Asymptotic Analysis of Amplify and Forward Relaying in a Parallel MIMO Relay Network

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***Abstract*— This paper considers the setup of a parallel MIMO relay network in which K relays, each equipped with N antennas, assist the transmitter and the receiver, each equipped with M antennas, in the half-duplex mode, under the assumption that $N \geq M$. This setup has been studied in the literature like in [1], [2], and [3]. In this paper, a simple scheme, the so-called Incremental Cooperative Beamforming, is introduced and shown to achieve the capacity of the network in the asymptotic case of $K \rightarrow \infty$ with a gap no more than $O\left(\frac{1}{\log(K)}\right)$. This result is shown to hold, as long as the power of the relays scales as $\omega\left(\frac{\log^9(K)}{K}\right)$. Finally, the asymptotic SNR behavior is studied and it is proved that the proposed scheme achieves the full multiplexing gain, regardless of the number of relays.**[1]

I. INTRODUCTION

The relay channel, which was first introduced by Van-der Meulen in 1971 [4], has been reconsidered a lot in recent years. The main idea is to employ some extra nodes in the network to aid the transmitter/receiver in sending/receiving the signal to/from the other end. In this way, the supplementary nodes act as (spatially) distributed antennas assisting the signal transmission and reception. Up to now, some promising results have been published on MIMO Multiple-Access and Broadcast channels in [5], [6], [7], [8], and [9]. However, there are still only a few results known concerning the MIMO relay networks. Moreover, no capacity-achieving strategy is known for the Gaussian relay channel.

Recently, several extensions of the relay channel have been considered, e.g. in [10]–[13]. Some of these extensions consider a multiple-relay scenario in which several nodes relay the message. The parallel relay channel is a special case of the multiple relay channel in which the relays transmit their data directly to the receiver. Besides studying the well-known "compress-and-forward" and "decode-and-forward" strategies, the authors in [10], [11] have also studied the "amplify-and-forward" strategy where the relays simply amplify and transmit their received data to the receiver. Despite its simplicity, the AF strategy achieves a good performance. In fact, [10] shows that AF outperforms other strategies in many scenarios. Moreover, [11] proves that AF achieves the capacity of the Gaussian (single antenna) parallel relay network as the number of relays increases.

References [1], [2] extend the work of [11] to the MIMO Rayleigh fading parallel relay network. Unlike the single antenna parallel relay scenario, in this case the AF multipliers are matrices rather than scalars. Hence, finding the optimum AF matrices becomes challenging. Reference [1] has proposed a coherent AF scheme, called "matched filtering", and proves that this scheme follows the capacity of the channel with a constant gap in terms of the number of relays in the asymptotic case of $K \rightarrow \infty$.

In this paper, we consider the AF strategy in the parallel MIMO relay network. We propose a new AF protocol called "Cooperative Beamforming Scheme" (CBS). Considering the uplink channel (from the transmitter to the relays) as a point-to-point channel, in CBS the relays cooperatively multiply the channel matrix with its left eigenvector matrix. The interesting point is that to perform such an operation, each relay only needs to know its corresponding sub-matrix of the beamforming matrix. For the outputs to be coherently added at the receiver end, each relay has to apply zero forcing beamforming to its corresponding downlink channel (the channel from each relay to the receiver). Here, the interesting result is that the overall channel from the transmitter to the receiver becomes diagonal and the overall Gaussian noise has independent components.

We show that the proposed scheme is optimum in the case of having negligible noise in the downlink channel. To enhance the performance of CBS in general scenarios, this work introduces a variant of CBS called "Incremental Cooperative Beamforming Scheme" (ICBS). In ICBS, the relays with ill-conditioned downlink channels are turned off. This strategy improves the overall point-to-point channel from the transmitter to the receiver. However, an interference term due to turning some of the relays off will be included in the equivalent point-to-point channel.

It is shown that for asymptotically large number of relays, one can simultaneously mitigate the downlink noise and the interference term due to the turned-off relays. As a result, the achievable rate of ICBS converges to the capacity of parallel MIMO relay network with a gap which scales as $O\left(\frac{1}{\log(K)}\right)$. This result is stronger than the result of [1] and [2] in which they show that their scheme can asymptotically ($K \rightarrow \infty$)

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achieve the capacity up to $O(1)$. Also, our numerical results show that the achievable rate of ICBS converges rapidly to the capacity, even for moderate number of relays. We also show that the same result can be achieved by ICBS, as long as the power of the relays scales as $\omega\left(\frac{P}{K}\log^9(K)\right)$ [2]. Finally, by analyzing the asymptotic SNR behavior of the proposed scheme, it is proved that, unlike the matched filtering scheme of Bcskei-Nabar-Oyman-Paulraj (BNOP) which results in a zero multiplexing gain, our proposed scheme achieves the full multiplexing gain, regardless of the number of relays.

The rest of the paper is organized as follows. In section II, the system model is introduced. In section III, the proposed AF scheme is described. Section IV is dedicated to the asymptotic analysis of the proposed scheme. Simulation results are presented in section V. Finally, section VI concludes the paper.

A. Notation

Throughout the paper, the superscripts ^{T},^{H} and * stand for matrix operations of transposition, conjugate transposition, and element-wise conjugation, respectively. Capital bold letters represent matrices, while lowercase bold letters and regular letters represent vectors and scalars, respectively. $\|\mathbf{v}\|$ denotes the norm of the vector $\mathbf{v}$ while $\|\mathbf{A}\|$ represents the frobenius norm of the matrix $\mathbf{A}$. $|\mathbf{A}|$ denotes the determinant of the matrix $\mathbf{A}$ while $\|\mathbf{A}\|_\star$ represents the maximum absolute value among the entries of $\mathbf{A}$. The notation $\mathbf{A}^\dagger$ stands for the pseudo inverse of the matrix $\mathbf{A}$. The notation $\mathbf{A} \preccurlyeq \mathbf{B}$ is equivalent to $\mathbf{B}-\mathbf{A}$ is a positive semi-definite matrix. For any functions $f(n)$ and $g(n)$, $f(n) = O(g(n))$ is equivalent to $\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| < \infty$, $f(n) = o(g(n))$ is equivalent to $\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| = 0$, $f(n) = \Omega(g(n))$ is equivalent to $\lim_{n\to\infty}\frac{f(n)}{g(n)} > 0$, $f(n) \gtrsim g(n)$ is equivalent to $\lim_{n\to\infty}\frac{f(n)}{g(n)} \geq 1$, $f(n) = \omega(g(n))$ is equivalent to $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty$, $f(n) \sim g(n)$ is equivalent to $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 1$ and $f(n) = \Theta(g(n))$ is equivalent to $\lim_{n\to\infty}\frac{f(n)}{g(n)} = c$, where $0 < c < \infty$.

II. System Model

The system model, as in [1], [2], and [3], is a parallel MIMO relay network with two-hop relaying and half-dulplexing between the uplink and downlink channels. In other words, the data transmission is performed in two time slots; in the first time slot, the signal is transmitted from the transmitter to the relays, and in the second time slot, the relays transmit data to the receiver. Note that there is no direct link between the transmitter and the receiver in this model. The transmitter and the receiver are equipped with M antennas and each of the relays is equipped with N antennas. Throughout the paper, we assume that $N \geq M$. The channel between the transmitter and the relays and the channel between the relays and the receiver are assumed to be frequency flat block Rayleigh fading. The channel from the transmitter to the kth relay, $1 \leq k \leq K$, is modeled as

$$\mathbf{r}_k = \mathbf{H}_k\mathbf{x} + \mathbf{n}_k, \tag{1}$$

and the downlink channel is modeled as

$$\mathbf{y} = \sum_{k=1}^{K}\mathbf{G}_k\mathbf{t}_k + \mathbf{z}, \tag{2}$$

where the channel matrices $\mathbf{H}_k$ and $\mathbf{G}_k$ are i.i.d. complex Gaussian matrices with zero mean and unit variance. $\mathbf{n}_k \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_N)$ and $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_M)$ are Additive White Gaussian Noise (AWGN) vectors, $\mathbf{r}_k$ and $\mathbf{t}_k$ are the kth relay's received and transmitted signal, respectively, and $\mathbf{x}$ and $\mathbf{y}$ are the transmitter's and the receiver's signal, respectively. $\mathbf{H}_k$ and $\mathbf{G}_k$ are of the sizes $N\times M$ and $M\times N$, respectively.

The task of amplify and forward (AF) relaying is to find the matrix $\mathbf{F}_k$ for each relay to be multiplied by its received signal to produce the relay's output as $\mathbf{t}_k = \mathbf{F}_k\mathbf{r}_k$. In addition, the power constraints $\mathbb{E}[\mathbf{x}^H\mathbf{x}] \leq P_s$ and $\mathbb{E}_{\mathbf{x},\mathbf{n}_k}[\mathbf{t}_k^H\mathbf{t}_k] \leq P_r$ must be satisfied for the transmitted signals of the transmitter and the relays, respectively. We assume $P_r = P_s = P$ throughout the paper, except in Theorem 2, where we study the case $P_r < P_s = P$.

III. Proposed Method

A. Cooperative Beamforming Scheme

The equivalent uplink channel can be represented as $\mathbf{H}^T = \left[\mathbf{H}_1^T|\mathbf{H}_2^T|\cdots|\mathbf{H}_K^T\right]^T$. By applying Singular Value Decomposition (SVD) to $\mathbf{H}$, we have $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^H$. Therefore, the diagonal matrix $\mathbf{\Lambda}$ has at most M nonzero diagonal entries corresponding to the nonzero singular values of $\mathbf{H}$. Consequently, we can rearrange the SVD such that $\mathbf{U}$ is of size $NK \times M$ while $\mathbf{V}$ and $\mathbf{\Lambda}$ are $M \times M$ matrices. $\mathbf{U}$ can be partitioned to $M \times N$ sub-matrices as $\mathbf{U} = \left[\mathbf{U}_1^T|\mathbf{U}_2^T|\cdots|\mathbf{U}_K^T\right]^T$. Suppose the kth relay multiplies its received signal by $\mathbf{F}_k = \alpha\mathbf{G}_k^\dagger\mathbf{U}_k^H$. At the receiver side, we have (figure 1(a))

$$\begin{aligned}\mathbf{y} &= \alpha\sum_{k=1}^{K}\mathbf{G}_k\mathbf{G}_k^\dagger\mathbf{U}_k^H\mathbf{r}_k + \mathbf{z}\\ &= \alpha\mathbf{U}^H\mathbf{r} + \mathbf{z}\\ &= \alpha\mathbf{U}^H\left(\mathbf{H}\mathbf{x} + \mathbf{n}\right) + \mathbf{z}\\ &= \alpha\left(\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^H\mathbf{x} + \mathbf{n}_u\right) + \mathbf{z},\end{aligned} \tag{3}$$

where $\mathbf{n} = \left[\mathbf{n}_1^T|\mathbf{n}_2^T|\cdots|\mathbf{n}_K^T\right]^T$, $\mathbf{r} = \left[\mathbf{r}_1^T|\mathbf{r}_2^T|\cdots|\mathbf{r}_K^T\right]^T$, and $\mathbf{n}_u = \mathbf{U}^H\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_M)$. If the transmitter beamforms its data vector as $\mathbf{x} = \mathbf{V}\mathbf{x}'$, the end-to-end channel becomes

$$\mathbf{y} = \alpha\left(\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{x}' + \mathbf{n}_u\right) + \mathbf{z}. \tag{4}$$

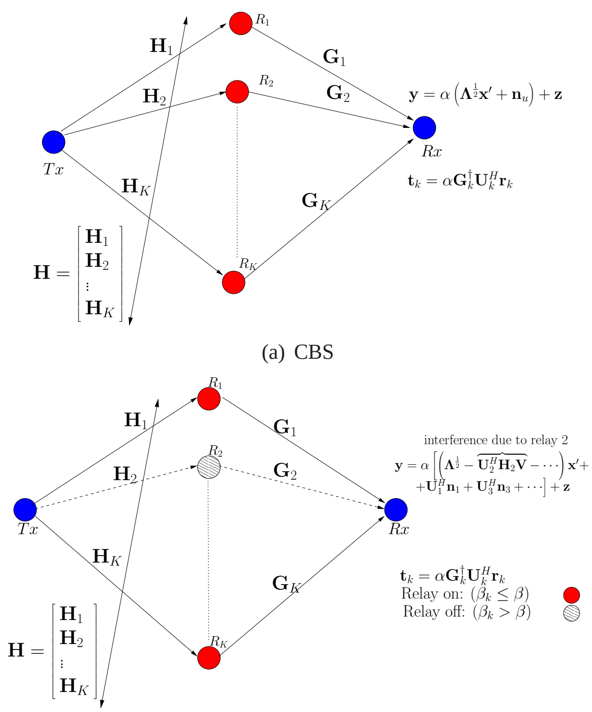
Equation (4) shows that the end-to-end channel is diagonal and the noise vector is white Gaussian. Note that the complexity of the decoder in such a channel is linear in terms of the number of transmitter's antennas, M, and also there is no interference among different data streams. Moreover, as it is shown in section IV, for $\alpha \to \infty$, the achievable rate

[2] $f(n) = \omega(g(n))$ is equivalent to $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \infty$

of such a scheme converges to the ergodic capacity of the Parallel MIMO relay network. The problem is that the value of α is dominated by

$$\alpha = \sqrt{\frac{P}{\max_k \mathbb{E}_{\mathbf{x},\mathbf{n}_k}\left[\left\|\mathbf{G}_k^{\dagger}\mathbf{U}_k^H\mathbf{r}_k\right\|^2\right]}}. \quad (5)$$

This guarantees that the output power of all relays is less than or equal to P. However, the value of α could be small in the cases where the downlink channel of any of the relays is ill conditioned. This means that while the output power of the worst relay (according to (5)) is equal to the maximum possible value, i.e. P, there may be many relays with the output power far less than P. This phenomenon degrades the performance, as in this case the downlink noise, $\mathbf{z}$, would be the dominant noise in (4).



(a) CBS

(b) ICBS

Fig. 1. Cooperative Beamforming Scheme and Incremental Cooperative Beamforming Schemes Schematics

B. Incremental Cooperative Beamforming Scheme (ICBS)

In this variant of CBS, we select a subset of relays which results in a high value of α. Defining $\beta_k \triangleq \mathbb{E}_{\mathbf{x},\mathbf{n}_k}\left[\left\|\mathbf{G}_k^{\dagger}\mathbf{U}_k^H\mathbf{r}_k\right\|^2\right]$, we activate the relays which satisfy $\beta_k \leq \beta$, where β is a predefined threshold. In this manner, it is guaranteed that $\alpha \geq \sqrt{\frac{P}{\beta}}$. This improvement in the value of α is realized at the expense of turning off some of the relays, creating interference in the equivalent point-to-point channel. More precisely, by defining $\mathcal{A} = \{k|\beta_k > \beta\}$, we have (figure 1(b))

$$\mathbf{y} = \alpha\left(\left(\mathbf{\Lambda}^{\frac{1}{2}} - \sum_{k\in\mathcal{A}}\mathbf{U}_k^H\mathbf{H}_k\mathbf{V}\right)\mathbf{x}' + \sum_{k\in\mathcal{A}^c}\mathbf{U}_k^H\mathbf{n}_k\right) + \mathbf{z}. \quad (6)$$

As (6) shows, by decreasing the value of β, one can guarantee a large value of α while increasing the gap of the equivalent channel matrix to $\mathbf{\Lambda}^{\frac{1}{2}}$. It will be shown in the next section that for large number of relays, it is possible to guarantee both having a large value of α and a small deviation from $\mathbf{\Lambda}^{\frac{1}{2}}$. Moreover, we show that by appropriately choosing the value of β, the rate of such a scheme would be at most $O\left(\frac{1}{\log(K)}\right)$ below the corresponding capacity.

C. A Note on CSI Assumption

In the BNOP scheme, it is assumed that each relay knows its corresponding forward and backward channels, i.e. $\mathbf{H}_k$ and $\mathbf{G}_k$, and at the receiver side, the effective signal power and the effective interference plus noise power are known for each antenna. However, in CBS and ICBS, it is assumed that the transmitter knows the uplink channel, i.e. $\mathbf{H}_1, \cdots, \mathbf{H}_K$, and sends the $N \times M$ matrix $\mathbf{U}_k$ to the k'th relay, $k = 1, \cdots, K$. This assumption is reasonable when the uplink channel is slow-fading; for example, in the case that the transmitter and all the relay nodes are fixed. Furthermore, similar to the BNOP scheme, we assume that each relay knows its forward channel, i.e. $\mathbf{G}_k$. In addition, in CBS, it is assumed that the value of α is set by negotiating between the relays through sending their corresponding β_k to the transmitter. This assumption is not required in ICBS, as the value of α can be set as $\alpha = \sqrt{\frac{P}{\beta}}$, where β is a predefined threshold. Finally, in both CBS and ICBS, it is assumed that the receiver has the perfect knowledge about the equivalent point-to-point channel from the transmitter to the receiver. This information can be obtained through sending pilot signals by the transmitter, amplified and forwarded at the relay nodes in the same manner as the information signal. In CBS, this assumption is equivalent to knowing the equivalent signal to noise ratio at each antenna.

IV. Asymptotic Analysis

In this section, we consider the asymptotic behavior ($K \to \infty$) of the achievable rate of ICBS. We show that by properly choosing the value of β, the achievable rate of ICBS converges rapidly to the capacity (the difference approaches zero as $O\left(\frac{1}{\log(K)}\right)$). The sequence of proof is as follows. In Lemma 1, we relate $\mathbb{P}\left[v > \xi\right]$ (the probability that the norm of interference term defined in equation (6) exceeds a certain threshold) to $\mathbb{P}[k \in \mathcal{A}]$ (the probability of turning off a relay) and $\mathbb{P}[\|\mathbf{U}_k\|^2 > \gamma]$ (the probability of having a sub-matrix with a large norm in the unitary matrix obtained from the SVD of $\mathbf{H}$). In Lemma 2, we bound $\mathbb{P}[\|\mathbf{U}_k\|^2 > \gamma]$. In Lemma 3, we bound $\mathbb{P}[k \in \mathcal{A}]$. As a result, in Lemma 4, we show that by properly choosing the value of β, with high probability, one can simultaneously reduce the effect of the interference to $o(K)$ and maintain a large value of α. In

Lemma 5, we show that with high probability, the minimum singular value of $\mathbf{H}$ scales as $O(K)$. Finally, in Theorem 1, we prove the main result by showing that the achievable rate of ICBS converges to the capacity of the uplink channel. As a consequence stated in corollary 1, the difference of the rates scales as $O(\frac{1}{\log(K)})$. As another consequence, the probability of outage $O(\frac{1}{\log(K)})$ below the ergodic capacity approaches zero as the number of relays increases. Using the proof of Lemma 4 and Theorem 1, Theorem 2 shows that as long as the power of relays behaves as $P_r(K) = \omega\left(\frac{P}{K}\log^9(K)\right)$, the same rate is achievable by ICBS. Finally, in Theorem 3, we study the asymptotic SNR behavior of CBS and ICBS, and show that, unlike the matched filtering scheme of BNOP, CBS and its variant achieve the full multiplexing gain, regardless of the number of relays.

Lemma 1 *Consider a parallel MIMO relay network with K relays using ICBS. We have*

$$\mathbb{P}[v > \xi] \quad \leq \quad \frac{MNK^2}{\xi}\left(\mathbb{P}[B_k] + \gamma\mathbb{P}[A_k]\right), \tag{7}$$

where $v = \left\|\sum_{k\in\mathcal{A}} \mathbf{U}_k^H \mathbf{H}_k\right\|^2$, $A_k \equiv (k \in \mathcal{A})$, and $B_k \equiv (\|\mathbf{U}_k\|^2 > \gamma)$.

Proof: See [14]. ■

Lemma 2 *Consider a $KN \times M$ Unitary matrix $\mathbf{U}$, where its columns $\mathbf{U}_i$, $i = 1, \cdots, M$, are isotropically distributed unit vectors in $\mathbb{C}^{NK\times 1}$. Let $\mathbf{W}$ be an arbitrary $N \times M$ sub-matrix of $\mathbf{U}$. Then, for $\gamma = \omega\left(\frac{1}{K}\right)$, as $K \to \infty$, we have*

$$\mathbb{P}\left[\|\mathbf{W}\|^2 \geq \gamma\right] = O\left((K\gamma)^{(N-1)} e^{-\frac{\gamma}{M}NK}\right) \tag{8}$$

Proof: See Appendix A of [14]. ■

Lemma 3 *For a small enough value of δ, we have*

$$\mathbb{P}[A_k] \leq \mathbb{P}[B_k] + c_1\sqrt{\delta} + c_2 e^{-\frac{d}{\sqrt{\delta}}}, \tag{9}$$

where $\delta = \frac{\gamma}{\beta}$, and c_1, c_2 and d are positive constant parameters independent of K, β, and γ.

Proof: See [14]. ■

Lemma 4 *By assigning $\beta = \frac{1}{\log(K)}$ and $\gamma = \frac{2\log(K)}{K}$, ICBS simultaneously achieves*

$$\alpha = \Omega\left(\sqrt{\log(K)}\right), \tag{10}$$

$$\mathbb{P}\left[v > \frac{K}{\log^2(K)}\right] = O\left(\frac{\log^4(K)}{\sqrt{K}}\right), \tag{11}$$

where v is defined in Lemma 1.

Proof: See [14]. ■

Lemma 5 *Let $\mathbf{A}$ be an $r \times s$ matrix whose entries are i.i.d complex Gaussian random variables with zero mean and unit variance. Assume that r is fixed and s tends to infinity. Then, with probability one $\lambda_{\min}(\mathbf{A}) \sim s$, or more precisely,*

$$\mathbb{P}\left[\lambda_{\min}(\mathbf{A}) \sim s\left(1 + O\left(\sqrt[4]{\frac{\log(s)}{s}}\right)\right)\right] \gtrsim 1 - O\left(\frac{1}{s\sqrt{\log(s)}}\right), \tag{12}$$

where $\lambda_{\min}(\mathbf{A})$ denotes the minimum singular value of $\mathbf{A}\mathbf{A}^H$.

Proof: See Appendix B of [14]. ■

Now, we prove the main theorem of this section.

Theorem 1 *By setting the threshold as $\beta = \frac{1}{\log(K)}$, the achievable rate of the proposed ICBS converges to the upper-bound capacity defined for the uplink channel. More precisely,*

$$\lim_{K\to\infty} C_u(K) - R_{ICBS}(K) = 0, \tag{13}$$

where

$$C_u(K) = \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\max_{\mathbf{Q},\mathrm{Tr}\{\mathbf{Q}\}\leq \mathrm{P}} \log\left(\left|\mathbf{I}_{KN} + \mathbf{H}\mathbf{Q}\mathbf{H}^H\right|\right)\right]$$

is the point to point ergodic capacity of the uplink channel and $R_{ICBS}(K)$ is the achievable rate of ICBS.

Proof: By applying the cut-set bound theorem [15] on the broadcast uplink channel, it can be easily verified [1], [2] that the point-to-point capacity of the uplink channel, $C_u(K)$, is an upper-bound on the capacity of the parallel MIMO relay network. Note that the factor $\frac{1}{2}$ in the expression of $C_u(K)$ is due to the half-duplex relaying. Define $C_{u^\star}(K) = \frac{M}{2}\log\left(\frac{KNP}{M}\right) + O\left(\sqrt{\frac{\log(K)}{K}}\right)$. We first show that $C_{u^\star}(K)$ is an upper-bound for $C_u(K)$, and then prove that a lower-bound for $R_{ICBS}(K)$ converges to $C_{u^\star}(K)$.

$$\begin{aligned}
C_u(K) &\overset{(a)}{=} \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\max_{\substack{\mathbf{Q}\\ \mathrm{Tr}\{\mathbf{Q}\}\leq P}} \log\left(\left|\mathbf{I}_M + \mathbf{H}^H\mathbf{H}\mathbf{Q}\right|\right)\right] \\
&\overset{(b)}{\leq} \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\max_{\substack{\mathbf{Q}\\ \mathrm{Tr}\{\mathbf{Q}\}\leq P}} M\log\left(1 + \frac{\mathrm{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}}{M}\right)\right] \\
&\overset{(c)}{\leq} \frac{M}{2}\max_{\substack{\mathbf{Q}\\ \mathrm{Tr}\{\mathbf{Q}\}\leq P}} \log\left(1 + \mathbb{E}_{\mathbf{H}}\left[\frac{\mathrm{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}}{M}\right]\right)
\end{aligned} \tag{14}$$

Here, (a) follows from the matrix determinant equality[3] , (b) results from the fact that for any positive semidefinite

[3]Assuming $\mathbf{A}$ and $\mathbf{B}$ to be $M \times N$ and $N \times M$ matrices respectively, we have $|\mathbf{I}_M + \mathbf{A}\mathbf{B}| = |\mathbf{I}_N + \mathbf{B}\mathbf{A}|$ [16].

matrix $\mathbf{A}$, we have $|\mathbf{A}| \leq \left(\frac{\text{Tr}\{\mathbf{A}\}}{M}\right)^M$, and (c) follows from the concavity of the logarithm function.

Let us define $\mathscr{D} = \mathscr{B} \bigcap \mathscr{C}$, in which $\mathscr{B}$ denotes the event that $\lambda_{\min}(\mathbf{H}) \sim KN\left[1 + O\left(\sqrt[4]{\frac{\log(K)}{K}}\right)\right]$ and $\mathscr{C}$ represents the event that $KNM\left[1 - \sqrt{\frac{2\log(K)}{K}}\right] < \text{Tr}\{\mathbf{H}^H\mathbf{H}\} < KNM\left[1 + \sqrt{\frac{2\log(K)}{K}}\right]$. Applying Lemma 5, we have $\mathbb{P}[\mathscr{B}^c] \lesssim O\left(\frac{1}{K\sqrt{\log(K)}}\right)$. Also, observing that the event $\mathscr{B}$ is defined in the same way as the event $\mathcal{D}_i$ in the proof of Lemma 5 (see [14]), and following the same steps by using Central Limit Theorem, we have $\mathbb{P}[\mathscr{C}^c] \sim O\left(\frac{1}{K\sqrt{\log(K)}}\right)$. Defining $\mathscr{E} \triangleq \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}\right]$, we bound $\mathscr{E}$ as follows

$$
\begin{aligned}
\mathscr{E} &\leq \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}\middle|\, \mathscr{D}\right] + \\
&\quad\; \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}\middle|\, \mathscr{D}^c\right]\mathbb{P}[\mathscr{D}^c] \\
&\overset{(a)}{\leq} \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}\middle|\, \mathscr{D}\right] + P \\
&\quad\; \left(\mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}\right] - \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}\middle|\, \mathscr{D}\right]\mathbb{P}\left[\mathscr{D}\right]\right) \\
&\overset{(b)}{\leq} \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}\middle|\, \mathscr{D}\right] + \\
&\quad\; PMNK\left(\sqrt{\frac{2\log(K)}{K}} + \mathbb{P}[\mathscr{D}^c]\right) \\
&\overset{(c)}{\lesssim} \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\mathbf{Q}\right\}\middle|\, \mathscr{D}\right] + \\
&\quad\; +PMNKO\left(\sqrt{\frac{\log(K)}{K}}\right) \\
&\overset{(d)}{\leq} \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\lambda_{\max}\left(\mathbf{H}\right)\mathbf{I}_M\mathbf{Q}\right\}\middle|\, \mathscr{D}\right] + \\
&\quad\; PMNKO\left(\sqrt{\frac{\log(K)}{K}}\right) \\
&\overset{(e)}{\leq} P\mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\} - (M-1)\lambda_{\min}\left(\mathbf{H}\right)\middle|\, \mathscr{D}\right] + \\
&\quad\; PMNKO\left(\sqrt{\frac{\log(K)}{K}}\right) \\
&\overset{(f)}{\lesssim} PKN\left[1 + O\left(\sqrt{\frac{\log(K)}{K}}\right)\right]. \qquad (15)
\end{aligned}
$$

Here, (a) follows from the generalization of the Cauchy-Schwarz inequality to the positive semidefinite matrices[4], $\text{Tr}\left\{\mathbf{Q}\right\} \leq P$, and also $\mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}\right] = \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}\middle|\, \mathscr{D}\right]\mathbb{P}[\mathscr{D}] + \mathbb{E}_{\mathbf{H}}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}\middle|\, \mathscr{D}^c\right]\mathbb{P}[\mathscr{D}^c]$, (b) follows from $\mathbb{E}\left[\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}\right] = MNK$ and the fact that conditioned on $\mathscr{D}$, we have $\text{Tr}\{\mathbf{H}^H\mathbf{H}\} > MNK\left[1 - \sqrt{\frac{2\log(K)}{K}}\right]$, (c) follows from the union bound on the probability and the fact that $\mathbb{P}[\mathscr{B}^c], \mathbb{P}[\mathscr{C}^c] \lesssim O\left(\frac{1}{K\sqrt{\log(K)}}\right)$, (d) follows from the fact that $\mathbf{H}^H\mathbf{H} \preccurlyeq \lambda_{\max}\left(\mathbf{H}\right)\mathbf{I}_M$, (e) results from the fact that $\lambda_{\max}\left(\mathbf{H}\right) \leq \sum_{i=1}^{M}\lambda_i\left(\mathbf{H}\right) - (M-1)\lambda_{\min}\left(\mathbf{H}\right) = \text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\} - (M-1)\lambda_{\min}\left(\mathbf{H}\right)$, and finally (f) follows from the fact that conditioned on $\mathscr{D}$, $\text{Tr}\left\{\mathbf{H}^H\mathbf{H}\right\}$ is upper-bounded by $KNM\left(1 + \sqrt{\frac{2\log(K)}{K}}\right)$ and $\lambda_{\min}\left(\mathbf{H}\right)$ is lower-bounded by $KN\left(1 - O\left(\sqrt[4]{\frac{\log(K)}{K}}\right)\right)$. Applying (14) and (15), we have

$$
\begin{aligned}
C_u(K) &\lesssim \frac{M}{2}\log\left(1 + KNP\left[1 + O\left(\sqrt{\frac{\log(K)}{K}}\right)\right]\right) \\
&\sim \frac{M}{2}\log\left(\frac{KNP}{M}\right) + O\left(\sqrt{\frac{\log(K)}{K}}\right) \\
&= C_{u^\star}(K) \qquad (16)
\end{aligned}
$$

Now, we lower-bound $R_{ICBS}(K)$. Rephrasing (6), we have

$$
\mathbf{y} = \alpha\mathbf{H}^\star\mathbf{x}' + \mathbf{n}^\star, \qquad (17)
$$

where

$$
\begin{aligned}
\mathbf{H}^\star &= \mathbf{\Lambda}^{\frac{1}{2}} - \sum_{k\in\mathcal{A}}\mathbf{U}_k^H\mathbf{H}_k\mathbf{V}, \\
\mathbf{n}^\star &= \alpha\sum_{k\in\mathcal{A}^c}\mathbf{U}_k^H\mathbf{n}_k + \mathbf{z} \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{P}_{\mathbf{n}^\star}\right),
\end{aligned}
$$

where $\mathbf{P}_{\mathbf{n}^\star} = \alpha^2\left(\sum_{k\in\mathcal{A}^c}\mathbf{U}_k^H\mathbf{U}_k\right) + \mathbf{I}_M$. The achievable rate of such a system is

$$
\begin{aligned}
R_{ICBS}(K) &= \frac{1}{2}\mathbb{E}_{\mathbf{H}}\Bigg[\\
&\quad \log\left(\left|\mathbf{I}_M + \alpha^2\frac{P}{M}\mathbf{H}^\star\mathbf{H}^{\star H}\mathbf{P}_{\mathbf{n}^\star}^{-1}\right|\right)\Bigg] \\
&\overset{(a)}{\geq} \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\log\left(\left|\frac{\alpha^2}{1+\alpha^2}\frac{P}{M}\mathbf{H}^\star\mathbf{H}^{\star H}\right|\right)\right] \\
&= \frac{M}{2}\log\left(\frac{\alpha^2}{1+\alpha^2}\right) + \\
&\quad \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\log\left(\left|\frac{P}{M}\mathbf{H}^\star\mathbf{H}^{\star H}\right|\right)\right], \qquad (18)
\end{aligned}
$$

where (a) follows from the fact that $\mathbf{P}_{\mathbf{n}^\star} = (\alpha^2+1)\mathbf{I}_M - \alpha^2\left(\sum_{k\in\mathcal{A}}\mathbf{U}_k^H\mathbf{U}_k\right)$ which results in $\mathbf{P}_{\mathbf{n}^\star}^{-1} \succcurlyeq \frac{1}{\alpha^2+1}\mathbf{I}_M$. For convenience, let $R_L(K) = \frac{1}{2}\mathbb{E}_{\mathbf{H}}\left[\log\left(\left|\frac{P}{M}\mathbf{H}^\star\mathbf{H}^{\star H}\right|\right)\right]$. Since α is lower-bounded by the inverse of the threshold as $\alpha \geq \sqrt{\frac{P}{\beta}}$, we have $\lim_{K\to\infty}\frac{M}{2}\log\left(\frac{\alpha^2}{1+\alpha^2}\right) = 0$, or equivalently

$$
\lim_{K\to\infty} R_{ICBS}(K) - R_L(K) \geq 0. \qquad (19)
$$

Define the events E_K and F_K as $E_K \equiv \left(\lambda_{\min}\left(\mathbf{H}\right) \gtrsim KN\left[1 + O\left(\sqrt[4]{\frac{\log K}{K}}\right)\right]\right)$ and $F_K \equiv \left(\left\|\mathbf{U}_{\mathcal{A}}^H\mathbf{H}_{\mathcal{A}}\right\|^2 \leq \frac{K}{\log^2(K)}\right)$. Consequently, we have

$$
\mathbb{P}\left[E_K, F_K\right] \overset{(a)}{\geq} 1 - \mathbb{P}[E_K^c] - \mathbb{P}[F_K^c] \overset{(b)}{\gtrsim} 1 + O\left(\frac{\log^4(K)}{\sqrt{K}}\right). \qquad (20)
$$

[4]Assuming $\mathbf{A}$ and $\mathbf{B}$ to be positive semidefinite matrices respectively, we have $\text{Tr}\left\{\mathbf{AB}\right\} \leq \text{Tr}\left\{\mathbf{A}\right\}\text{Tr}\left\{\mathbf{B}\right\}$ [17].

Here, (a) follows from union bound inequality and (b) follows from Lemmas 4 and 5. Assume the diagonal entries of $\mathbf{\Lambda}$ are ordered as $\lambda_1(\mathbf{H}) \geq \lambda_2(\mathbf{H}) \geq \cdots \geq \lambda_M(\mathbf{H})$. Thus, $R_L(K)$ can be lower bounded as

$$\begin{aligned}
R_L(K) &\geq \mathbb{P}\left[E_K, F_K\right] \mathbb{E}_{\mathbf{H}} \Bigg[\\
&\qquad \log\left(\left|\sqrt{\frac{P}{M}}\left(\mathbf{\Lambda}^{\frac{1}{2}} - \mathbf{U}_{\mathcal{A}}^H \mathbf{H}_{\mathcal{A}} \mathbf{V}\right)\right|\right) \Bigg| E_K, F_K \Bigg] \\
&\overset{(a)}{\geq} \mathbb{P}\left[E_K, F_K\right] \mathbb{E}_{\mathbf{H}} \Bigg[\log\Bigg(\left(\frac{P}{M}\right)^{\frac{M}{2}} \Bigg(\prod_{i=1}^{M} \lambda_i^{\frac{1}{2}}(\mathbf{H}) \\
&\qquad - \sum_{i=1}^{M} i! \binom{M}{i} \left\|\mathbf{U}_{\mathcal{A}}^H \mathbf{H}_{\mathcal{A}} \mathbf{V}\right\|_\star^i \prod_{j=1}^{M-i} \lambda_j^{\frac{1}{2}}(\mathbf{H})\Bigg)\Bigg) \\
&\qquad \Bigg| E_K, F_K \Bigg] \\
&\overset{(b)}{\geq} \mathbb{P}\left[E_K, F_K\right] \mathbb{E}_{\mathbf{H}} \Bigg[\log\Bigg(\left(\frac{P}{M}\right)^{\frac{M}{2}} \prod_{i=1}^{M} \lambda_i^{\frac{1}{2}}(\mathbf{H}) \\
&\qquad \Bigg(1 - \sum_{i=1}^{M} i! \binom{M}{i} \left(\frac{\left\|\mathbf{U}_{\mathcal{A}}^H \mathbf{H}_{\mathcal{A}}\right\|^2}{\lambda_{\min}(\mathbf{H})}\right)^{\frac{i}{2}}\Bigg)\Bigg) \\
&\qquad \Bigg| E_K, F_K \Bigg] \\
&\overset{(c)}{\gtrsim} \mathbb{P}\left[E_K, F_K\right] \mathbb{E}_{\mathbf{H}} \Bigg[\log\Bigg(\left(\frac{P}{M}\right)^{\frac{M}{2}} \prod_{i=1}^{M} \lambda_i^{\frac{1}{2}}(\mathbf{H}) \cdot \\
&\qquad \cdot \Bigg(1 - \sum_{i=1}^{M} i! \binom{M}{i} \Big(N \log^2(K) \Big[1 + \\
&\qquad + O\left(\sqrt[4]{\frac{\log K}{K}}\right)\Big]\Big)^{\frac{-i}{2}}\Bigg)\Bigg) \Bigg| E_K, F_K \Bigg] \\
&\overset{(d)}{\gtrsim} \left\{\frac{M}{2} \log\left(\frac{KNP}{M}\right) + O\left(\frac{1}{\log(K)}\right)\right\} \\
&\qquad \mathbb{P}\left[E_K, F_K\right] \\
&\overset{(e)}{\gtrsim} \frac{M}{2} \log\left(\frac{KNP}{M}\right) + O\left(\frac{1}{\log(K)}\right). \qquad (21)
\end{aligned}$$

Here, (a) follows from an upper-bound on the determinant expansion [5] of $\mathbf{\Lambda}^{\frac{1}{2}} - \mathbf{U}_{\mathcal{A}}^H \mathbf{H}_{\mathcal{A}} \mathbf{V}$, expanded over all possible set entries between $\mathbf{\Lambda}$ and $\mathbf{U}_{\mathcal{A}}^H \mathbf{H}_{\mathcal{A}} \mathbf{V}$, (b) follows from the fact that the Frobenius norm of a matrix is an upper-bound on the square of the maximum absolute value among its entries and also $\forall i : \lambda_i(\mathbf{H}) \geq \lambda_{\min}(\mathbf{H})$, (c) follows from the fact that the expectation is derived conditioned on the events E_K and F_K, (d) holds due to the fact that conditioned on E_K, we have $\lambda_i(\mathbf{H}) \gtrsim KN\left[1 + O\left(\sqrt[4]{\frac{\log K}{K}}\right)\right]$ and $\log\left(1 + O\left(\sqrt[4]{\frac{\log(K)}{K}}\right)\right) \sim O\left(\sqrt[4]{\frac{\log(K)}{K}}\right) \sim o\left(\frac{1}{\log^2(K)}\right)$, and finally, (e) results from (20). Comparing (16), (19), and (21), completes the proof. ■

[5] $\det(A) = \sum_\pi (-1)^{\sigma(\pi)} a_{1\pi_1} a_{2\pi_2} \cdots a_{n\pi_n} \leq \sum_\pi |a_{1\pi_1} a_{2\pi_2} \cdots a_{n\pi_n}|$, where σ is the parity function of permutation.

Corollary 1 *Achievable rate of ICBS is at most $O\left(\frac{1}{\log(K)}\right)$ below the upper-bound corresponding to the cut-set defined on the point-to-point uplink channel, i.e. $C_u(K)$.*

Proof: See [14]. ■

Apart from increasing the rate, using parallel relays also increases the reliability of the transmission. As the following corollary shows, the probability of outage when sending information at the rate $O\left(\frac{1}{\log(K)}\right)$ below the ergodic capacity approaches zero, as $K \to \infty$.

Corollary 2 *Consider the parallel MIMO relay network and ICBS with the threshold value $\beta = \frac{1}{\log(K)}$. We have*

$$\mathbb{P}\left[\frac{1}{2}\log\left(\left|\mathbf{I}_M + \alpha^2 \frac{P}{M} \mathbf{H}^\star \mathbf{H}^{\star H} \mathbf{P}_{\mathbf{n}^\star}^{-1}\right|\right) \lesssim C_u(K) + O\left(\frac{1}{\log(K)}\right)\right] \sim O\left(\frac{\log^4(K)}{\sqrt{K}}\right).$$

Proof: See [14]. ■

Another interesting result is that by increasing the number of relays, each relay can operate with a much lower power as compared to the transmitter, while the scheme achieves the optimum rate. This shows another benefit of using many parallel relays in the network.

Theorem 2 *Up to the point that $P_r(K) = \omega\left(\frac{P}{K}\log^9(K)\right)$, the achievable rate of ICBS satisfies*

$$\begin{aligned}
\lim_{K\to\infty} R_{ICBS}(K) - C_u(K) &= \\
\lim_{K\to\infty} R_{ICBS}(K) - \frac{M}{2}\log\left(\frac{KNP}{M}\right) &= 0.
\end{aligned}$$

Proof: See [14]. ■

Theorem 3 *The proposed Cooperative Beamforming scheme and its variant achieve the maximum multiplexing gain of the relay channel. More precisely:*

$$\lim_{P\to\infty} \frac{R_{CBS}(P)}{\log(P)} = \frac{M}{2}, \qquad (22)$$

and $\frac{M}{2}$ is the maximum achievable multiplexing gain of the underlying half duplex system. (Here $R_{CBS}(P)$ is the achievable rate of the proposed scheme for the given power constraint P.)

Proof: See [14]. ■

V. SIMULATION RESULTS

Figure 2 shows the simulation results for the achievable rate of ICBS, BNOP matched filtering scheme [1], and the upper-bound of the capacity based on the uplink Cut-Set for varying number of relays. The number of transmitting and receiving antennas in the relays, the transmitter, and the receiver is $M = N = 2$, and the SNR is $P_s = P_r = 10dB$. While both of the schemes demonstrate logarithmic scaling of rate in terms of K, we observe that there is a significant gap between the BNOP scheme and our scheme, reflecting the gap of $O(1)$ in the achievable rate of [1]. On the other hand, the gap between ICBS and the upper-bound rapidly approaches zero due to the term $O\left(\frac{1}{\log(K)}\right)$ predicted in Corollary 2.

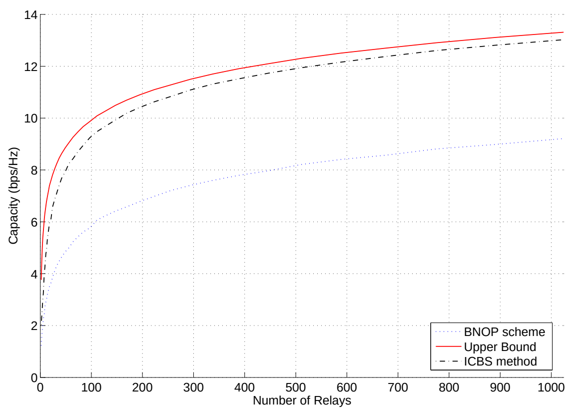


Fig. 2. Upper-bound of the capacity, ICBS, and BNOP matched filtering Scheme vs. number of relays in parallel MIMO relay network

VI. CONCLUSION

A simple new scheme, Cooperative Beamforming Scheme (CBS), based on Amplify and Forward (AF) strategy is introduced in a parallel MIMO relay network. A variant of CBS, called Incremental Cooperative Beamforming Scheme (ICBS) is shown to achieve the capacity of parallel MIMO relay network for $K \to \infty$. The scheme is shown to rapidly approach the upper-bound of the capacity with a gap no more than $O\left(\frac{1}{\log(K)}\right)$. As a result, it is shown that the capacity of a parallel MIMO relay network is $C(K) = \frac{M}{2}\log\left(1 + \frac{KNP}{M}\right) + O\left(\frac{1}{\log(K)}\right)$ in terms of the number of relays, K. Moreover, it is shown that as the number of relays increases, the relays in ICBS can operate using much less power without any performance degradation. Finally, the proposed scheme is shown to achieve the maximum multiplexing gain regardless of the number of relays. The simulation results confirm the validity of the theoretical arguments.

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