Capacity Bounds for the Gaussian Interference Channel

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***Abstract*—The capacity region of the two-user Gaussian Interference Channel (IC) is studied. Two classes of channels are considered: weak and mixed Gaussian IC. For the weak Gaussian IC, a new outer bound on the capacity region is obtained that outperforms previously known outer bounds. The sum capacity for a certain range of channel parameters is derived. For this range, it is proved that using Gaussian codebooks and treating interference as noise is optimal. It is shown that when Gaussian codebooks are used, the full Han-Kobayashi (HK) achievable rate region can be obtained by using the naive HK achievable scheme over three frequency bands. For the mixed Gaussian IC, a new outer bound is obtained that outperforms previously known outer bounds. For this case, the sum capacity for the entire range of channel parameters is derived. It is proved that the full HK achievable rate region using Gaussian codebooks is equivalent to that of the one-sided Gaussian IC for a particular range of channel parameters.**

I. Introduction

In this paper, we focus on the two-user Gaussian IC which can be represented in standard form as [1]

$$\begin{aligned} y_1 &= x_1 + \sqrt{a}x_2 + z_1, \\ y_2 &= \sqrt{b}x_1 + x_2 + z_2, \end{aligned} \tag{1}$$

where x_i and y_i denote the input and output alphabets of User $i \in \{1,2\}$, respectively, and $z_1 \sim \mathcal{N}(0,1)$, $z_2 \sim \mathcal{N}(0,1)$ are standard Gaussian random variables. Constants $a \geq 0$ and $b \geq 0$ represent the gains of the interference links. Furthermore, Transmitter i, $i \in \{1,2\}$, is subject to the power constraint P_i. The capacity region of the Gaussian IC defined in [2] is denoted by $\mathscr{C}$.

Depending on the values of a and b, the two-user Gaussian IC is classified into weak, strong, mixed, one-sided, and degraded Gaussian IC. In Figure 1, regions in ab-plane together with their associated names are shown. We use $\gamma(x)$ as an abbreviation for the function $0.5\log_2(1+x)$.

II. Preliminaries

A. Support Functions

Throughout this paper, we use the following facts from convex analysis. There is a one to one correspondence between any closed convex set and its support function [3]. The support function of any set $D \in \Re^m$ is a function $\sigma_D : \Re^m \to \Re$ defined as

$$\sigma_D(\mathbf{c}) = \sup\{\mathbf{c}^t\mathbf{R} | \mathbf{R} \in D\}. \tag{2}$$

[1]**An earlier version of this work containing all the results is reported in *Library and Archives Canada Technical Report* *UW-ECE 2007-26, Aug. 2007* (see http://www.cst.uwaterloo.ca/pub_tech_rep.html for details).**

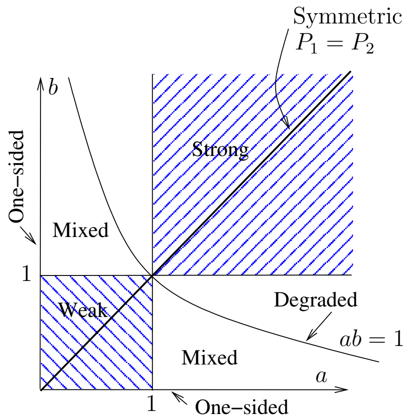

Fig. 1. Classes of the two-user ICs.

Clearly, if the set D is compact, then the sup is attained and can be replaced by max. In this case, the solutions of (2) correspond to the boundary points of D [3]. The following relation is the dual of (2) and holds when D is closed and convex

$$D = \{\mathbf{R} | \mathbf{c}^t\mathbf{R} \leq \sigma_D(\mathbf{c}), \forall\ \mathbf{c}\}. \tag{3}$$

For any two closed convex sets D and D', $D \subseteq D'$, if and only if $\sigma_D \leq \sigma_{D'}$.

B. Methods for Enlarging an Achievable Region

Assume an achievable scheme for an M-user channel with the power constraint $\mathbf{P} = [P_1, P_2, \ldots, P_M]$ is given and can be represented as

$$D_0(\mathbf{P}, \Theta) = \{\mathbf{R} | A\mathbf{R} \leq \Psi(\mathbf{P}, \Theta)\}, \tag{4}$$

where A is a $K \times M$ matrix and $\Theta \in [0,1]^M$. The support function of D_0 is a function of $\mathbf{P}$, Θ, and $\mathbf{c}$. Hence, we have

$$\sigma_{D_0}(\mathbf{c}, \mathbf{P}, \Theta) = \max\{\mathbf{c}^t\mathbf{R} | A\mathbf{R} \leq \Psi(\mathbf{P}, \Theta)\}. \tag{5}$$

For fixed $\mathbf{P}$ and Θ, (5) is a linear program. Using strong duality of linear programming, we obtain

$$\sigma_{D_0}(\mathbf{c}, \mathbf{P}, \Theta) = \min\{\mathbf{y}^t\Psi(\mathbf{P}, \Theta) | A^t\mathbf{y} = \mathbf{c}, \mathbf{y} \geq 0\}. \tag{6}$$

In general, $\hat{\mathbf{y}}$, the minimizer of (6), is a function of $\mathbf{P}$, Θ, and $\mathbf{c}$. We say D_0 possesses *the unique minimizer property* if $\hat{\mathbf{y}}$ merely depends on $\mathbf{c}$, for all $\mathbf{c}$. In this case, we have

$$\sigma_{D_0}(\mathbf{c}, \mathbf{P}, \Theta) = \hat{\mathbf{y}}^t(\mathbf{c})\Psi(\mathbf{P}, \Theta), \tag{7}$$

where $A^t\hat{\mathbf{y}} = \mathbf{c}$. This condition means that for any $\mathbf{c}$ the extreme point of D_0 maximizing the objective $\mathbf{c}^t\mathbf{R}$ is an extreme point obtained by intersecting a set of specific hyperplanes. A necessary condition for D_0 to possess the unique minimizer property is that each inequality in describing D_0 is either redundant or active for all $\mathbf{P}$ and Θ.

Since D_0 is a convex region, the convex hull operation does not lead to a new enlarged region. However, if the extreme points of the region are not a concave function of $\mathbf{P}$, it is possible to enlarge D_0 by using two different methods which are explained next. The first method is based on using the time sharing parameter. Let us denote the corresponding region as D which can be written as

$$D = \{\mathbf{R}|A\mathbf{R} \leq \sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i, \Theta_i), \sum_{i=1}^{q} \lambda_i \mathbf{P}_i \leq \mathbf{P}, \sum_{i=1}^{q} \lambda_i = 1, \lambda_i \geq 0, \Theta_i \in [0,1]^M \ \forall i\}, \quad (8)$$

where $q \in \mathbb{N}$. In the second method, we make use of TD/FD to enlarge the achievable rate region. This results in an achievable region D_2 represented as

$$D_2 = \{\mathbf{R} = \sum_{i=1}^{q'} \lambda_i \mathbf{R}_i | A\mathbf{R_i} \leq \Psi(\mathbf{P}_i, \Theta_i), \sum_{i=1}^{q'} \lambda_i \mathbf{P}_i \leq \mathbf{P}, \sum_{i=1}^{q'} \lambda_i = 1, \lambda_i \geq 0, \Theta_i \in [0,1]^M \ \forall i\}, \quad (9)$$

where $q' \in \mathbb{N}$. We refer to this method as concavification. It can be readily shown that D and D_2 are closed and convex, and $D_2 \subseteq D$. We are interested in situations where the inverse inclusion holds.

C. Admissible Channels

Definition 1 (Admissible Channel): An IC $\mathscr{C}'$ with input letter x_i and output letter $\tilde{y}_i$ for User $i \in \{1, 2\}$ is an admissible channel for the two-user Gaussian IC if there exist two deterministic functions $\hat{y}_1^n = f_1(\tilde{y}_1^n)$ and $\hat{y}_2^n = f_2(\tilde{y}_2^n)$ such that

$$I(x_1^n; y_1^n) \leq I(x_1^n; \hat{y}_1^n), \quad (10)$$
$$I(x_2^n; y_2^n) \leq I(x_2^n; \hat{y}_2^n) \quad (11)$$

hold for all $p(x_1^n)p(x_2^n)$ and for all $n \in \mathbb{N}$. $\mathscr{E}$ denotes the collection of all admissible channels.

It is easy to show that $\mathscr{C} \subseteq \mathscr{C}'$ for all admissible channels $\mathscr{C}'$. Hence, to obtain an outer bound, we need to find the intersection of the capacity regions of all admissible channels. Nonetheless, it may happen that finding the capacity region of an admissible channel is as hard as that of the original one (in fact, based on the definition, the channel itself is one of its admissible channels). Hence, we need to find classes of admissible channels, say $\mathscr{F}$, which possess two important properties. First, their capacity regions are close to $\mathscr{C}$. Second, either their exact capacity regions are computable or there exist good outer bounds for them.

Having $\mathscr{F}$, we can obtain an upper bound on the support function of $\mathscr{C}$ by solving the following optimization problems for all $1 \leq \mu$:

$$\sigma_{\mathscr{C}}(\mu, 1) \leq \min_{\mathscr{C}' \in \mathscr{F}} \max_{(R_1,R_2)\in\mathscr{C}'} \mu R_1 + R_2, \quad (12)$$

$$\sigma_{\mathscr{C}}(1, \mu) \leq \min_{\mathscr{C}' \in \mathscr{F}} \max_{(R_1,R_2)\in\mathscr{C}'} R_1 + \mu R_2. \quad (13)$$

III. Main Results and Related Works

In this section, we provide the summary of the results obtained for the weak and mixed Gaussian IC. The results are categorized in three subsections: new outer bounds, sum capacities, and the HK inner bound. In the following section, we provide proofs for the sum capacity results. To see the rest of the proofs, interested readers are referred to [4].

A. New Outer Bounds

For the weak Gaussian IC, there are two outer bounds that are tighter than the other known bounds. The first one, due to Kramer [5], is obtained by relying on the fact that the capacity region of the Gaussian IC is inside the capacity regions of the two underlying one-sided Gaussian ICs. Even though the capacity region of the one-sided Gaussian IC is unknown, there exists an outer bound for this channel that can be used instead.

The second outer bound, due to Etkin *et al.* [6], is obtained by using Genie aided technique to upper bound different linear combinations of rates that appear in the HK achievable region [7].

To obtain a new outer bound, we derive an upper bound on all linear combinations of the rates. To this end, we introduce three classes of admissible channels, namely Class A1, Class A2, and Class B (see [4]). Using these classes, we obtain two functions $W(\mu_1)$ and $\tilde{W}(\mu_2)$ that upper bound $\sigma_{\mathscr{C}}(\mu_1, 1)$ and $\sigma_{\mathscr{C}}(1, \mu_2)$, respectively.

Theorem 1 (New Outer Bound): For any rate pair (R_1, R_2) achievable for the two-user weak Gaussian IC, the inequalities

$$\mu_1 R_1 + R_2 \leq W(\mu_1), \quad (14)$$
$$R_1 + \mu_2 R_2 \leq \tilde{W}(\mu_2), \quad (15)$$

hold for all $1 \leq \mu_1, \mu_2$.

For the mixed Gaussian IC, the best outer bound to date, due to Etkin *et al.* [6], is obtained by using the Genie aided technique. The capacity region of the mixed Gaussian IC is inside the intersection of the capacity regions of the two underlying one-sided Gaussian ICs. Removing the link between Transmitter 1 and Receiver 2 results in a weak one-sided Gaussian IC whose outer bound E_1 is the collection of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma\left(\frac{(1-\beta)P'}{\beta P' + 1/a}\right), \quad (16)$$
$$R_2 \leq \gamma(\beta P'), \quad (17)$$

for all $\beta \in [0, \beta_{\max}]$, where $P' = P_1/a + P_2$ and $\beta_{\max} = \frac{P_2}{P'(1+P_1)}$. On the other hand, removing the link between Transmitter 2 and Receiver 1 results in a strong one-sided

Gaussian IC whose capacity region E_2 is fully characterized as the collection of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma(bP_1), \quad (18)$$

$$R_2 \leq \gamma(P_2), \quad (19)$$

$$R_1 + R_2 \leq \gamma(bP_1 + P_2). \quad (20)$$

Moreover, by introducing a new class of admissible channels, which is called Class C [4], we can obtain a function $W_{\text{mixed}}(\mu)$ that upper bounds $\sigma_{\mathscr{C}}(\mu, 1)$. Hence, we have the following theorem that provides an outer bound on the capacity region of the mixed Gaussian IC.

Theorem 2: For any rate pair (R_1, R_2) achievable for the two-user mixed Gaussian IC, $(R_1, R_2) \in E_1 \bigcap E_2$. Moreover, the inequality

$$\mu R_1 + R_2 \leq W_{\text{mixed}}(\mu) \quad (21)$$

holds for all $1 \leq \mu$.

B. Sum Capacities

Theorem 3: The sum capacity of the two-user Gaussian IC is

$$\mathscr{C}_{sum} = \gamma\left(\frac{P_1}{1 + aP_2}\right) + \gamma\left(\frac{P_2}{1 + bP_1}\right), \quad (22)$$

for the range of parameters satisfying

$$\sqrt{b}P_1 + \sqrt{a}P_2 \leq \frac{1 - \sqrt{a} - \sqrt{b}}{\sqrt{ab}}. \quad (23)$$

Remark 1: The above sum capacity result for the weak Gaussian IC (see also [4]) has been established independently in [8] and [9].

Theorem 4: The sum capacity of the mixed Gaussian IC with $a < 1$ and $b \geq 1$ can be stated as

$$\mathscr{C}_{sum} = \gamma(P_2) + \min\left\{\gamma\left(\frac{P_1}{1 + aP_2}\right), \gamma\left(\frac{bP_1}{1 + P_2}\right)\right\}. \quad (24)$$

Remark 2: In an independent work [8], the sum capacity of the mixed Gaussian IC is obtained for a certain range of parameters, whereas in the above theorem, we characterize the sum capacity of this channel for the entire range of its parameters (see also [4]).

C. Han Kobayashi Inner Bound

The following theorems concern about D and D_2 and their properties.

Theorem 5: If D_0 possesses the unique minimizer property, then $D = D_2$.

Theorem 6: The cardinality of the time sharing parameter q in (8) is less than $M + K + 1$, where M and K are the dimensions of $\mathbf{P}$ and $\Psi(\mathbf{P})$, respectively. Moreover, if $\Psi(\mathbf{P})$ is a continuous function of $\mathbf{P}$, then $q \leq M + K$.

Theorem 7: To characterize boundary points of D_2, it suffices to set $q' \leq M + 1$.

Surprising fact about Theorem 7 is that the upper bound for q' is independent of the number of inequalities in the description of the achievable rate region.

The best inner bound for the two-user Gaussian IC is the full HK achievable region denoted by $\mathscr{C}_{HK}$ [7]. Despite having a single letter formula, $\mathscr{C}_{HK}$ is not fully characterized yet. In fact, finding the optimum distributions achieving boundary points of $\mathscr{C}_{HK}$ is still an open problem. We define $\mathscr{G}_0$ as the naive HK achievable region (we use a shorter description of $\mathscr{G}_0$ obtained in [10]). $\mathscr{G}_0$ can be represented in a matrix form as $\mathscr{G}_0 = \{\mathbf{R} | A\mathbf{R} \leq \Psi(P_1, P_2, \alpha, \beta)\}$ (see [4]. Now, $\mathscr{G}_0$ can be enlarged by using time-sharing and concavification. Let us define these two regions as $\mathscr{G}$ and $\mathscr{G}_2$. Clearly, the chain of inclusions $\mathscr{G}_0 \subseteq \mathscr{G}_2 \subseteq \mathscr{G} \subseteq \mathscr{C}_{HK} \subseteq \mathscr{C}$ always holds. Now, we can state the main theorem of this subsection which comes from the fact that $\mathscr{G}_0$ possesses the unique minimizer property.

Theorem 8: For the two-user weak Gaussian IC, time-sharing and concavification result in the same region. In other words, $\mathscr{G}$ can be fully characterized by using TD/FD and allocating power over three different dimensions.

Theorem 9: For the mixed Gaussian IC satisfying $1 \leq ab$, region $\mathscr{G}$ is equivalent to that of the one sided Gaussian IC obtained from removing the stronger interfering link.

IV. Proofs for the Sum Capacity Results

A. Sum Capacity Result for the Weak Gaussian IC

To obtain the sum capacity result in Theorem 3, a class of admissible channel, say Class B, is defined as the set of channels modeled by

$$\begin{aligned} \tilde{y}_{11} &= x_1 + z_{11}, \\ \tilde{y}_{12} &= x_1 + \sqrt{a'}x_2 + z_{12}, \\ \tilde{y}_{21} &= x_2 + \sqrt{b'}x_1 + z_{21}, \\ \tilde{y}_{22} &= x_2 + z_{22}, \end{aligned} \quad (25)$$

where $\tilde{y}_{11}$ and $\tilde{y}_{12}$ are the signals at the first receiver, $\tilde{y}_{21}$ and $\tilde{y}_{22}$ are the signals at the second receiver, and z_{ij} is additive Gaussian noise with variance N_{ij} for $i, j \in \{1, 2\}$. Transmitter 1 and 2 are subject to the power constraints P_1 and P_2, respectively.

Let us consider two linear deterministic functions f_1 and f_2 with parameters $0 \leq g_1$ and $0 \leq g_2$, resp., as follows

$$f_1(\tilde{y}_{11}^n, \tilde{y}_{12}^n) = (1 - \sqrt{g_1})\tilde{y}_{11}^n + \sqrt{g_1}\tilde{y}_{12}^n, \quad (26)$$

$$f_2(\tilde{y}_{22}^n, \tilde{y}_{21}^n) = (1 - \sqrt{g_2})\tilde{y}_{22}^n + \sqrt{g_2}\tilde{y}_{21}^n. \quad (27)$$

To satisfy (10) and (11), it suffices to have

$$\begin{aligned} a'g_1 &= a, \\ b'g_2 &= b, \\ (1 - \sqrt{g_1})^2 N_{11} + g_1 N_{12} &= 1, \\ (1 - \sqrt{g_2})^2 N_{22} + g_2 N_{21} &= 1. \end{aligned} \quad (28)$$

We further add the following two constraints to the equality conditions in (28):

$$\begin{aligned} b'N_{11} &\leq N_{21}, \\ a'N_{22} &\leq N_{12}. \end{aligned} \quad (29)$$

Although adding more constraints reduces the number of the admissible channels, it helps us to prove the following theorem which is first observed by Etkin *et al.* in [6].

Theorem 10: The sum capacities of channels in Class B are attained when transmitters use Gaussian codebooks and receivers treat the interference as noise. In this case, the sum capacity is

$$\mathscr{C}'_{\text{sum}} = \gamma\left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) + \gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right).$$

Proof: See [4] for the proof. □

By using Theorem 10, we introduce the following optimization problem that upper bound the sum capacity of the weak Gaussian IC.

$$\begin{aligned} W = &\min \gamma\left(\frac{(1-\sqrt{g_1})^2 P_1}{1-S_1} + \frac{g_1 P_1}{aP_2 + S_1}\right) \\ &+ \gamma\left(\frac{(1-\sqrt{g_2})^2 P_2}{1-S_2} + \frac{g_2 P_2}{bP_1 + S_2}\right) \\ &\text{subject to: } \frac{b(1-S_1)}{(1-\sqrt{g_1})^2} \leq S_2 < 1 \\ &\frac{a(1-S_2)}{(1-\sqrt{g_2})^2} \leq S_1 < 1 \\ &0 < [g_1, g_2]. \end{aligned} \tag{30}$$

where $S_1 = g_1 N_{12}$ and $S_2 = g_2 N_{21}$. By first minimizing with respect to g_1 and g_2, the optimization problem (30) can be decomposed as

$$\begin{aligned} W = &\min W_1 + W_2 \\ &\text{subject to: } 0 < S_1 < 1,\ 0 < S_2 < 1. \end{aligned} \tag{31}$$

where W_1 is defined as

$$\begin{aligned} W_1 = &\min_{g_1} \gamma\left(\frac{(1-\sqrt{g_1})^2 P_1}{1-S_1} + \frac{g_1 P_1}{aP_2 + S_1}\right) \\ &\text{subject to: } \frac{b(1-S_1)}{S_2} \leq (1-\sqrt{g_1})^2,\ 0 < g_1. \end{aligned} \tag{32}$$

Similarly, W_2 is defined as

$$\begin{aligned} W_2 = &\min_{g_2} \gamma\left(\frac{(1-\sqrt{g_2})^2 P_2}{1-S_2} + \frac{g_2 P_2}{bP_1 + S_2}\right) \\ &\text{subject to: } \frac{a(1-S_2)}{S_1} \leq (1-\sqrt{g_2})^2,\ 0 < g_2. \end{aligned} \tag{33}$$

It is easy to show that if $\sqrt{b}(1+aP_2) \leq \sqrt{S_2(1-S_1)}$ and $\sqrt{a}(1+bP_1) \leq \sqrt{S_1(1-S_2)}$ then

$$W = \gamma\left(\frac{P_1}{1+aP_2}\right) + \gamma\left(\frac{P_2}{1+bP_1}\right), \tag{34}$$

which is achievable by treating interference as noise. In what follows, we prove that it is possible to find appropriate S_1 and S_2 for a certain range of parameters.

Let us fix a and b, and define D as

$$\begin{aligned} D = \{(P_1, P_2) | P_1 \leq \frac{\sqrt{S_1(1-S_2)}}{b\sqrt{a}} - \frac{1}{b}, , 0 < S1, S_2 < 1, \\ P_2 \leq \frac{\sqrt{S_2(1-S_1)}}{a\sqrt{b}} - \frac{1}{a}\}. \end{aligned} \tag{35}$$

In fact, if D is feasible then there exist a, b, P_1, and P_2 such that the sum capacity of the channel is attained by treating interference as noise. We claim that $D = D'$, where D' is defined as

$$D' = \left\{(P_1, P_2) | \sqrt{b}P_1 + \sqrt{a}P_2 \leq \frac{1-\sqrt{a}-\sqrt{b}}{\sqrt{ab}}\right\}. \tag{36}$$

To show $D' \subseteq D$, we set $S_1 = 1 - S_2$ in (35) to obtain

$$\left\{(P_1, P_2) | P_1 \leq \frac{S_1}{b\sqrt{a}} - \frac{1}{b}, P_2 \leq \frac{1-S_1}{a\sqrt{b}} - \frac{1}{a}, 0 < S_1 < 1\right\}$$

It is easy to show that the above set is another representation of the region D'. Hence, we have $D' \subseteq D$.

To show $D \subseteq D'$, it suffices to prove that for any $(P_1, P_2) \in D$, $\sqrt{b}P_1 + \sqrt{a}P_2 \leq \frac{1-\sqrt{a}-\sqrt{b}}{\sqrt{ab}}$ holds. To this end, we introduce the following maximization problem

$$J = \max_{(P_1, P_2) \in D} \sqrt{b}P_1 + \sqrt{a}P_2, \tag{37}$$

which can be written as

$$J = \max_{(S_1, S_2) \in (0,1)^2} \frac{\sqrt{S_1(1-S_2)} + \sqrt{S_2(1-S_1)}}{\sqrt{ab}} - \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}.$$

It is easy to show that the solution to the above optimization problem is

$$J = \frac{1}{\sqrt{ab}} - \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}. \tag{38}$$

Hence, we deduce that $D \subseteq D'$. This completes the sum capacity result in Theorem 3.

As an example, let us consider the symmetric Gaussian IC. In this case, the constraint in (23) becomes

$$P \leq \frac{1-2\sqrt{a}}{2a\sqrt{a}}. \tag{39}$$

For a fixed P and all $0 \leq a \leq 1$, the upper bound in (30) and the lower bound when receivers treat the interference as noise are plotted in Figure 2. We observe that up to a certain value of a, the upper bound coincides with the lower bound.

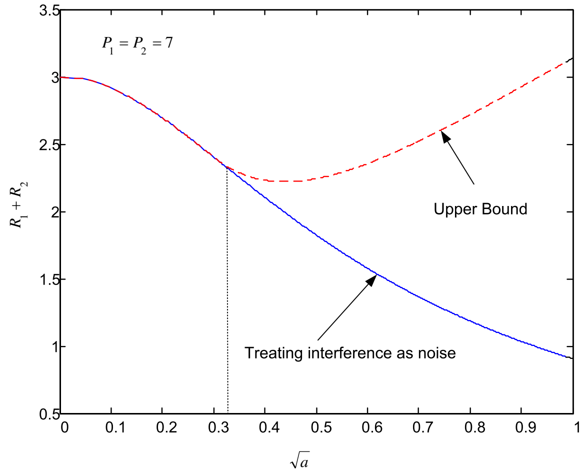

Fig. 2. The upper bound obtained by solving (30). The lower bound is obtained by using the simple scheme of considering the interference as Gaussian noise.

B. Sum capacity of the Mixed Gaussian IC

In this subsection, we prove Theorem 4. To this end, we need to prove the achievaility and converse for the theorem.

Achievability part: Transmitter 1 sends a common message to both receivers, while the first user's signal is considered as noise at both receivers. In this case, the rate

$$R_1 = \min\left\{\gamma\left(\frac{P_1}{1+aP_2}\right), \gamma\left(\frac{bP_1}{1+P_2}\right)\right\} \quad (40)$$

is achievable. At Receiver 2, the signal from Transmitter 1 can be decoded and removed. Therefore, User 2 is left with a channel without interference and it can communicate at its maximum rate which is

$$R_2 = \gamma(P_2). \quad (41)$$

By adding (40) and (41), we obtain the desired result.

Converse part: From Theorem 2, we know that the capacity region of the Gaussian mixed IC is inside the intersection of E_1 and E_2. Hence, we can obtain two upper bounds on the sum rate. By using E_1, we have

$$\mathscr{C}_{sum} \leq \gamma(P_2) + \gamma\left(\frac{P_1}{1+aP_2}\right). \quad (42)$$

By using E_2, we have

$$\mathscr{C}_{sum} \leq \gamma\left(bP_1 + P_2\right), \quad (43)$$

which equivalently can be written as

$$\mathscr{C}_{sum} \leq \gamma(P_2) + \gamma\left(\frac{bP_1}{1+P_2}\right). \quad (44)$$

By taking the minimum of the right hand sides of (42) and (44), we obtain

$$\mathscr{C}_{sum} \leq \gamma\left(P_2\right) + \min\left\{\gamma\left(\frac{P_1}{1+aP_2}\right), \gamma\left(\frac{bP_1}{1+P_2}\right)\right\}. \quad (45)$$

This completes the proof.

V. Simulation Results

In Figure 3, different bounds for the symmetric weak Gaussian IC are plotted. As depicted in figures, the new outer bound is tighter than previously known outer bounds.

Different bounds are compared for the mixed Gaussian ICs for $P_1 = 7$, $P_2 = 7$, $a = 0.4$, and $b = 1.5$ in Figure 4. As it can be seen and proved, the inner bound and the outer bound coincide on a facet for this case. It is important to note that, surprisingly, this facet is obtainable when the second transmitter uses both the common message and the private message.

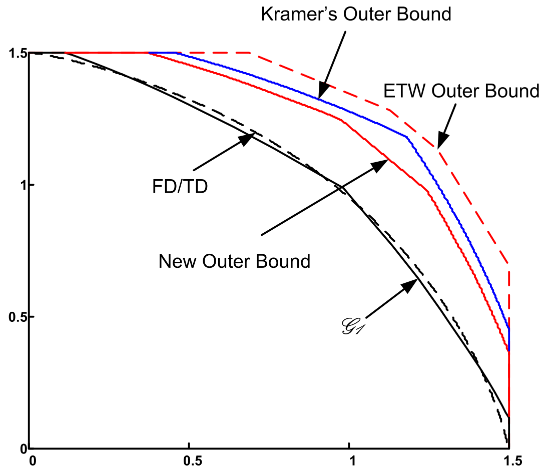

Fig. 3. Comparison between different bounds for the symmetric weak Gaussian IC when $P = 7$ and $a = 0.2$

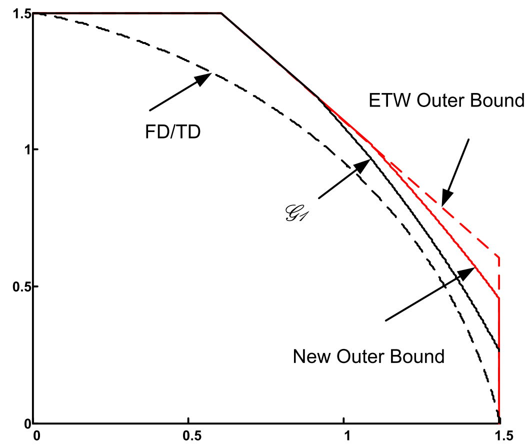

Fig. 4. Comparison between different bounds for the mixed Gaussian IC when $1 + P_2 > b + abP_2$ and $1 - a \leq abP_1$ (Case II) for $P_1 = 7$, $P_2 = 7$, $a = 0.4$, and $b = 1.5$.

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