An Analytical Method for Performance Analysis of Binary Linear Block Codes

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Abstract — **An analytical method for performance evaluation of binary linear block codes using an Additive White Gaussian Noise (AWGN) channel model with Binary Phase Shift Keying (BPSK) modulation is presented. We focus on the probability distribution function (*pdf*) of the bit Log-Likelihood Ratio (LLR) which is expressed in terms of the Gram-Charlier series expansion. This expansion requires knowledge of the statistical moments of the bit LLR. We introduce an analytical method for calculating these moments. It is shown that the estimate of the bit error probability provided by the proposed method will asymptotically converge to the true bit error performance.**

I. Introduction

Assume that a binary linear code $\mathcal{C}$ with code-words of length N is given. We use notation $\mathbf{c}^i = (c_1^i, c_2^i, \ldots, c_N^i)$ to refer to a code-word and its elements. We partition the code into a sub-code C_k^0 and its coset C_k^1 according to the value of k^{th} bit position of its code-words $\mathbf{c}^i$.

$$\forall \mathbf{c}^i \in \mathcal{C}: \text{ if } c_k^i = 0 \Longrightarrow \mathbf{c}^i \in C_k^0$$
$$\text{if } c_k^i = 1 \Longrightarrow \mathbf{c}^i \in C_k^1$$
$$C_k^0 \cup C_k^1 = \mathcal{C}, \quad C_k^0 \cap C_k^1 = \varnothing$$

If we modulate a code-word $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_N)$ using BPSK modulation and send it through an AWGN channel we will receive $\mathbf{x} = \mathbf{m}(\tilde{\mathbf{c}}) + \mathbf{n}$, where $\mathbf{n} = (n_1, n_2, ..., n_N)$ is an independent, identically distributed Gaussian noise vector which has zero mean elements of variance σ^2. The Log-Likelihood-Ratio (LLR) of the k^{th} bit position is defined by the following equation,

$$LLR(k) = \log \frac{P(\tilde{c}_k = 1|\mathbf{x})}{P(\tilde{c}_k = 0|\mathbf{x})} = \log \frac{\sum_{\mathbf{c}^i \epsilon C_k^1} \exp\left[\frac{\mathbf{n}.\mathbf{m}(\mathbf{c}^i) + \mathbf{m}(\tilde{\mathbf{c}}).\mathbf{m}(\mathbf{c}^i)}{\sigma^2}\right]}{\sum_{\mathbf{c}^i \epsilon C_k^0} \exp\left[\frac{\mathbf{n}.\mathbf{m}(\mathbf{c}^i) + \mathbf{m}(\tilde{\mathbf{c}}).\mathbf{m}(\mathbf{c}^i)}{\sigma^2}\right]} \tag{1}$$

where $\tilde{c}_k$ is the value of k^{th} bit in the transmitted code-word.

II. Gram-Charlier Expansion of *pdf*

As the *pdf* of bit LLR is approximately Gaussian [2–4], the appropriate orthogonal basis can be normal Gaussian *pdf* and its derivatives. One can expand the *pdf* of random variable Y, $f_Y(y)$ using the following formula which is called the Gram-Charlier series expansion [5],

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \sum_{i=0}^{\infty} C_i T_i(y) \tag{2}$$

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where, $T_i(y)$ is Tchebychev-Hermite polynomial and C_i can be calculated using moments of Y [5]. We present an analytical method using Taylor series expansion of (1) to compute the statistical moments of the bit LLR.

III. Computing moments

The definition of m^{th} order moment is,

$$\mu_m = E[H^m(\mathbf{n})] \tag{3}$$

where $E[.]$ stands for expectation and $H(\mathbf{n})$ specifies $LLR(k)$ as a function of noise vector $\mathbf{n}$. To compute (3), we use the Taylor series expansion of $H^m(\mathbf{n})$ and average this expansion with respect to different components of $\mathbf{n}$. We present an analytical method to compute the desired moments. This is based on some straight-forward recursive calculations involving certain weight enumerating functions of the code. Finally replacing the results in (3) and integrating, we present a closed form expression for computing bit error performance. It is shown that the estimate of the bit error probability provided by the proposed method will asymptotically converge to the true bit error performance. As an example, we have applied the proposed method to compute the bit error performance of a (15,11) Cyclic code (refer to Fig.1).

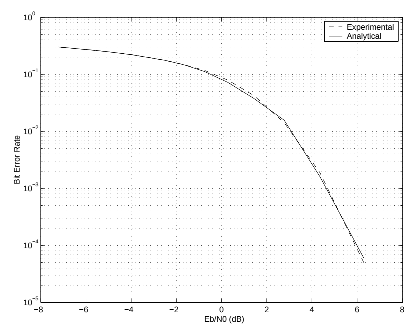

Figure 1: Comparison of analytical and experimental BER

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