Interference-Limited versus Noise-Limited Communication Over Dense Wireless Networks

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***Abstract*— A network of n wireless communication links is considered. Rayleigh fading is assumed to be the dominant factor affecting the strength of the channels between nodes. In previous works it is shown that the maximum throughput of this network over all link activation strategies scales as $\log n$. However, it is achieved by assigning a vanishingly small rate to each active link. The objective of this paper is to analyze the achievable throughput of the network when the data rate of each active link is constrained to be a constant $\lambda > 0$. A link activation strategy is proposed and analyzed using random graph theory. In the interference-limited regime, a throughput scaling as $\tau \log n$ is achievable, where the scaling factor τ approaches 1 as $\lambda \to 0$ or $\lambda \to \infty$. This implies the asymptotic optimality of the proposed scheme. In the noise-limited regime, it is shown that rate-per-links scaling as $\log(\Delta_0 \rho)$ are achievable, where Δ_0 is a constant and ρ is the transmit signal to noise ratio. However, in this case the throughput decreases by a factor of $\log \log n$ as compared to the interference-limited regime.**

I. Introduction

Most of the works stdying the scaling laws of wireless networks consider a channel model in which the signal power decays according to a distance-based attenuation law [1]–[7]. However, the power attenuation laws may not be valid when the receiver is not in the far field of the transmitter as in the dense networks. In addition, in a wireless environment the presence of obstacles and scatterers adds some randomness to the received signal. This random behaviour of the channel, known as fading, can drastically change the scaling laws of a network [8], [9]. To study the effect of fading, in [9]–[11], fading is assumed to be the dominant factor affecting the strength of the channels between nodes . We follow the same model in this paper.

A common attribute of [9], [10] and most of the previous works, e.g. [1], [2], [8], is that maximizing the throughput of the network leads to rate-per-links that decrease to zero as the number of nodes grows. However, it is practically appealing to assign constant rates to active communication links. In a recent work by the authors [10], a suboptimal strategy is proposed that renders a rate $\lambda > 0$ to the active links and achieves a throughput of $\tau \log n$. However, the scaling factor τ is a decreasing function of λ approaching zero as $\lambda \to \infty$.

In this paper, we build on the method of [10] to develop an efficient link activation strategy. Utilizing random graph theory, it is demonstrated that the proposed method is capable of providing constant rate-per-links. Moreover, by deriving an upper bound, we show that the proposed method is asymptotically optimum. To achieve its optimum performance, the proposed method works in an interference-limited regime.

A natural question is whether it is possible to have rate-per-links which are determined by the *SNR*, yet of order $\Theta(1)$. We address this issue in this paper and show that the answer is affirmative. However, the throughput in this scenario scales down by a factor of $\log \log n$ as compared to the interference-limited regime.

Due to the space limitation, some of the proofs have been omitted or summarized. The details can be found in [12].

Notation: $\mathcal{N}_n$ represents the set of natural numbers less than or equal to n; log is the natural logarithm function; $\mathrm{P}(A)$ denotes the probability of event A; an event A_n holds *asymptotically almost surely* (a.a.s) if $\mathrm{P}(A_n) \to 1$ as $n \to \infty$; for any functions $f(n)$ and $h(n)$, $h(n) = o(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| = 0$, $h(n) = O(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| < \infty$, $h(n) = \Theta(f(n))$ is equivalent to $\lim_{n\to\infty} |h(n)/f(n)| = c$, where $0 < c < \infty$, and $h(n) \sim f(n)$ is equivalent to $\lim_{n\to\infty} h(n)/f(n) = 1$.

II. Network Model and Objectives

We consider a wireless network with n pairs of transmitters and receivers. Each transmitter aims to send data to its corresponding receiver. These n communication links are indexed by the elements of $\mathcal{N}_n$. The transmit power of link i is denoted by p_i and satisfies $p_i \in \{0, P\}$, where P is a constant. Each link with $p_i = P$ is called an active link and the set of active links is denoted by $\mathcal{A}$. The received power from transmitter j at the receiver i equals $g_{ji} p_j$. In this paper, we assume the channel coefficients are i.i.d. random variables drawn from an exponential distribution with pdf $f(x) = e^{-x} u(x)$. This model corresponds to a Rayleigh fading channel.

We consider an additive white Gaussian noise (AWGN) with limited variance η at the receiver i. The transmit *signal-to-noise ratio* (*SNR*) of the network is defined as $\rho = \frac{P}{\eta}$. Since the transmissions occur simultaneously within the same environment, the signal from each transmitter acts as interference for other links. Assuming Gaussian signal transmission from all links, the distribution of the interference will be Gaussian as well. Thus, the maximum supportable rate of active link i

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is obtained as $r_i(\mathcal{A}) = \log(1 + \gamma_i(\mathcal{A}))$, where

$$\gamma_i(\mathcal{A}) = \frac{g_{ii}}{1/\rho + \sum_{\substack{j\in\mathcal{A}\\ j\neq i}} g_{ji}} \tag{1}$$

is the *SINR* of active link i. The throughput of the network is defined as

$$T(\mathcal{A}) = \sum_{i\in\mathcal{A}} r_i(\mathcal{A}). \tag{2}$$

The number of active links is defined as $k = |\mathcal{A}|$. Based on the previous results [10], we find it useful to define the following quantities. The scaling factors of the throughput and of the number of active links are defined as

$$\begin{aligned} \tau &= \lim_{n\to\infty} \frac{T}{\log n}, \\ \kappa &= \lim_{n\to\infty} \frac{k}{\log n}. \end{aligned}$$

We consider a scenario in which all active links transmit with a same rate λ. Hence, the problem of throughput maximization becomes equivalent to maximizing the number of active links subject to a constraint on the rate of active links, i.e.,

$$\begin{array}{cc} \max\limits_{\mathcal{A}\subseteq\mathcal{N}} & |\mathcal{A}| \\ \text{s.t.} & r_i(\mathcal{A}) \geq \lambda, \quad \forall i \in \mathcal{A} \end{array}. \tag{3}$$

This problem is nonconvex, integral, and probabilistic in nature. However, we provide a link activation strategy which is a.a.s. optimum. It is shown that the optimum result is achieved when the network operates in an interference-limited regime. In Section V, we turn attention to the noise-limited regime. In this case, it is shown that there is a penalty in the throughput scaling law.

III. Upper Bounds

In this section, we obtain an upper bound on the optimum solution of (3). This upper bound can be either presented as an upper bound on the throughput or as an upper bound on the number of active links.

Lemma 1: Assume T^* and κ^* are the maximum throughput and the corresponding scaling factor of the number of active links obtained from problem (3). Then, a.a.s. we have

$$\begin{aligned} T^* &< \log n - \log\log n + c, \\ \kappa^* &< \frac{1}{\lambda}, \end{aligned}$$

for some constant c.

Proof: Assume $\mathcal{L}$ is the event that there exists at least one set $\mathcal{A} \subseteq \mathcal{N}$ with $|\mathcal{A}| = k$ such that the constraints in (3) are satisfied and Z is the number of such sets. Also, assume $\bar{\gamma}$ is a quantity that satisfies $\lambda = \log(1 + \bar{\gamma})$. We have

$$\begin{aligned} \mathrm{P}(\mathcal{L}) &= \mathrm{P}(Z \geq 1) \\ &\overset{(a)}{\leq} \binom{n}{k} \left(\mathrm{P}(\gamma_i \geq \bar{\gamma})\right)^k \\ &= \binom{n}{k} \frac{e^{-\bar{\gamma}k/\rho}}{(1+\bar{\gamma})^{k(k-1)}} \\ &\overset{(b)}{\leq} \left(\frac{ne}{k}\right)^k \frac{e^{-\bar{\gamma}k/\rho}}{(1+\bar{\gamma})^{k(k-1)}} \\ &= e^{k(\log n - \log k - \lambda k + \lambda + 1 - \bar{\gamma}/\rho)}, \end{aligned}$$

where (a) is due to the Markov's inequality and (b) is the result of applying the Stirling's approximation for the factorial. It can be verified that there exists a constant c such that if $k\lambda = \log n - \log\log n + c$, then, the above upper bound approaches zero. Hence, for the event $\mathcal{L}$ to have non-zero probability, the upper bounds in the lemma should hold. ■

IV. Optimum Link Activation Strategy

In this section, we propose and analyze a link activation strategy and prove its optimality.

For the threshold values Δ and δ

i. Choose the largest set $\mathcal{A}_1 \subseteq \mathcal{N}_n$ such that $g_{ii} > \Delta$ for all $i \in \mathcal{A}_1$.
ii. Choose the largest set $\mathcal{A}_2 \subseteq \mathcal{A}_1$ such that $g_{ij} \leq \delta$ and $g_{ji} \leq \delta$ for all $i, j \in \mathcal{A}_2$.

The set of active links is $\mathcal{A} = \mathcal{A}_2$.

We aim to find Δ and δ such that the throughput is maximized subject to the rate constraints of the active links.

For simplicity, we take the notation $k_i = |\mathcal{A}_i|$ for $i = 1, 2$. Without loss of generality, assume $\mathcal{A}_i = \{1, \cdots, k_i\}$. From (2), it can be shown that the throughput provided by the proposed link activation strategy a.a.s. satisfies

$$T_2 \geq k_2 \log\left(1 + \frac{\Delta}{\hat{\mu}k_2}\right) \qquad a.a.s.$$

where

$$\hat{\mu} = \mathrm{E}\{g_{ji} | g_{ji} \leq \delta\} = 1 - \frac{\delta e^{-\delta}}{1 - e^{-\delta}}.$$

The following discussion shows that k_2, which is a random variable, is highly concentrated around a deterministic value.

Construct an undirected graph $G(\mathcal{A}_1, \boldsymbol{E})$ with vertex set $\mathcal{A}_1$ and the adjacency matrix $\boldsymbol{E} = [e_{ij}]$ defined as

$$e_{ij} = \begin{cases} 1 & ; \quad g_{ij} \leq \delta \text{ and } g_{ji} \leq \delta \\ 0 & ; \quad \text{otherwise} \end{cases}.$$

It can be easily verified that the probability of having an edge between vertices i and j equals

$$p = \left(1 - e^{-\delta}\right)^2. \tag{4}$$

The definition of G implies that $G \in \mathcal{G}(k_1, p)$, where $\mathcal{G}(k_1, p)$ is the family of k_1-vertex *random graphs* with edge probability p.

In the second phase of the proposed strategy, we are interested to choose the largest subset of the links in $\mathcal{A}_1$ whose cross channel coefficients are smaller than δ. This is equivalent to choose the largest complete subgraph of G. The size of the largest complete subgraph of G is called its clique number. The following lemma about the clique number is a well-known result in random graph theory [13]. The lemma is presented in a format that suits what we need in this paper.

Lemma 2: For a fixed $0 < p < 1$, the clique number $\mathrm{cl}(G)$ of $G \in \mathcal{G}(m, p)$ a.a.s. scales as

$$\mathrm{cl}(G) \sim \frac{2\log m}{\log 1/p}$$

According to this lemma, k_2 can be easily obtained in terms of k_1. Hence, noting that $k_1 \sim ne^{-\Delta}$, we obtain

$$k_2 \sim 2\log_b ne^{-\Delta},$$

where $b = (1 - e^{-\delta})^{-2}$. The next lemma indicates how to choose the thresholds Δ and δ to obtain a throughput proportional to $\log n$ and a constant average rate-per-link.

Lemma 3: Assume the threshold Δ is chosen to be $\Delta = (1-\alpha)\log n(1+o(1))$ for some $\alpha > 0$ and δ is a constant. Then, a.a.s. we have

$$\tau = \frac{-\alpha}{\log(1-e^{-\delta})}\log\left(1 - \frac{(1-\alpha)\log(1-e^{-\delta})}{\alpha\left(1-\frac{\delta e^{-\delta}}{1-e^{-\delta}}\right)}\right)$$

$$\kappa = \frac{-\alpha}{\log(1-e^{-\delta})}.$$

According to this lemma, a constant average rate-per-link $\bar{r} = \tau/\kappa$ is achievable. However, it is not trivial whether under the specified conditions the constraints of (3) are satisfied. The following lemma addresses this issue.

Lemma 4: Assume $\Delta = (1-\alpha)\log n$ for some $\alpha > 0$ and δ is fixed. Then, for the proposed strategy, a.a.s. we have

$$|r_i - \lambda| < c\sqrt{\frac{\log\log n}{\log n}}(1+o(1)), \quad \forall i \in \mathcal{A},$$

for some constant $c > 0$, where

$$\lambda = \log\left(1 - \frac{(1-\alpha)\log(1-e^{-\delta})}{\alpha\hat{\mu}}\right).$$

Lemma 4 expresses the guaranteed rate for all links in terms of the parameters α and δ. The next step is to choose these parameters such that the guaranteed rate equals λ and the achieved throughput, or equivalently the number of supported links, is maximized. Assume $\bar{\gamma}$ is a quantity that satisfies $\lambda = \log(1+\bar{\gamma})$. Then, the throughput maximization problem is converted to the following optimization problem

$$\max_{\alpha,\,\delta} \quad \tau$$

$$\text{s.t.} \quad -\frac{(1-\alpha)\log(1-e^{-\delta})}{\alpha\left(1-\frac{\delta e^{-\delta}}{1-e^{-\delta}}\right)} = \bar{\gamma}.$$

This problem has a unique solution that can be found numerically. Figure 1 shows the maximum scaling factor of the number of supported links versus the demanded rate-per-link. The same quantity for the decentralized method proposed in [10] and the upper bound from Lemma 1 are also plotted for comparison. It is seen that the performance of the proposed strategy is very close to the upper bound. Indeed, if κ^* is the maximum scaling factor of the number of active links, it can be analytically shown that

$$\lim_{\lambda\to\infty} (\kappa - \kappa^*) = 0,$$

$$\lim_{\lambda\to 0} \frac{\kappa}{\kappa^*} = 1.$$

This proves the asymptotic optimality of the proposed strategy.

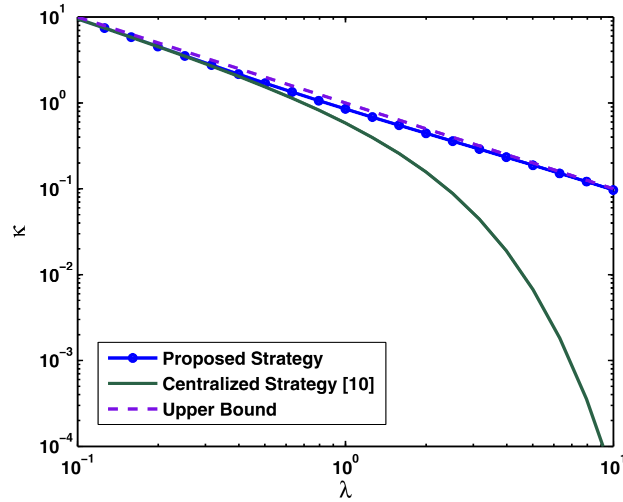

Fig. 1. Tradeoff between rate-per-link and the number of active links.

V. Noise-Limited Regime

In the previous section, we considered an interference-limited regime in which the noise power is negligible in comparison with the interference power. In this case, the achievable throughput is not a function of the network *SNR*. In other words, changing the transmission powers does not affect the supportable rate of each link. However, in a practical scenario, it is appealing to have rates which are proportional to $\log\rho$. This way, the transmission rates can be easily adjusted by changing the transmission powers. In this section, we show how to realize such a situation by the using proposed Strategy.

To have a noise-limited regime, the following conditions should be met:

i. The average interference power should approach zero, i.e.,

$$\frac{1}{k_2}\sum_{i=1}^{k_2} I_i \to 0, \quad a.a.s. \tag{5}$$

where $I_i = \sum_{\substack{j=1 \\ j\neq i}}^{k_2} g_{ji}$ is the interference seen by link i.

ii. The threshold Δ should be a constant, i.e.,

$$\Delta = \Delta_0 = \Theta(1).$$

With these conditions, the achievable throughput a.a.s. satisfies

$$T \geq k_2\log(\Delta_0\rho), \quad a.a.s. \tag{6}$$

which is proportional to $\log\gamma$ as desired. Now, we should find the maximum value of k_2 for which condition (5) holds.

Lemma 5: The condition (5) is a.a.s. equivalent to

$$(k_2 - 1)\frac{\delta}{2} \to 0$$

For convenience, we express the condition stated in Lemma 5 as

$$k_2\delta = \epsilon(n), \tag{7}$$

for some $\epsilon(n) \to 0$. The next step is to see how k_2 scales with n and δ. First, note that since $\delta \to 0$, (4) can be rewritten as

$$p = \delta^2 + O(\delta^3), \tag{8}$$

which approaches zero as well. As discussed in the previous section, k_2 is the clique number of a random graph $\mathcal{G}(k_1, p)$. However, a critical question is whether the result of Lemma 2 is valid when p approaches zero. The following lemma addresses this issue and shows that for some cases that $p \to 0$, the result of Lemma 2 remains valid.

Lemma 6: Let $p = p(m)$ be such that $p = o(1)$ and $p = \omega(m^{-a})$ for all $a > 0$. For fixed $\epsilon > 0$ the clique number $\mathrm{cl}(G)$ of $G \in \mathcal{G}(m, p)$ a.a.s. satisfies $\lfloor s \rfloor \leq \mathrm{cl}(G) \leq \lfloor s \rfloor + 1$, where $s = 2\log_b m - 2\log_b \log_b m + 1 - 4\log_b 2 - \frac{\epsilon}{\log b}$ and $b = 1/p$.

Proof: See the Appendix. ■

By using this lemma, (8), and $k_1 \sim ne^{-\Delta}$, the number of active links becomes

$$k_2 \sim -\frac{\log n}{\log \delta}. \tag{9}$$

Thus, condition (7) can be rewritten as

$$-\frac{\delta}{\log \delta}\log n = \epsilon(n). \tag{10}$$

Assuming $|\log \epsilon(n)| \ll \log\log n$, it can be verified that the solution of (10) is

$$\delta = \frac{\epsilon(n)\log\log n}{\log n}(1 + o(1)).$$

With this value of δ the number of active links is obtained from (9) as

$$k_2 \sim \frac{\log n}{\log\log n}.$$

Consequently, from (6), the throughput of the network is lower bounded as

$$T \geq \frac{\log n}{\log\log n}\log(\Delta_0\rho).$$

It is observed that the price for operating in the noise-limited regime is a decrease in the throughput by a factor of $\log\log n$.

VI. Conclusion

In this paper, we considered a wireless network with Rayleigh fading channels in which each link can be active and transmit with power P or inactive and silent. Each active link transmits with a constant rate λ. The maximum throughput of the network over all link activation strategies was sought. It was shown that the maximum number of active links at most scales as $\frac{1}{\lambda}\log n$. A link activation strategy that reaches this upper bound in an interference-limited regime was proposed. It was also shown that the price for operating in a noise-limited regime is a decrease by a factor of $\log\log n$ in the throughput.

Appendix

Sketch of the Proof of Lemma 6

Assume Y_r denotes the number of complete subgraphs of size r in $\boldsymbol{G}$. The mean and variance of Y_r are obtained as

$$\mu_r = \mathrm{E}(Y_r) = \binom{m}{r}p^{\binom{r}{2}},$$

$$\sigma_r^2 = \sum_{l=2}^{r}\binom{m}{r}\binom{r}{l}\binom{m-r}{r-l}p^{2\binom{r}{2}}(b^{\binom{l}{2}} - 1),$$

where $b = 1/p$.

It can be shown that if $r \geq s + 2\log_b 2e + \epsilon(m)$, for some $\epsilon(m) = o(1)$, then we have $\mu_r \to 0$ as $m \to \infty$. Thus, from the Markov's inequality we obtain

$$\mathrm{P}\{Y_r \geq 1\} \leq \mu_r \to 0 \quad \text{for} \quad r \geq s + 2\log_b 2e + \epsilon(m). \tag{11}$$

Also, for $r \leq s$ it can be verified that $\frac{\sigma_r^2}{\mu_r^2} \to 0$ as $m \to \infty$. Hence, from the Chebyshev's inequality we obtain

$$\mathrm{P}\{Y_r = 0\} \leq \frac{\sigma_r^2}{\mu_r^2} \to 0 \quad \text{for} \quad r \leq s. \tag{12}$$

The lemma is the result of (11) and (12).

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