Characterization of Rate Region in Interference Channels with Constrained Power

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***Abstract*— In this paper, an n-user Gaussian interference channel under arbitrary linear power constraints is considered. Using Perron-Frobenius theorem, a closed-form expression for the boundary points of the rate region of such a channel is derived. This is a generalization of the well-known result on the maximum rate that some interfering links can simultaneously achieve when the power is unbounded. Moreover, this result is extended to the time-varying channels with constraints on the average power.**

I. INTRODUCTION

Channel sharing is known as an efficient scheme to increase the spectral efficiency of the wireless systems. While such a scheme increases the capacity and the coverage area of systems, it suffers from the interference among the concurrent links (co-channel interference). Consequently, the signal-to-interference-plus-noise-ratio (SINR) of the links are upper-bounded, even if there is no constraint on the transmit powers.

There have been some efforts to evaluate the maximum achievable SINR in the interference channels. In [1], the maximum achievable SINR of a system with no constraint on the power is expressed in terms of the Perron-Frobenius (PF) eigenvalue of a non-negative matrix. Then, this expression is utilized to develop an SINR-balancing scheme for satellite networks. This formulation for the maximum achievable SINR is deployed in many other wireless communication applications such as [2], [3] afterwards. In [4], the maximum achievable SINR is obtained based on the PF-eigenvalue of an $(n+1)\times(n+1)$ primitive non-negative matrix, when the *total power* is constrained.

Recently, the rate region of interference channels and its properties has been investigated in the literature. In [5], it is shown that the capacity region when the power is unbounded is convex. The capacity region in [5] is defined as the set of feasible processing gains while for a constant bandwidth, the processing gain is inversely proportional to the rate. In [6], some topological properties of the capacity region (with the aforementioned definition) of CDMA systems are investigated for the cases when there are constraints on the power of individual users and when there is no constraint on the power. The authors in [6] show that the boundary of the capacity region with one user's power fixed and the rest unbounded is a shift of the boundary of some capacity region with modified parameters, but unlimited power. However, this result is not in a closed form and cannot be extended for the other forms of power constraints.

It is shown that the feasible SINR region is not convex, in general [7]–[9]. In [10], it is shown that in the case of unlimited power, the feasible SINR region is log-convex. The authors in [5] also consider a CDMA system without power constraints, and show that the feasible inverse-SINR region is a convex set. In [7], it is proved that the feasible QoS region is a convex set, if the SINR is a log-convex function of the corresponding QoS parameter. Reference [11] shows that under a total power constraint, the infeasible SINR region is not convex.

In this paper, we extend the result on the maximum achievable SINR in [1] to the systems with certain constraints on the power of transmitters. This result yields a closed-form solution for the rate region of the systems with constraints on the power, in terms of the PF-eigenvalue of an $n\times n$ irreducible matrix. The approach that we use is more general as compared to the scheme used in [4] in the sense that it is easily applied to the systems with different power constrains. In addition, the resulting closed-form solution has a direct relationship with the solution for the systems with unbounded power presented in [1]. This result is extended to a time-varying system, where the channel gain is selected from a limited-cardinality set, and the average power of users are subject to some upper-bounds.

Notation: All boldface letters indicate column vectors (lower case) or matrices (upper case). x_{ij} and $\mathbf{x}_i$ represent the entry (i,j) and column i of the matrix $\mathbf{X}$, respectively. A matrix $\mathbf{X}_{n\times m}$ is called *non-negative* if $x_{ij}\geq 0 \quad \forall i,j\in\{1,\dots,n\}$, and denoted by $\mathbf{X}\geq\mathbf{0}$. Also, we have $\mathbf{X}\geq\mathbf{Y}\iff\mathbf{X}-\mathbf{Y}\geq\mathbf{0}$, where $\mathbf{X},\mathbf{Y}$ and $\mathbf{0}$ are non-negative matrices of compatible dimensions [12]. $\det(\mathbf{X})$, $\mathrm{Tr}(\mathbf{X})$, $\mathbf{X}'$, and $|\mathbf{X}|$ denote the determinant, the trace, the transpose, and the norm of the matrix $\mathbf{X}$, respectively. $\mathbf{I}$ is an identity matrix with compatible size. $\otimes$ represents the Kronecker product operator. $\mathrm{diag}(\mathbf{x})$ is a diagonal matrix whose main diagonal is $\mathbf{x}$. We define the reciprocal of polynomial $\mathrm{q}(x)$ of degree m as $x^m\mathrm{q}(\frac{1}{x})$. $\psi(\mathbf{X},\mathbf{y},\mathcal{S})$ is a matrix defined as a function of three parameters, which are respectively a matrix, a vector

[1]Financial support provided by Nortel and the corresponding matching funds by the Natural Sciences and Engineering Research Council of Canada (NSERC), and Ontario Centres of Excellence (OCE) are gratefully acknowledged.

and a set of indices,

$$\psi(\mathbf{X},\mathbf{y},\mathcal{S})=\mathbf{Z}=[\mathbf{z}_j],\quad \mathbf{z}_j=\begin{cases}\mathbf{x}_j+\mathbf{y} & j\in\mathcal{S}\\ \mathbf{x}_j & \text{otherwise}\end{cases}$$

II. Problem Formulation

A Gaussian interference channel, including n links (users), is represented by the gain matrix $\mathbf{G}=[g_{ij}]_{n\times n}$ where g_{ij} is the attenuation of the power from transmitter j to receiver i. This attenuation can be the result of fading, shadowing, or the processing gain of the CDMA system. A white Gaussian noise with zero mean and variance σ_i^2 is added to each signal at the receiver i terminal. In many applications, the QoS of the system is measured by an increasing function of SINR. In an interference channel, SINR of each user, denoted by γ_i, is

$$\gamma_i=\frac{g_{ii}p_i}{\sigma_i^2+\sum_{\substack{j=1\\ j\neq i}}^{n}g_{ij}p_j},\quad \forall i\in\{1,\ldots,n\},$$

where p_i is the power of transmitter i. In addition, in practice, the power vector $\mathbf{p}$ is subject to a set of constraints. The main goal is to find the maximum SINR which can be obtained by all users in the presence of such constraints. To this end, we solve the following optimization problem

$$\max\gamma \tag{1}$$
$$\text{s.t.}\quad \gamma_i\geq\mu_i\gamma \tag{2}$$
$$\mathbf{p}\geq\mathbf{0} \tag{3}$$
$$\sum_{i\in\Omega}p_i\leq\overline{p}_\Omega, \tag{4}$$

where $\Omega\subseteq\{1,\ldots,n\}$ with k elements and $\boldsymbol{\mu}$ is a given vector with $\mu_i\geq 0$ and $|\boldsymbol{\mu}|=1$. As we will see, the solution can be easily extended for the case of multiple power constraints of the form $\sum_{i\in\Omega}p_i\leq\overline{p}_\Omega$ for different $\Omega\subseteq\{1,\ldots n\}$. $\boldsymbol{\mu}$ provides the flexibility of satisfying different rate services for different users. According to Fig. 1, the solution of (1) yields the maximum achievable SINR in the direction of vector $\boldsymbol{\mu}$. Although the numerical solution of this problem is already obtained through geometric programming [13], [14], we propose a different approach which leads to a closed-form result.

By defining the normalized gain matrix $\mathbf{A}$ as

$$\mathbf{A}=[a_{ij}]_{n\times n},\quad a_{ij}=\begin{cases}\dfrac{g_{ij}}{g_{ii}} & i\neq j\\ 0 & i=j\end{cases}$$

the constraint (2) is rewritten as

$$\frac{p_i}{\eta_i+\sum_{j=1}^{n}\mu_i a_{ij}p_j}\geq\gamma,\quad\forall i\in\{1,\ldots,n\}, \tag{5}$$

where

$$\eta_i=\frac{\mu_i\sigma_i^2}{g_{ii}},\quad \boldsymbol{\eta}=[\eta_i]_{n\times 1}. \tag{6}$$

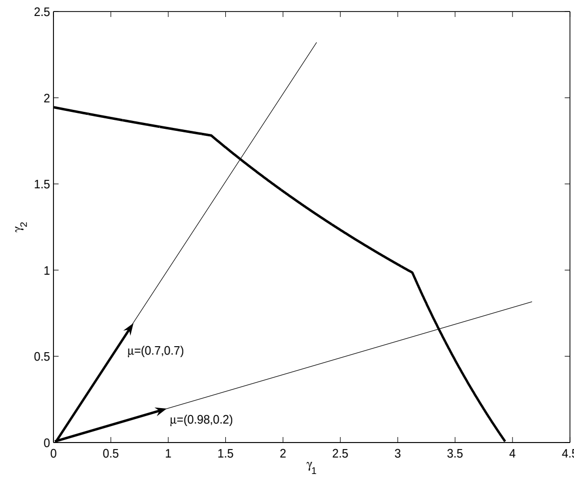

Fig. 1. The boundary of SINR Region for an interference channel with 2 users

Since we are interested in maximizing the minimum SINR, if SINR of one user is more than that of the others, it can reduce its power to other users' advantage, and finally the minimum SINR is improved. Therefore, equality holds in (5). After reformulating the problem in a matrix form we will have

$$\Big(\frac{1}{\gamma}\mathbf{I}-\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\Big)\mathbf{p}=\boldsymbol{\eta}. \tag{7}$$

The objective is to find the maximum γ while the system of linear equations in (7) yields a power satisfying the constraints on the power vector (3), (4).

When there is no constraint on the power vector (rather than trivial constraint of $\mathbf{p}\geq\mathbf{0}$), the maximum achievable SINR, γ^*, is characterized based on the Perron-Frobenius theorem as

$$\gamma^*=\frac{1}{\lambda^*\big(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big)}. \tag{8}$$

where λ^* is the Perron-Frobenius eigenvalue of the associated matrix [12]. This result was deployed in the communication systems for SINR-balancing ($\mu_1=\mu_2=\ldots=\mu_n$) in [1] for the first time.

We find the maximum achievable SINR, considering certain upper-bounds on the power of transmitters in the following sections.

III. SINR Region Characterization

We define $\mathbf{F}$ as

$$\mathbf{F}=\mathbf{I}-\gamma\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}. \tag{9}$$

Then, the system of linear equations in (7) is reformulated as

$$\mathbf{F}\mathbf{p}=\gamma\boldsymbol{\eta}, \tag{10}$$

where $\boldsymbol{\eta}$ is defined in (6). According to the Cramer's rule, the solution to (10) is obtained by

$$p_i=\frac{\det(\mathbf{H}^{(i)})}{\det(\mathbf{F})},$$

where $\mathbf{H}^{(i)} = [\mathbf{h}_j^{(i)}]_{n\times n}, \quad \mathbf{h}_j^{(i)} = \begin{cases} \gamma\boldsymbol{\eta} & j = i \\ \mathbf{f}_j & j \neq i \end{cases}$.
Defining $\mathrm{h}^{(i)}(\gamma) = \det(\mathbf{H}^{(i)})$ and $\mathrm{f}(\gamma) = \det(\mathbf{F})$, we have

$$p_i = \frac{\mathrm{h}^{(i)}(\gamma)}{\mathrm{f}(\gamma)}.$$

Therefore, the constraint in (4) can be written as

$$\frac{\sum_{i\in\Omega} \mathrm{h}^{(i)}(\gamma)}{\mathrm{f}(\gamma)} \leq \overline{p}_\Omega. \tag{11}$$

Defining $\mathrm{u}_\Omega(\gamma) = \overline{p}_\Omega \mathrm{f}(\gamma) - \sum_{i\in\Omega} \mathrm{h}^{(i)}(\gamma)$, the inequality (11) is equivalent to

$$\frac{\mathrm{u}_\Omega(\gamma)}{\mathrm{f}(\gamma)} \geq 0. \tag{12}$$

We desire to find the largest possible interval where both the numerator and the denominator have the same sign. It can be shown that this interval is connected and adjacent to zero. Apparently, $\mathrm{u}_\Omega(0) > 0$, and $\mathrm{f}(0) > 0$. Consequently,

$$\exists \quad \epsilon > 0 \quad : \quad \mathrm{f}(\epsilon) > 0 \quad \text{and} \quad \mathrm{u}_\Omega(\epsilon) > 0.$$

Therefore, both the numerator and the denominator are positive in the positive neighborhood of zero. For satisfying (12), we have to find the smallest positive real simple root of the numerator and the denominator, $r(\mathrm{u}_\Omega)$ and $r(\mathrm{f})$, and take the minimum of the two as

$$\hat{\gamma} = \min\{r(\mathrm{f}), r(\mathrm{u}_\Omega)\}. \tag{13}$$

For the sake of simplicity, without loss of generality, we assume that $\Omega = \{1, \ldots, k\}$, $k \leq n$, i.e., the first k users are subject to the total power constraint. For the numerator we have

$$\begin{aligned} \mathrm{u}_\Omega(\gamma) &= \overline{p}_\Omega \det(\mathbf{F}) - \sum_{i=1}^{k} \det(\mathbf{H}^{(i)}) \\ &= \overline{p}_\Omega \Big(\det(\mathbf{F}) - \sum_{i=1}^{k} \det(\hat{\mathbf{H}}^{(i)}) \Big), \end{aligned} \tag{14}$$

where $\hat{\mathbf{H}}^{(i)}$ is defined as

$$\hat{\mathbf{H}}^{(i)} = [\hat{\mathbf{h}}_j^{(i)}]_{n\times n}, \quad \hat{\mathbf{h}}_j^{(i)} = \begin{cases} \dfrac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega} & j = i \\ \mathbf{f}_j & j \neq i \end{cases}.$$

Lemma 1 *If square matrices* $\mathbf{X}$ *and* $\mathbf{Y}$ *differ only in column* i*, i.e.,* $\begin{cases} \mathbf{x}_j \neq \mathbf{y}_j & j = i \\ \mathbf{x}_j = \mathbf{y}_j & j \neq i \end{cases}$*, then*

$$\begin{aligned} \det\big(\mathbf{X}\big) + \det(\mathbf{Y}\big) &= \det\big(\psi(\mathbf{X}, \mathbf{y}_i, \{i\})\big) \\ &= \det\big(\psi(\mathbf{Y}, \mathbf{x}_i, \{i\})\big). \end{aligned}$$

Equation (14) is rewritten as

$$\mathrm{u}_\Omega(\gamma) = \overline{p}_\Omega \Big(\det(\mathbf{F}) - \det(\hat{\mathbf{H}}^{(1)}) - \sum_{i=2}^{k} \det(\hat{\mathbf{H}}^{(i)}) \Big). \tag{15}$$

Since $\mathbf{F}$ and $\hat{\mathbf{H}}^{(1)}$ are the same except for the first column, using Lemma 1 , we will have

$$\det(\mathbf{F}) - \det(\hat{\mathbf{H}}^{(1)}) = \det\big(\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1\})\big). \tag{16}$$

On the the other hand, using the fact that addition or substraction of columns does not change the value of the determinant, we will have

$$\det(\hat{\mathbf{H}}^{(i)}) = \det\big(\psi(\hat{\mathbf{H}}^{(i)}, -\hat{\mathbf{h}}_i^{(i)}, \{1, \ldots, i-1\})\big). \tag{17}$$

Then, using (16) and (17) and regarding $\hat{\mathbf{h}}_i^{(i)} = \frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}$, we can rewrite (15) as

$$\begin{aligned} \mathrm{u}_\Omega(\gamma) = \overline{p}_\Omega \Big(&\det\big(\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1\})\big) \\ &- \sum_{i=2}^{k} \det(\psi(\hat{\mathbf{H}}^{(i)}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1, \ldots, i-1\})) \Big). \end{aligned} \tag{18}$$

Since $\mathbf{F}$ and $\hat{\mathbf{H}}^{(i)}$ are the same except for the column i, we can easily see that $\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1, \ldots, i-1\})$ and $\psi(\hat{\mathbf{H}}^{(i)}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1, \ldots, i-1\})$ are the same except for the i^{th} column. Therefore,

$$\begin{aligned} &\det\big(\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1, \ldots, i-1\})\big) \\ &- \det\big(\psi(\hat{\mathbf{H}}^{(i)}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1, \ldots, i-1\})\big) \\ &= \det\big(\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \{1, \ldots, i\})\big). \end{aligned}$$

Applying this result to (18) successively yields the following lemma.

Lemma 2 $\mathrm{u}_\Omega(\gamma) = \overline{p}_\Omega \det\big(\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big)$.

We utilize the result in Lemma 2 to find the smallest positive simple root of u_Ω using Perron-Frobenius theorem. This theorem states some properties about the eigenvalues of an irreducible matrix. A square non-negative matrix $\mathbf{X}$ is said to be irreducible if for every pair i, j of its index set, there exists a positive integer $m \equiv m(i, j)$ such that $x_{ij}^{(m)} > 0$ which $x_{ij}^{(m)}$ is the ij^{th} element of X^m [12].

Theorem 1 *[12] (The Perron-Frobenius Theorem for irreducible matrices) Suppose* $\mathbf{X}$ *is an* $m \times m$ *irreducible non-negative matrix. Then there exists an eigenvalue* $\lambda^*(\mathbf{X})$ *(Perron-Frobenius eigenvalue or PF-eigenvalue) such that*

(i) $\lambda^*(\mathbf{X}) > 0$ *and it is real.*
(ii) *there is a positive vector* $\mathbf{v}$ *such that* $\mathbf{X}\mathbf{v} = \lambda^*(\mathbf{X})\mathbf{v}$.
(iii) $\lambda^*(\mathbf{X}) \geq |\lambda(\mathbf{X})|$ *for any eigenvalue* $\lambda(\mathbf{X}) \neq \lambda^*(\mathbf{X})$.
(iv) *If* $\mathbf{X} \geq \mathbf{Y} \geq \mathbf{0}$*, then* $\lambda^*(\mathbf{X}) \geq |\lambda(\mathbf{Y})|$ *for any eigenvalue of* $\mathbf{Y}$.
(v) $\lambda^*(\mathbf{X})$ *is a simple root of the characteristic polynomial of* $\mathbf{X}$.

Lemma 3 *The smallest positive root of* $\mathrm{u}_\Omega(\gamma)$ *is*

$$r(\mathrm{u}_\Omega) = \frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big)}.$$

Proof:

$$\begin{aligned}\mathrm{u}_\Omega(\gamma) &= \overline{p}_\Omega \det\big(\psi(\mathbf{F}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big)\\ &= \overline{p}_\Omega \det\big(\psi(\mathbf{I} - \gamma\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, -\frac{\gamma\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big)\\ &= \overline{p}_\Omega\gamma^n \det\big(\psi(\frac{1}{\gamma}\mathbf{I} - \mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, -\frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big)\\ &= \overline{p}_\Omega\gamma^n \det\big(\frac{1}{\gamma}\mathbf{I} - \psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big).\end{aligned}$$

Consequently, $\frac{\mathrm{u}_\Omega(\gamma)}{\overline{p}_\Omega\gamma^n}$ is the reciprocal of the characteristic polynomial of $\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)$. Therefore, the roots of this polynomial are equal to the inverse of the eigenvalues of $\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)$. On the other hand, according to Theorem 1, since $\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)$ is an irreducible matrix, the PF-eigenvalue of this matrix is real and positive and has the largest norm among all eigenvalues. Also it is the simple root of the characteristic polynomial of the aforementioned matrix. Therefore, the inverse of this eigenvalue gives the smallest positive simple root of $\mathrm{u}_\Omega(\gamma)$ and the claim is proved. ■

For the denominator using (9), we have

$$\begin{aligned}\mathrm{f}(\gamma) &= \det(\mathbf{F}) = \det\big(\mathbf{I} - \gamma\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big)\\ &= \gamma^n \det\big(\frac{1}{\gamma}\mathbf{I} - \mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big).\end{aligned} \tag{19}$$

Therefore, $\mathrm{f}(\gamma)$ is the reciprocal of the characteristic polynomial of $\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}$. On the other hand, according to Theorem 1, the PF-eigenvalue of $\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}$, is real and positive. It also has the largest magnitude (norm) among the eigenvalues of the matrix and it is the simple root of the characteristic polynomial of the associated matrix. Therefore, $\lambda^*\big(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big)$ is the inverse of the smallest positive simple root of $\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}$. Thus,

$$r(\mathrm{f}) = \frac{1}{\lambda^*\big(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big)}. \tag{20}$$

On the other hand, according to (8), $r(\mathrm{f})$ is also the maximum achievable SINR for the system with unbounded powers satisfying constraint (3). Consequently, using (13), (20) and Lemma(3), the maximum achievable SINR to satisfy all constraints on the power (constraints (3) and (4)) is

$$\begin{aligned}\gamma^* &= \min\{r(\mathrm{f}), r(\mathrm{u}^{(i)})\}\\ &= \min\{\frac{1}{\lambda^*\big(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big)}, \frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_i}, \{i\})\big)}\}.\end{aligned}$$

Since $\psi\big(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_i}, \Omega\big) \geq \mathrm{diag}(\boldsymbol{\mu})\mathbf{A}$ and both are irreducible, using Theorem 1 we have

$$\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_i}, \{i\})\big) \geq \lambda^*\big(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}\big),$$

and consequently the maximum achievable γ for a system with constraint on the total power of any subset of the users is achieved. This discussion leads to the following theorem.

Theorem 2 *The maximum achievable* γ *in an interference channel with* n *links and gain matrix* $\mathbf{A}$*, where power vector is subject to the following constraints,*

$$\mathbf{p} \geq \mathbf{0}, \quad \sum_{i\in\Omega} p_i \leq \overline{p}_\Omega$$

is equal to

$$\gamma^* = \frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_\Omega}, \Omega)\big)},$$

where $\Omega \subseteq \{1, \ldots, n\}$ *is an arbitrary subset of the users.*

When multiple constraints on power exist, it is obvious that the maximum achievable SINR is the minimum of the maximum achievable SINR when each of the constraints is applied separately, i.e.,

$$\gamma^* = \min_i \gamma_i^*,$$

where γ_i^* is the maximum achievable SINR for the constraint i on power. The following corollary yields the maximum achievable SINR when the power of individual users and the total power are constrained.

Corollary 1 *The maximum achievable* γ *in* (1)*, where power vector is subject to the following constraints,*

$$\mathbf{p} \geq \mathbf{0}, \quad \mathbf{p} \leq \overline{\mathbf{p}}, \quad \sum_{i=1}^{n} p_i \leq \overline{p}_t$$

is equal to $\gamma^* =$

$$\begin{aligned}&\min\{\frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_t}, \{1, \ldots, n\})\big)},\\ &\frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_1}, \{1\})\big)}, \frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_2}, \{2\})\big)},\\ &\ldots, \frac{1}{\lambda^*\big(\psi(\mathrm{diag}(\boldsymbol{\mu})\mathbf{A}, \frac{\boldsymbol{\eta}}{\overline{p}_n}, \{n\})\big)}\}.\end{aligned} \tag{21}$$

The boundary of the SINR region in any direction can be obtained by choosing $\boldsymbol{\mu}$, accordingly. Due to the explicit relationship between the SINR and the rate in Gaussian channels, obtaining the SINR region in these channels amounts to the rate region characterization. As an example, Fig. 2 depicts the rate region of a system with the gain matrix $\mathbf{G}$ as

$$\mathbf{G} = \begin{bmatrix} 0.6791 & 0.0999 \\ 0.0411 & 0.6864 \end{bmatrix},$$

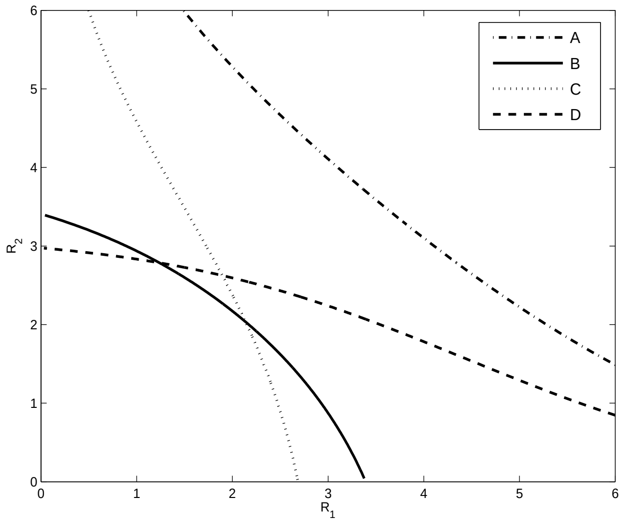

Fig. 2. The rate region for a 2-user interference channel with the following constraints on the power, A: $p_1 \geq 0$, $p_2 \geq 0$, B: $p_1 + p_2 \leq \bar{p}_t$, $p_1 \geq 0$, $p_2 \geq 0$ C: $0 \leq p_1 \leq \bar{p}_1$, $p_2 \geq 0$, D: $0 \leq p_2 \leq \bar{p}_2$, $p_1 \geq 0$

while the power of individual users and the total power are upper-bounded as $\bar{p}_1 = 0.8, \quad \bar{p}_2 = 1, \quad \bar{p}_t = 1.4$, and $\sigma_1^2 = \sigma_2^2 = 10^{-1}$.

IV. Time-Varying Channel

So far, we have assumed that the channel gains are fixed with time. However, in practice, channel gains vary with time due to the users' movement or changing the environment conditions.

In this section, we consider an interference channel with n co-channel links whose channel gain matrix is randomly selected from a finite set $\{\mathbf{G}_1, \ldots, \mathbf{G}_l\}$ with probability $\rho_1, \ldots, \rho_l$, respectively. The matrix $\mathbf{A}_i$ denotes the normalized gain matrix in the state i, $i \in \{1, \ldots, l\}$. The objective is to find the maximum γ which is achievable by all users in all channel states, while the average power of the users are constrained, i.e.,

$$\begin{aligned} & \max \gamma \\ \text{s.t.} \quad & \gamma_{j,i} \geq \mu_j \gamma, \quad \forall j \in \Omega, i \in \{1, \ldots, l\} \\ & p_{j,i} \geq 0, \quad \forall j \in \Omega, i \in \{1, \ldots, l\} \\ & E[\sum_{j \in \Omega} p_{j,i}] \leq \bar{p}_\Omega, \end{aligned}$$

where $\gamma_{j,i}$ and $p_{j,i}$ are the SINR and the power of transmitter j, respectively when the channel gain matrix is $\mathbf{G}_i$. We define an expanded system including ln users with block diagonal matrices $\mathbf{G}$ and $\mathbf{A}$ as the channel gain matrix and the normalized gain matrix, respectively. In the matrices $\mathbf{G}$ and $\mathbf{A}$, the i^{th} matrix on the diagonal is $\mathbf{G}_i$ and $\mathbf{A}_i$, respectively. Applying a similar technique as before to this system, we can obtain the maximum achievable SINR.

Theorem 3 *The maximum achievable γ in a time-varying interference channel with n links and probability vector $\boldsymbol{\rho}_{l\times 1}$, with the following constraints on power,*

$$p_{j,i} \geq 0, \forall j \in \Omega, i \in \{1, \ldots, l\}, \quad E[\sum_{j \in \Omega} p_{j,i}] \leq \bar{p}_\Omega$$

is equal to $\gamma^* = \dfrac{1}{\lambda^*(Q)}$, *where*

$$\begin{aligned} Q =& \mathrm{diag}(\mathbf{1}_{l\times 1} \otimes \boldsymbol{\mu})\mathbf{A} \\ & + \sum_{i=1}^{l} \psi(\mathbf{0}_{ln\times ln}, \frac{\rho_i \boldsymbol{\eta}}{\bar{p}_\Omega}, \{j + (i-1)n : j \in \Omega\}), \end{aligned}$$

and $\eta_{j+(i-1)n} = \dfrac{\mu_j \sigma_j^2}{g_{j+(i-1)n, j+(i-1)n}}, j \in \Omega, i \in \{1, \ldots, l\}.$

For the proof refer to [15].

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