How much feedback is required in MIMO Broadcast Channels?

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***Abstract*— In this paper, a downlink communication system, in which a Base Station (BS) equipped with M antennas communicates with N users each equipped with K receive antennas is considered. We study the minimum required amount of feedback at the BS, in order to achieve the maximum sum-rate capacity. First, we define the amount of feedback as the average number of users who send information to the BS. In the asymptotic case of $N \to \infty$, we show that with finite amount of feedback, it is not possible to achieve the maximum sum-rate. Indeed, in order to reduce the gap between the achieve sum-rate and the optimum value to zero, a minimum feedback of $\ln \ln \ln N$ is asymptotically necessary. Then, we consider a practical scenario, in which the amount of feedback is defined as the average number of bits which is sent to the BS. We show that to achieve the maximum sum-rate, infinite amount of feedback is required. Moreover, the minimum amount of feedback, in order to reduce the gap to the optimum sum-rate to zero, scales as $\Theta(\ln \ln \ln N)$, which is achievable by the random beam-forming scheme proposed in [12].**

I. Introduction

Multiple-Input Multiple-Output (MIMO) systems have proved their ability to achieve high bit rates in a scattering wireless network. In a point-to-point scenario, it has been shown that the capacity scales linearly with the minimum number of transmit and receive antennas, regardless of the availability of Channel State Information (CSI) at the transmitter [1] [2]. This linear increase is so-called *multiplexing gain*.

In a MIMO Broadcast Channel (MIMO-BC), a BS equipped with multiple antennas communicates with several multiple-antenna users. Recently, there has been a lot of interest in characterizing the capacity region of this channel [3], [4], [5], [6]. In these works, it has been shown that the sum-rate capacity of MIMO-BC grows linearly with the minimum number of transmit and receive antennas, provided that both the transmitter and the receivers have perfect CSI. Indeed, in a network with a large number of users, the BS can increase the throughput by selecting the best set of users to communicate with. This results in the so-called *multiuser diversity gain* [7], [8].

Unlike the point-to-point scenario, in MIMO-BC, it is crucial for the transmitter to have CSI. It has been shown that MIMO-BC without CSI at the BS is degraded [9]. Moreover, for the case of single antenna users, multiplexing gain reduces to one, and multiuser diversity gain disappears [10] [11].

Due to the weak performance of having no CSI at the BS, some authors have considered MIMO-BC with partial CSI [10] [12] [13] [14]. Reference [12] proposes a downlink transmission scheme based on random beam-forming relying on partial CSI at the transmitter. In this scheme, the BS randomly constructs M orthogonal beams and transmits data to the users with the maximum Signal to Interference plus Noise Ratio (SINR) for each beam. Therefore, only the value of maximum SINR, and the index of the beam for which the maximum SINR is achieved, are fed back to the BS for each user. This significantly reduces the amount of feedback. Reference [12] shows that when the number of users tends to infinity, the optimum sum-rate throughput can be achieved. Reference [10] considers a downlink channel where the receivers have perfect CSI, but the transmitter only has quantized information regarding the channel instantiation. This reference shows that the full multiplexing gain can be achieved with partial CSI if the quality of the CSI is increased linearly with SNR. A similar result is obtained in [13], by showing that achieving the asymptotic sum-rate capacity of the MIMO-BC requires the transmitter to know the fading levels with infinite precision. More precisely, this reference shows that with finite feedback rate, the maximum achievable multiplexing gain is 1. In fact, both [10] and [13] study the performance degradation of MIMO-BC due to the imperfect CSI, at high SNR regime. The size of the network (the number of users) is assumed to be fixed in these references.

In [14], we have considered a downlink scheme based on zero-forcing beam-forming and have proved that when the number of users, N, tends to infinity, the maximum sum-rate capacity is achievable with the amount of feedback scaling as $[\ln N]^M$. Two essential questions arises here; i) Is it possible to achieve the maximum sum-rate capacity with a finite feedback in a large network ($N \to \infty$), at a fixed SNR? ii) If not, what is the minimum feedback rate (in terms of N), in order to achieve the sum-rate capacity of the system?

In this paper, we aim to answer the above questions. First, we define the amount of feedback as the average number of users who send information to the BS. Our results show that it is not possible to achieve the maximum sum-rate with a finite amount of feedback. Moreover, to reduce the gap between the achieved sum-rate and the optimum value to zero, the amount of feedback must be greater than $\ln\ln\ln N$. In the second part, we define the amount of feedback as the number of information bits sent to the BS. Our analysis shows that the minimum amount of feedback, in order to reduce the gap to the optimum sum-rate to zero, scales as $\Theta(\ln\ln\ln N)$, which can be achieved using random beam-forming scheme proposed in [12].

The rest of the paper is organized as follows. In section II, we introduce the system model. Sections III is devoted to asymptotic analysis of the amount of feedback.

Throughout this paper, the norm of the vectors and the Frobenius norm of the matrices are denoted by $\|.\|$, the Hermitian operation is denoted by $(.)^*$, and the determinant and the trace operations are denoted by $\det(.)$ and $\mathrm{Tr}(.)$, respectively. $\mathbb{E}\{.\}$ represents the expectation, notation "ln" is used for the natural logarithm, and the rates are expressed in *nats*. RH(.) represents the right hand side of the equations, and Prob$\{\mathcal{A}\}$ denotes the probability of event $\mathcal{A}$. For given functions $f(N)$ and $g(N)$, $f(N) = O(g(N))$ is equivalent to $\lim_{N\to\infty}\left|\frac{f(N)}{g(N)}\right| < \infty$, $f(N) = o(g(N))$ is equivalent to $\lim_{N\to\infty}\left|\frac{f(N)}{g(N)}\right| = 0$, $f(N) = \Omega(g(N))$ is equivalent to $\lim_{N\to\infty}\frac{f(N)}{g(N)} > 0$, $f(N) = \omega(g(N))$ is equivalent to $\lim_{N\to\infty}\frac{f(N)}{g(N)} = \infty$, and $f(N) = \Theta(g(N))$ is equivalent to $\lim_{N\to\infty}\frac{f(N)}{g(N)} = c$, where $0 < c < \infty$.

II. System Model

In this work, a MIMO-BC in which a BS equipped with M antennas communicates with N users, each equipped with K antennas, is considered. The channel between each user and the BS is modeled as a zero-mean circularly symmetric Gaussian matrix (Rayleigh fading). The received vector by user k can be written as

$$\boldsymbol{y}_k = \boldsymbol{H}_k\boldsymbol{x} + \boldsymbol{n}_k, \tag{1}$$

where $\boldsymbol{x} \in \mathbb{C}^{M\times 1}$ is the transmitted signal, $\boldsymbol{H}_k \in \mathbb{C}^{K\times M}$ is the channel matrix from the transmitter to the kth user (assumed to be known at the receiver side), and $\boldsymbol{n}_k \in \mathbb{C}^{K\times 1} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I}_K)$ is the noise vector at this receiver. We assume that the transmitter has an average power constraint P, i.e. $\mathbb{E}\{\mathrm{Tr}(\boldsymbol{x}\boldsymbol{x}^*)\} \leq P$. We consider a block fading model in which each $\boldsymbol{H}_k$ is constant for the duration of a frame. The frame itself is assumed to be long enough to allow communication at rates close to the capacity.

III. Asymptotic Analysis

A. The average number of users send feedback to the BS

In this section, we define the amount of feedback as the average number of users who send feedback to the BS. In the following theorems, we provide the necessary and the sufficient condition in order to achieve $\lim_{N\to\infty}\frac{\mathcal{R}_S}{\mathcal{R}_{\mathrm{Opt}}} = 1$, and $\lim_{N\to\infty}\mathcal{R}_{\mathrm{Opt}} - \mathcal{R}_S = 0$, for any user selection strategy S, respectively:

Theorem 1 *Consider a MIMO-BC with N users ($N \to \infty$), which utilizes a user selection strategy S. Let $\mathcal{N}_S$ be the number of users who send information to the BS in this strategy. Then, the necessary and sufficient condition to achieve $\lim_{N\to\infty}\frac{\mathcal{R}_S}{\mathcal{R}_{\mathrm{Opt}}} = 1$, is having*

$$\mathbb{E}\{\mathcal{N}_S\} \sim \omega(1). \tag{2}$$

Proof- *Necessary Condition-* Let us denote $\mathcal{G}_S$ as the set of users who send information to the BS, when using strategy S. Define $p_S(k)$ as the probability that user k belongs to $\mathcal{G}_S$. Since we consider a homogenous network, this probability is independent of k, and we denote it by p_S. Therefore, $\mathcal{N}_S = |\mathcal{G}_S|$ is a Binomial random variable with parameter p_S, and we have $\mathbb{E}\{\mathcal{N}_S\} = Np_S$.

Let us define

$$\mathcal{R}_1 = \mathbb{E}\left\{\max_{\substack{\boldsymbol{Q}_n \\ \sum \mathrm{Tr}(\boldsymbol{Q}_n)=\mathrm{P}}} \ln\det\left(\boldsymbol{I}_M + \sum_{n=1}^{N}\boldsymbol{H}_n^*\boldsymbol{Q}_n\boldsymbol{H}_n\right)\middle|\, \mathcal{A}_S\right\},$$

and

$$\mathcal{R}_2 = \mathbb{E}\left\{\max_{\substack{\boldsymbol{Q}_n \\ \sum \mathrm{Tr}(\boldsymbol{Q}_n)=\mathrm{P}}} \ln\det\left(\boldsymbol{I}_M + \sum_{n=1}^{N}\boldsymbol{H}_n^*\boldsymbol{Q}_n\boldsymbol{H}_n\right)\middle|\, \mathcal{A}_S^C\right\},$$

where $\mathcal{A}_S$ is the event that $|\mathcal{G}_S| = 0$, and $\mathcal{A}_S^C$ is the complement of $\mathcal{A}_S$. We have

$$\begin{aligned}\mathcal{R}_S &\leq \mathrm{Prob}\{\mathcal{A}_S\}\mathcal{R}_{\mathcal{A}_S}^{\mathrm{NCSI}} + \mathrm{Prob}\{\mathcal{A}_S^C\}\mathcal{R}_2 \\ &= (1-p_S)^N\mathcal{R}_{\mathcal{A}_S}^{\mathrm{NCSI}} + \left[1-(1-p_S)^N\right]\mathcal{R}_2, \end{aligned}\tag{3}$$

where $\mathcal{R}_S$ denotes the achievable sum-rate by the strategy S, $\mathcal{R}_{\mathcal{A}_S}^{\mathrm{NCSI}}$ stands for the sum-rate of MIMO-BC when no CSI is available at the BS, conditioned on $\mathcal{A}_S$. The above equation comes from the fact that with probability $(1-p_S)^N$ no users are selected and hence, the resulting sum-rate is upper-bounded by $\mathcal{R}_{\mathcal{A}_S}^{\mathrm{NCSI}}$. Using (3), and having

$$\mathcal{R}_{\mathrm{Opt}} = \mathrm{Prob}\{\mathcal{A}_{\mathrm{S}}\}\mathcal{R}_1 + \mathrm{Prob}\{\mathcal{A}_{\mathrm{S}}^{\mathrm{C}}\}\mathcal{R}_2, \tag{4}$$

where $\mathcal{R}_{\mathrm{Opt}}$ is the maximum achievable sum-rate in MIMO-BC, we can write

$$\mathcal{R}_{\mathrm{Opt}} - \mathcal{R}_{\mathrm{S}} \geq (1-p_S)^N(\mathcal{R}_1 - \mathcal{R}_{\mathcal{A}_S}^{\mathrm{NCSI}}). \tag{5}$$

It can be easily shown that

$$\mathcal{R}_1 \geq \mathbb{E}\left\{\ln\left(1 + P\max_{j,k}\|\boldsymbol{H}_{j,k}\|^2\right)\middle|\, \mathcal{A}_S\right\}, \tag{6}$$

where $\boldsymbol{H}_{j,k}$ denotes the jth row of $\boldsymbol{H}_k$. The right hand side of (6) can be lower-bounded as,

$$\begin{aligned}\mathrm{RH}(6) \geq\ & \mathbb{E}\left\{\ln\left(1 + P\max_{j,k}\|\boldsymbol{H}_{j,k}\|^2\right)\middle|\, \mathcal{A}_S, \mathcal{M}_t\right\} \times \\ & \mathrm{Prob}\left\{\mathcal{M}_t|\mathcal{A}_S\right\}, \end{aligned}\tag{7}$$

where $\mathcal{M}_t$ is the event that $\max_{j,k} \|\boldsymbol{H}_{j,k}\|^2 > t$, for a chosen t. Hence,

$$\begin{aligned} \text{RH}(6) &\geq \ln(1+Pt)\frac{\text{Prob}\{\mathcal{A}_S, \mathcal{M}_t\}}{\text{Prob}\{\mathcal{A}_S\}} \\ &\geq \ln(1+Pt)\frac{1-\text{Prob}\{\mathcal{A}_S^C\}-\text{Prob}\{\mathcal{M}_t^C\}}{\text{Prob}\{\mathcal{A}_S\}} \\ &= \ln(1+Pt)\left(1-\frac{\text{Prob}\{\mathcal{M}_t^C\}}{\text{Prob}\{\mathcal{A}_S\}}\right), \end{aligned} \quad (8)$$

where $\mathcal{M}_t^C$ is the complement of $\mathcal{M}_t$. $\text{Prob}\{\mathcal{M}_t^C\}$ can be computed as

$$\begin{aligned} \text{Prob}\{\mathcal{M}_t^C\} &= \text{Prob}\left\{\max_{k,j} \|\boldsymbol{H}_{k,j}\|^2 \leq t\right\} \\ &= (1-e^{-t})^{NK}. \end{aligned} \quad (9)$$

Now, assume that

$$\mathbb{E}\{\mathcal{N}_S\} = Np_S \nsim \omega(1), \quad (10)$$

i.e., $Np_S \sim O(1)$. Choosing $t = \frac{\ln N}{2}$, from (9), we obtain

$$\text{Prob}\{\mathcal{M}_t^C\} \sim e^{-K\sqrt{N}}. \quad (11)$$

Indeed, noting $\text{Prob}\{\mathcal{A}_S\} = (1-p_S)^N$ and $Np_S \sim O(1)$, we have

$$\text{Prob}\{\mathcal{A}_S\} \sim \Theta(1). \quad (12)$$

Substituting (11) and (12) in (8) yields

$$\begin{aligned} \text{RH}(6) &\geq \ln\left(1+\frac{P}{2}\ln N\right)\left(1-\Theta(e^{-K\sqrt{N}})\right) \\ &\sim \ln\ln N + O(1). \end{aligned} \quad (13)$$

Indeed, similar to [9], it can be shown that

$$\begin{aligned} \mathcal{R}_{\mathcal{A}_S}^{\text{NCSI}} &= \mathbb{E}_{\boldsymbol{H}_k|\mathcal{A}_S}\left\{\ln\det\left[\boldsymbol{I}+\frac{P}{M}\boldsymbol{H}_k\boldsymbol{H}_k^*\right]\middle|\mathcal{A}_S\right\} \\ &\leq M\mathbb{E}_{\boldsymbol{H}_k|\mathcal{A}_S}\left\{\ln\left(1+\frac{P}{M}\|\boldsymbol{H}_k\|^2\right)\middle|\mathcal{A}_S\right\} \\ &\leq M\ln\left(1+\frac{P}{M}\mathbb{E}_{\boldsymbol{H}_k|\mathcal{A}_S}\left\{\|\boldsymbol{H}_k\|^2\middle|\mathcal{A}_S\right\}\right) \\ &\leq M\ln\left(1+\frac{P}{M}\frac{\mathbb{E}_{\boldsymbol{H}_k}\left\{\|\boldsymbol{H}_k\|^2\right\}}{\text{Prob}\{\mathcal{A}_S\}}\right) \\ &= M\ln\left(1+\frac{PK}{\text{Prob}\{\mathcal{A}_S\}}\right) \\ &\overset{(12)}{\sim} \Theta(1). \end{aligned} \quad (14)$$

Combining (6), (13), and (14), and substituting in (5), under the assumption of (10), we get

$$\begin{aligned} \mathcal{R}_{\text{Opt}} - \mathcal{R}_S &\geq \left(1-\frac{O(1)}{N}\right)^N [\ln\ln N + O(1)] \\ &\sim e^{-O(1)}\ln\ln N. \\ \Rightarrow \frac{\mathcal{R}_S}{\mathcal{R}_{\text{Opt}}} &\leq 1-\frac{e^{-O(1)}\ln\ln N}{\mathcal{R}_{\text{Opt}}}. \end{aligned} \quad (15)$$

As a result, noting that $\mathcal{R}_{\text{Opt}} \sim M\ln\ln N$, we obtain

$$\mathbb{E}\{\mathcal{N}_S\} \nsim \omega(1) \Rightarrow \lim_{N\to\infty}\frac{\mathcal{R}_S}{\mathcal{R}_{\text{Opt}}} \neq 1. \quad (16)$$

Sufficient Condition- Let us define the strategy S as selecting M users randomly among the following set:

$$\mathcal{G}_S = \{k|\lambda_{\max}(\boldsymbol{H}_k) > t\}, \quad (17)$$

where $\lambda_{\max}(\boldsymbol{H}_k)$ is the maximum singular value of $\boldsymbol{H}_k\boldsymbol{H}_k^*$, and t is a threshold value. After selecting the users, the BS performs zero-forcing beam-forming, where the coordinates are chosen as the eigenvectors, corresponding to the maximum singular values of the selected users. In [15], it has been shown that for a $K\times M$ matrix $\boldsymbol{A}$, whose elements are i.i.d Gaussian, we have

$$p_S \triangleq \text{Prob}\{\lambda_{\max}(\boldsymbol{A}) > t\} = \frac{t^{M+K-2}e^{-t}(1+O(t^{-1}))}{\Gamma(M)\Gamma(K)}. \quad (18)$$

Hence,

$$\begin{aligned} \mathbb{E}\{\mathcal{N}_S\} &= Np_S \\ &= N\frac{t^{M+K-2}e^{-t}(1+O(t^{-1}))}{\Gamma(M)\Gamma(K)}. \end{aligned} \quad (19)$$

Having $\mathbb{E}\{\mathcal{N}_S\} \sim \omega(1)$, yields,

$$t \sim \ln N + (M+K-2)\ln\ln N - \omega(1). \quad (20)$$

Utilizing zero-forcing beam-forming at the BS, and defining

$$\mathcal{R}^* \triangleq M\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{\ln\left(1+\frac{P}{\text{Tr}\left\{[\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}}]^{-1}\right\}}\right)\middle| |\mathcal{G}_S| \geq M\right\},$$

we can write

$$\mathcal{R}_S \geq \mathcal{R}^*\text{Prob}\{|\mathcal{G}_S| \geq M\}, \quad (21)$$

where $\boldsymbol{\mathcal{H}} = \left[\boldsymbol{g}_{s_1,\max}^T|\boldsymbol{g}_{s_2,\max}^T|\cdots|\boldsymbol{g}_{s_m,\max}^T\right]^T$ in which $\boldsymbol{g}_{s_i,\max} = \sqrt{\lambda_{\max}(\boldsymbol{H}_{s_i})}\boldsymbol{V}_{s_i,\max}^*$, $i = 1,\cdots,m$ $(m \leq M)$, and $\boldsymbol{V}_{s_i,\max}$ is the eigenvector corresponding to maximum singular value of the ith selected user (s_i).

$\eta_S \triangleq \text{Prob}\{|\mathcal{G}_S| \geq M\}$ can be computed as follows:

$$\begin{aligned} \eta_S &= 1-\text{Prob}\{|\mathcal{G}_S| < M\} \\ &= 1-\sum_{m=0}^{M-1}\binom{N}{m}p_S^m(1-p_S)^{N-m} \\ &\geq 1-\sum_{m=0}^{M-1}\frac{(Np_S)^m}{m!}e^{-(N-m)p_S}. \end{aligned} \quad (22)$$

Since $Np_S \sim \omega(1)$, we have $\eta_S \sim 1-o(1)$.

Indeed, we can lower-bound $\mathcal{R}^*$ as

$$\mathcal{R}^* \geq M\ln P - M\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{X(\boldsymbol{\mathcal{H}})|\ |\mathcal{G}_S| \geq M\right\}, \quad (23)$$

where $X(\boldsymbol{\mathcal{H}}) \triangleq \ln\left(\text{Tr}\left\{[\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}}]^{-1}\right\}\right)$. In [16], it has been shown that

$$\mathbb{E}_{\boldsymbol{\mathcal{H}}}\left\{X(\boldsymbol{\mathcal{H}})|\ |\mathcal{G}_S| \geq M\right\} \leq \ln\frac{M}{t} + (M-1)\ln(2M^2). \quad (24)$$

Using the above equation and (23), and selecting $t > \ln N$, yields,

$$\mathcal{R}^* \geq M \ln\left(\frac{P\ln N}{M}\right) - M(M-1)\ln(2M^2). \qquad (25)$$

Substituting $\mathcal{R}^*$ and η_S in (21), and having the fact that $\mathcal{R}_{\rm Opt} \sim M\ln\ln N$ [12], yields

$$\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{R}_{\rm Opt}} = 1. \qquad (26)$$

■

Theorem 2 *For any user selection strategy S, the necessary condition to achieve* $\lim_{N\to\infty} \mathcal{R}_{\rm Opt} - \mathcal{R}_S = 0$ *is having*

$$\mathbb{E}\{\mathcal{N}_S\} \sim \ln\ln\ln N + \omega(1). \qquad (27)$$

Proof - Assume that

$$\mathbb{E}\{\mathcal{N}_S\} \nsim \ln\ln\ln N + \omega(1). \qquad (28)$$

In other words, $\mathbb{E}\{\mathcal{N}_S\} \sim \ln\ln\ln N + O(1)$, or $\mathbb{E}\{\mathcal{N}_S\} < \ln\ln\ln N$. Similar to (5), we can write

$$\mathcal{R}_{\rm Opt} - \mathcal{R}_S \quad \geq \quad (1-p_S)^N[\mathcal{R}_1 - \mathcal{R}^{\rm NCSI}_{\mathcal{A}_S}]. \qquad (29)$$

Following the same approach as in Theorem 1, under the assumption of (28), we can show that $\mathcal{R}_1 \gtrsim \ln\ln N + O(1)$, and $\mathcal{R}^{\rm NCSI}_{\mathcal{A}_S} \sim O(\ln\ln\ln N)$. Hence,

$$\begin{aligned}\mathcal{R}_{\rm Opt} - \mathcal{R}_S &\geq (1-p_S)^N\left[\ln\ln N + O(\ln\ln\ln N)\right] \\ &\sim e^{-\mathbb{E}\{\mathcal{N}_S\}[1+O(p_S)]}\left[\ln\ln N + O(\ln\ln\ln N)\right] \\ &\sim e^{-(\mathbb{E}\{\mathcal{N}_S\}-\ln\ln\ln N)}\left[1+o(1)\right]. \qquad (30)\end{aligned}$$

In the case of $\mathbb{E}\{\mathcal{N}_S\} \sim \ln\ln\ln N + O(1)$, we have RH(30) $\sim e^{-O(1)}\left[1+o(1)\right]$, and in the case of $\mathbb{E}\{\mathcal{N}_S\} < \ln\ln\ln N$, we have RH(30) $\sim \Upsilon\left[1+o(1)\right]$, where $\Upsilon > 1$. As a result,

$$\mathbb{E}\{\mathcal{N}_S\} \nsim \ln\ln\ln N + \omega(1) \Rightarrow \lim_{N\to\infty} \mathcal{R}_{\rm Opt} - \mathcal{R}_S \neq 0. \quad (31)$$

■

Theorem 3 *The sufficient condition to achieve* $\lim_{N\to\infty} \mathcal{R}_{\rm opt} - \mathcal{R}_S = 0$ *is having*

$$\mathbb{E}\{\mathcal{N}_S\} \sim M\ln\ln\ln N + \omega(1). \qquad (32)$$

Proof - Consider the random beam-forming strategy, introduced in [12]. In this strategy, the BS randomly constructs M orthogonal beams and transmits data to the users with the maximum SINR for each beam. Assuming each user's antenna as a separate user, we define the following set:

$$\mathcal{G}^{(m)}_{\rm RBF} = \{k | \exists i, \quad {\rm SINR}^{(m)}_{k,i} > t\}, \quad m = 1, \cdots, M, \qquad (33)$$

where ${\rm SINR}^{(m)}_{k,i}$ is the received SINR over the ith antenna of the kth user, for the mth transmitted beam. $\mathcal{G}_{\rm RBF} = \bigcup_{m=1}^{M} \mathcal{G}^{(m)}_{\rm RBF}$ is the set of users who send feedback to the BS. The achievable sum-rate by this scheme, denoted by $\mathcal{R}_{\rm RBF}$, is lower-bounded as,

$$\begin{aligned}\mathcal{R}_{\rm RBF} &\geq M\ln(1+t){\rm Prob}\left\{\bigcap_{m=1}^{M}\mathcal{A}_m\right\} \\ &\geq M\ln(1+t)\left(1-\sum_{m=1}^{M}{\rm Prob}\{\mathcal{A}_m^C\}\right), \quad (34)\end{aligned}$$

where $\mathcal{A}_m$ is the event that $|\mathcal{G}^{(m)}_{\rm RBF}| \geq 1$, and $\mathcal{A}_m^C$ is the complement of $\mathcal{A}_m$.

For a randomly chosen user k, we define

$$\begin{aligned}p_k^{(m)} &\triangleq {\rm Prob}\{k \in \mathcal{G}^{(m)}_{\rm RBF}\} \\ &= {\rm Prob}\left\{\bigcup_{i=1}^{K}\mathcal{B}^{(m)}_{k,i}\right\} \\ &\leq \sum_{i=1}^{K}\eta^{(m)}_{k,i}, \qquad (35)\end{aligned}$$

where $\mathcal{B}^{(m)}_{k,i}$ is the event that ${\rm SINR}^{(m)}_{k,i} > t$, and $\eta^{(m)}_{k,i} \triangleq {\rm Prob}\{\mathcal{B}^{(m)}_{k,i}\}$, which is independent of k, i, m, and we denote it by η. Indeed, $p_k^{(m)}$ is independent of k, m, and is denoted by p. Hence, $p \leq K\eta$.

To evaluate the right hand side of (34), first we compute ${\rm Prob}\{\mathcal{A}_m^C\}$ as follows:

$$\begin{aligned}{\rm Prob}\{\mathcal{A}_m^C\} &= (1-\eta)^{KN} \\ &\leq \left(1-\frac{p}{K}\right)^{KN}. \qquad (36)\end{aligned}$$

Therefore,

$$\begin{aligned}{\rm RH}(34) &\geq M(1+t)\left[1-M\left(1-\frac{p}{K}\right)^{KN}\right] \\ &\geq M(1+t)[1-Me^{-Np}]. \qquad (37)\end{aligned}$$

Under the condition of (32), and knowing the fact that $p = \frac{e^{-Mt/P}}{(1+t)^{M-1}}$ [12], we can write

$$\begin{aligned}\mathcal{R}_{\rm RBF} \geq\; & M\ln\left(1+\frac{P}{M}\ln N + O(\ln\ln N)\right) \times \\ & \left(1-Me^{-Np}\right). \qquad (38)\end{aligned}$$

Using the above equation and having the facts that $\mathcal{R}_{\rm Opt} \sim M\ln\left(1+\frac{P}{M}\ln N + O(\ln\ln N)\right)$ [12], $\mathbb{E}\{\mathcal{N}_{\rm RBF}\} \leq MNp$, and $\mathbb{E}\{\mathcal{N}_{\rm RBF}\} \sim M\ln\ln\ln N + \omega(1)$, we can write

$$\begin{aligned}\mathcal{R}_{\rm Opt} - \mathcal{R}_{\rm RBF} &\leq O\left(\frac{\ln\ln N}{\ln N}\right) + \\ &\quad M^2 e^{-\left(\frac{\mathbb{E}\{\mathcal{N}_{\rm RBF}\}}{M} - \ln\ln\ln N\right)}[1+o(1)] \\ &\sim o(1). \qquad (39)\end{aligned}$$

Consequently, $\lim_{N\to\infty} \mathcal{R}_{\rm Opt} - \mathcal{R}_{\rm RBF} = 0$.

■

B. Amount of bits fed back to the BS

In this section, we study the minimum amount of feedback required at the BS (in terms of bits), in order to achieve the optimum sum-rate capacity. It is assumed that the number of bits per each user is an integer value.

Theorem 4 *The necessary and sufficient condition to achieve* $\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{R}_{\text{Opt}}} = 1$ *for any users selection strategy* S*, is having*

$$\mathcal{F}_S \sim \omega(1), \tag{40}$$

where $\mathcal{F}_S$ *is the average number of bits fed back to the BS.*

Proof- *Necessary condition-* From Theorem 1, it is realized that the average number of users in order to achieve $\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{R}_{\text{Opt}}} = 1$ must be infinite. Since the number of bits sent to the BS by each user in $\mathcal{G}_S$ is at least $\Theta(1)$, to achieve $\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{R}_{\text{Opt}}} = 1$, the total number of bits sent to the BS must be infinite.

Sufficient Condition- Consider the scheme described in the proof of the sufficient condition, in Theorem 1. Assume that each selected user quantizes the eigenvector corresponding to their maximum singular value, using Random Vector Quantization (RVQ), and sends this quantized value to the BS. Similar to the proof of Theorem 1 in [10], it can be shown that

$$\mathcal{R}_S - \mathcal{R}_S^{\text{Q}} \leq \ln(1 + \frac{P}{M} t^* . 2^{-\frac{N_{FB}}{M-1}}), \tag{41}$$

where $\mathcal{R}_S^{\text{Q}}$ is the achievable sum-rate of the proposed strategy, when quantization is used, N_{FB} is the number of quantized bits for each selected user, and $t^* = \ln N + (M + K - 2) \ln \ln N$.

The total number of bits sent to the BS is computed from

$$\mathcal{F}_S = M \mathcal{N}_S N_{FB}. \tag{42}$$

Assume that $\mathcal{F}_S(N) = \varphi(N) \sim \omega(1)$. Let $\mathcal{N}_S(N) = N_{FB}(N) = \Psi(N)$, where $\Psi(N) \triangleq \sqrt{\frac{\varphi(N)}{M}}$. It follows from Theorem 1 that

$$\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{R}_{\text{Opt}}} = 1. \tag{43}$$

Indeed, from (41), it is realized that having $N_{FB} \sim \omega(1)$ yields

$$\lim_{N\to\infty} \frac{\mathcal{R}_S}{\mathcal{R}_S^{\text{Q}}} = 1. \tag{44}$$

Combining (43) and (44), we have

$$\lim_{N\to\infty} \frac{\mathcal{R}_S^{\text{Q}}}{\mathcal{R}_{\text{Opt}}} = 1, \tag{45}$$

for any function $\varphi(N) \sim \omega(1)$.

■

Theorem 5 *The necessary and sufficient condition to achieve* $\lim_{N\to\infty} \mathcal{R}_{\text{Opt}} - \mathcal{R}_S = 0$*, is having*

$$\mathcal{F}_S \sim \Theta(\ln \ln \ln N) + \omega(1). \tag{46}$$

Proof- *Necessary condition-* The necessary condition, easily follows from Theorem 2, and the fact that each user in the set $\mathcal{G}_S$ must send at least $\Theta(1)$ bits to the BS.

Sufficient condition- Consider the random beam-forming scheme. In the proof of Theorem 3, it can be observed that given any function $f(N) \triangleq \mathbb{E}\{\mathcal{N}_S\} \sim M \ln \ln \ln N + \omega(1)$, one can find a threshold t, which is the solution to the following equation:

$$\frac{e^{-Mt/P}}{(1+t)^{M-1}} = \frac{f(N)}{MN}. \tag{47}$$

Since the users in $\mathcal{G}_{\text{RBF}}^{(m)}$ only need to send the index m to the BS, for each user $\lceil \log_2(M) \rceil$ bits are required. Consequently, it is possible to achieve $\lim_{N\to\infty} \mathcal{R}_{\text{Opt}} - \mathcal{R}_S = 0$ with number of feedback bits scaling as $M \lceil \log_2(M) \rceil \ln \ln \ln N + \omega(1)$.

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