is essentially determined by the number of shells (and not by the number of points). Due to the additivity property of cost, the number of shells is usually much lower than the number of points. This serves as the basis for a set of computational and addressing techniques described in [3], [4].

In an unsymmetrical shaping problem, the objective is to minimize the average energy of the constellation subject to: (i) fixed total rate, (ii) fixed number of points for the one-D subconstellations (fixed CER_u), and (iii) some constraints on the resulting power spectrum. To take the effect of the spectral constraints into account, the cost is considered as: "energy plus some Lagrange multipliers times the spectral constraints".

Assume that the spectral constraints are of the general form: $\mathbf{L}[S(\omega)] \leq e$ where $\mathbf{L}$ is a linear operator, i.e.,

$$\mathbf{L}\left[\sum_{i=0}^{N-1} \lambda_i S_i(\omega)\right] = \sum_{i=0}^{N-1} \lambda_i \mathbf{L}\left[S_i(\omega)\right]. \qquad (16)$$

The key point is that a spectral constraint of the form given in (16) can be decomposed as the sum of some components associated with the one-D subspaces. This property is the basis for some of the most powerful techniques known in shaping.

Assume that the shells in a given subspace are indexed in the order of increasing cost. Some of the shaping techniques discussed in [3], [4] are based on the property that the cost of a given shell is an affine function of its index (cost of the ith shell is equal to $c_0 + i\Delta$). This results in a recursive merging rule in the cartesian product of the shells. In the following, we provide a link between the present shaping problem and those techniques.

The cost of the pth point along the ith dimension is considered as, $E_p(1 + \sum_l \xi_l \mathbf{L}_l[S_i])$ where E_p is the energy of the point and ξ_l's are the set of the Lagrange multipliers. For the half integer grid, we have, $E_p = (p + 0.5)^2$. The one-D points along each dimension are aggregated into K macro-shells with a fixed increment in cost, Δ. Assume that the costs of the one-D points are in the range $c \in [c_{\min}, c_{\max}]$. We set $\Delta = (c_{\max} - c_{\min})/K$. The costs of the points in the ith macro-shell satisfy, $c_{\min} \leq c \leq c_{\min} + \Delta$ for $i = 0$ and $c_{\min} + i\Delta < c \leq c_{\min} + (i+1)\Delta$ for $i = 1, \ldots, K-1$. The cost of a macro-shell is considered as the average cost of its points. Obviously, some of the macro-shells may remain empty.

The higher-dimensional macro-shells are the set of the high dimensional points with fixed sum of the indices. This results in a recursive merging rule for the macro-shells. There are $n(K-1)+1$ macro-shells in the n-D subspaces. The final constellation is selected as the set of the N-D macro-shells with the indices $0 \leq M \leq M_{\max}$ where $M_{\max}$ is selected such that the total cardinality is equal to T. The Lagrange multipliers are selected to satisfy the spectral constraints.

Addressing can be achieved using the decomposition methods discussed in [4]. The key point is that the addressing of a set which is equal to the cartesian product of two other sets can be achieved independently along those sets. Recursive structure of the macro-shells provides the required framework for such an addressing decomposition. In the present problem, it is more efficient to use a technique discussed in [4] which is based on a recursive aggregation of macro-shells of *equal cardinality*. This technique does not rely on any specific property of the shell indices.

3.2.1 Shaping performance

Consider a discrete set of points $\psi \in \Psi$. A cost $c(\psi)$ is assigned to each point $\psi \in \Psi$. The weight distribution of Ψ is defined as:

$$W_\Psi(q) = \sum_{\psi \in \Psi} q^{c(\psi)} = \sum_v C_\Psi(v) q^v, \qquad (17)$$

where $C_\Psi(v)$ is the number of points of Ψ of a cost v.

In most cases (including those discussed here), the cost in a cartesian product space has an additive property. In this case, it is easy to show that the weight distribution of the cartesian product of some sets is obtained by multiplying their weight distributions. This property is used in the following to compute the shaping performance.

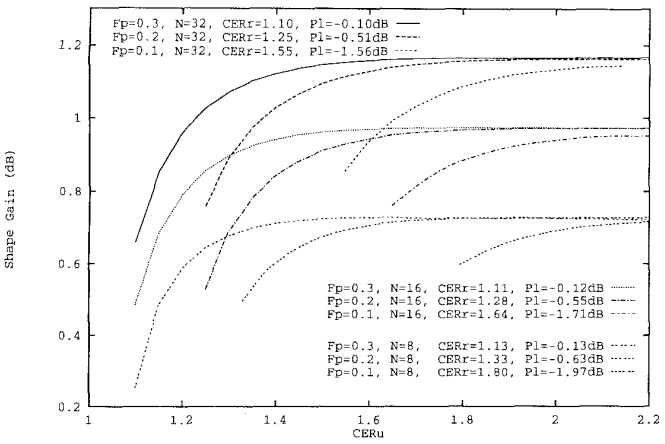

Figure 3: Examples of the optimum tradeoff curves in spaces of a finite dimensionality, sine basis, F_p-constraint with $f_c = 0.2$, $K = 256$, $\mathrm{CER}_u \geq \mathrm{CER}_r$. The constellation expansion due to the boundary shaping can be computed using $\mathrm{RCE} = \mathrm{CER}_u / \mathrm{CER}_r$.

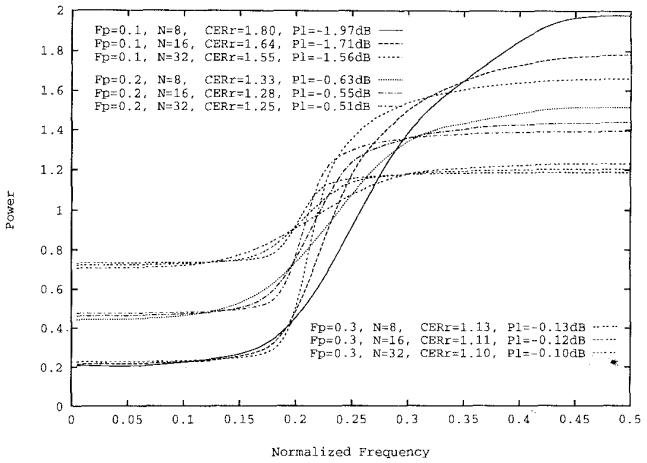

Figure 4: Examples of the power spectrums achieved in spaces of a finite dimensionality, sine basis, F_p-constraint with $f_c = 0.2$.

Assume that the cardinality of the jth macro-shell in the ith one-D subspace is equal to $C_i(j)$. By replacing q by a proper transform operator, one can relate (17) to the Discrete Fourier Transform (DFT) of the sequence $\mathbf{C}_i = [C_i(0), \ldots, C_i(K-1)]$. Using this fact, in conjunction with the multiplicativity property of the weight distributions, it is easy to show that the probability induced on a given macro-shell, say M, along a given dimension, say d, is equal to:

$$P_d(M) = \frac{1}{T} \sum_{m=0}^{M_{\max}-M} \mathrm{DFT}_m^{-1} \left[\prod_{i=0, i\neq d}^{N-1} \mathrm{DFT}(\mathbf{C}_i) \right], \quad (18)$$

where DFT is the discrete Fourier transform of length $N(K-1)+1$, multiplication of DFT's is achieved on an element-by-element basis, DFT_m^{-1} is the mth element of the corresponding inverse DFT and T is the cardinality of the constellation. The induced probabilities are used to compute the average energy along different dimensions. Knowing the average energies, computation of the shaping performance and the power spectrum is straight-forward. Figure (3) shows some examples of the corresponding optimum tradeoff curves subject to the F_p-constraint. Figure (4) shows the corresponding power spectrums.

Note: In the case of the F_p-constraint, the approach based on the Lagrange multipliers corresponds to using a rectangular window for giving different weights to the energy in different frequency bands. More generally, we can consider:

$$\mathbf{L}[S_i(\omega)] = \int_{-\pi}^{\pi} W_0(\omega) S_i(\omega) d\omega, \quad (19)$$

where $W_0(\omega)$ is an arbitrary weighting window. Spectral shaping is achieved by adjusting $W_0(\omega)$.

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