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Application of Cumulant Method In Performance Evaluation of Turbo-Like Codes

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Abstract In this article, a new method for performance evaluation of Turbo-like codes is presented. This is based on estimating the Probability Density Function (pdf) of the bit Log-Likelihood-Ratio (LLR) using higher order statistics. We do not restrict ourselves to any specific model for the pdf and try to estimate it directly using a Cumulant matching method. Numerical results show a close agreement between the proposed method and simulations. The complexity of this method is similar to the Monte-Carlo simulation with the advantage of providing similar accuracy using significantly fewer samples.

I. Introduction

For any arbitrary random variable, the logarithm of its characteristics function may be approximated using a Taylor series expansion. The coefficients of this series expansion are known as *Cumulants* or higher order statistics. Cumulants have been widely used in a variety of applications including analysis of digital communications systems.

The problem of performance evaluation of coherent optical communication systems is considered in [1], where a solution based on estimating the Cumulants of the noise process is presented. A condition is derived to quantify under what system conditions a Gaussian Probability Density Function (pdf) is a good approximation. A discrete-time method is proposed in [2] for estimating the impulse response of a frequency selective digital modulated communication channel. This method is based on estimating the Cumulants up to the fourth order. Parameters of a moving average model are estimated in [3], using second and third order Cumulant matching. This estimation is further improved in [4]. Cumulants of symmetric distributions like uniform, triangular, and Gaussian are estimated in [5] using a robust estimation technique. The application of Edgeworth series and higher-order statistics to the discrete-time detection of a known constant signal in multivariate non-Gaussian noise is considered in [6]. A numerical algorithm based on knowledge of the noise Cumulants is presented in order to analyze the finite-sample size performance of the sub-optimum detectors.

Non-Gaussian sources are modeled in [7] using Gaussian mixture densities. It is shown that in high Signal to Noise Ratio (SNR) regions, this method outperforms the Cumulant based algorithms for parameter estimation. The problem of blind equalization and estimation of digital communication finite impulse response channels is considered in [8]. The channel parameters are estimated by nonlinear optimization of a quadratic Cumulant matching criterion involving second and fourth order Cumulants. This problem is later considered in [9] for partial-response signals. A method for phase recovery in Quadrature Amplitude Modulation (QAM) communication systems based on higher order statistics is presented in [10]. A relation is derived between the phase error and the fourth order Cumulant of the output.

Since the higher order Cumulant-based criteria can be multi-modal, conventional gradient search techniques require a good initial estimate to avoid converging to local minima. This problem is solved in [11], where a novel scheme based on genetic algorithms is employed to optimize the Cumulant fitting cost function. A method based on higher order statistics is proposed in [12] to mitigate the performance degradation caused by multi-path propagation in a mobile radio communication system. It is shown that an over-determined system of linear equations (involving only Cumulants of the received baseband signal) can be obtained to perform non-iterative deconvolution. The study of chaotic communication systems with Additive White Gaussian Noise (AWGN) interference is considered in [13] by employing suitable Cumulant analysis tools.

In this paper we present a method based on using the Cumulants of the bit Log-Likelihood-Ratio (LLR) versus its moments as used in [14]. The first two Cumulants of the normal density are its mean and variance and the higher order Cumulants are zero. Since the pdf of the bit LLR is nearly normal [15]–[17], it is expected that its higher order Cumulants are fairly small. This allows for easy truncation of the series expansion of the pdf in terms of its Cumulants.

This paper is organized as follows. The problem is modeled in Section II. In Section III, the Cumulant matching method, which is used to find the parameters of the proposed model, is described. The accuracy of this method is investigated in Section IV. The numerical results and conclusion are presented in Section V and Section VI, respectively.

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II. Preliminaries

A common tool to express the bit probabilities in bit decoding algorithms is based on the so-called LLR. The LLR of the k^{th} bit position is defined by the following equation:

$$LLR(k) = \log \frac{P(c_k = 1|\mathbf{x})}{P(c_k = 0|\mathbf{x})}, \quad (1)$$

where c_k is the value of the k^{th} bit in the transmitted code-word, $\mathbf{x}$ is the received vector, and $\log$ represents the natural logarithm. Let us define the random variable $Y = LLR(k)$ and let its *pdf* be denoted as $f(y)$. It is proved in [18] that the *pdf* of the bit LLR is independent of the transmitted code-word, as long as the value of the bit position under consideration remains unchanged. By using this result and without loss of generality, we consider the case of sending the all-zero code-word. The received bit is decoded to 0 (or 1), if the corresponding LLR is negative (or positive). Therefore, the following integral simplifies the remaining Bit Error Rate (BER) calculations:

$$P_e = \int_0^{\infty} f(y)dy. \quad (2)$$

III. Cumulants Matching

The characteristic function of a random variable Y with its *pdf* denoted as $f(y)$ is defined as

$$\Phi(t) = \int_{-\infty}^{+\infty} f(y)e^{ity}dy. \quad (3)$$

The Cumulants of the random variable Y, denoted as k_m, are the coefficients of the following series expansion:

$$\log \Phi(t) = \sum_{m=0}^{\infty} k_m \frac{(it)^m}{m!}. \quad (4)$$

The first few Cumulants can be expressed in terms of the raw moments as follows [19]:

$$k_0 = 1, \quad (5)$$
$$k_1 = \mu_1, \quad (6)$$
$$k_2 = \mu_2 - \mu_1^2, \quad (7)$$
$$k_3 = 2\mu_1^3 - 3\mu_1\mu_2 + \mu_3, \quad (8)$$

where μ_i is the i^{th} raw moment of random variable Y with n samples y_j defined as

$$\mu_i = \frac{1}{n}\sum_{j=1}^{n} y_j^i \quad (9)$$

The K-statistics are the unique symmetric unbiased estimators of the Cumulants [20]. Thus,

$$E[K_m] = k_m, \quad (10)$$

where the notation K_m is used for the m^{th} K-statistic of a given density. In addition, the variance,

$$V[K_m] = E[(K_m - k_m)^2], \quad (11)$$

is a minimum compared to all other unbiased estimators [21], [22]. In other words, the K-statistics are the Uniformly Minimum Variance Unbiased Estimators (UMVUE) of the Cumulants. The first few K-statistics are as follows:

$$K_1 = \frac{S_1}{n}, \quad (12)$$
$$K_2 = \frac{nS_2 - S_1^2}{n(n-1)}, \quad (13)$$
$$K_3 = \frac{2S_1^3 - 3nS_1S_2 + n^2S_3}{n(n-1)(n-2)}, \quad (14)$$

where n is the number of samples (denoted by y_i) used in the estimation, and

$$S_r = \sum_{i=1}^{n} y_i^r. \quad (15)$$

A combinatorial method for computing higher orders of the K-statistics is presented in [23]. Once the first few Cumulants are estimated by using the K-statistics, the characteristic function of the bit LLR can be approximated by using (4). Following that, the *pdf* of the bit LLR can be approximated by taking the Inverse Fourier Transform (IFT) of $\Phi(t)$.

$$f(y) = \int_{-\infty}^{+\infty} \Phi(t)e^{-ity}dt \quad (16)$$

IV. Accuracy Analysis

The cumulative distribution function (CDF) of the bit LLR is defined as

$$F(T) = \int_{-\infty}^{T} f(y)dy. \quad (17)$$

We are interested in computing the error probability

$$P_e = \int_0^{\infty} f(y)dy = 1 - \int_{-\infty}^{0} f(y)dy = 1 - F(0). \quad (18)$$

Taking the IFT of the characteristic function, $f(y) = \text{IFT}\{\Phi(t)\}$, noting (4), and using properties of IFT for integral of a function, we have

$$F(T) = \text{IFT}\left\{\frac{1}{it}\exp[\sum_{m=0}^{\infty} k_m \frac{(it)^m}{m!}]\right\}. \quad (19)$$

A small error, Δk_m in estimating each Cumulant, results in an error, $\Delta F(T)$ in CDF:

$$F(T) + \Delta F(T) = \text{IFT}\left\{\frac{1}{it}\exp[\sum_{m=0}^{\infty}(k_m + \Delta k_m)\frac{(it)^m}{m!}]\right\} \quad (20)$$

$$= \text{IFT}\left\{\frac{1}{it}\exp[\sum_{m=0}^{\infty} k_m \frac{(it)^m}{m!}]\exp[\sum_{n=0}^{\infty}\Delta k_n \frac{(it)^n}{n!}]\right\} \quad (21)$$

$$\simeq \text{IFT}\left\{\frac{1}{it}\exp[\sum_{m=0}^{\infty} k_m \frac{(it)^m}{m!}](1 + \sum_{n=0}^{\infty}\Delta k_n \frac{(it)^n}{n!})\right\} \quad (22)$$

$$= F(T) + \text{IFT}\left\{\frac{1}{it}\exp[\sum_{m=1}^{\infty} k_m \frac{(it)^m}{m!}]\sum_{n=1}^{\infty}\Delta k_n \frac{(it)^n}{n!}\right\}, \quad (23)$$

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taking $\Delta k_0 = 0$. This means,

$$\Delta F(T) \simeq \text{IFT}\left\{\frac{1}{it}\exp[\sum_{m=1}^{\infty} k_m \frac{(it)^m}{m!}]\sum_{n=1}^{\infty}\Delta k_n \frac{(it)^n}{n!}\right\} \tag{24}$$

$$= \sum_{n=1}^{\infty}\Delta k_n \frac{i^{n-1}}{n!}\text{IFT}\left\{t^{n-1}\exp[\sum_{m=1}^{\infty} k_m \frac{(it)^m}{m!}]\right\} \tag{25}$$

$$= \sum_{n=1}^{\infty}\Delta k_n \frac{i^{n-1}}{n!} f^{(n-1)}(T), \tag{26}$$

where $f^{(n)}(T)$ is the n^{th} derivative of $f(y)$ at point $y = T$ for $n > 0$. To have a consistent notation, we define $f^{(0)}(T) = f(T)$. This results in the following relationship between the error in computing P_e and the error in estimating Cumulants:

$$\Delta P_e \simeq \sum_{n=1}^{\infty}\Delta k_n \frac{i^{n-1}}{n!} f^{(n-1)}(0). \tag{27}$$

In order to simplify (27), we assume that the derivatives, $f^{(n)}(y)$, are similar to the derivatives of the normal density. This is based on the fact that pdf of the bit LLR is close to the normal density. Thus we suppose that

$$f^{(n)}(y) \simeq (-1)^n e^{-y^2/2} T_n(y), \tag{28}$$

where $T_n(y)$ is the Hermite polynomial [14] of order n, defined as

$$T_n(y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j n!}{2^j (n-2j)! j!} y^{n-2j}. \tag{29}$$

This approximation results in the following equation:

$$\Delta P_e \simeq \sum_{l=0}^{\infty} \frac{\Delta k_{2l+1}}{(2l+1)2^l l!} \tag{30}$$

Numerical values presented in Table-3, section V have been calculated using (30).

V. Numerical Results

The proposed algorithm is compared with Monte-Carlo (MC) simulation in this section. A Turbo-code of length 100 and rate $1/2$ is used to perform the simulations as seen in Figure 1. It is evident that increasing the number of Cumulants (the order of approximation) that are involved from two to four significantly improves the approximation. A short code is used to assure the accuracy of MC simulation and hence provide a reliable comparison benchmark. The relationship between interval Probabilities of the Point Estimates (PPE) and the number of samples n is computed using numerical methods (complete description of these methods are available in the Journal version of this paper [24]) and demonstrated in Table-1 and Table-2. It is evident that the proposed method is still accurate even by using fewer samples compared to the MC simulations.

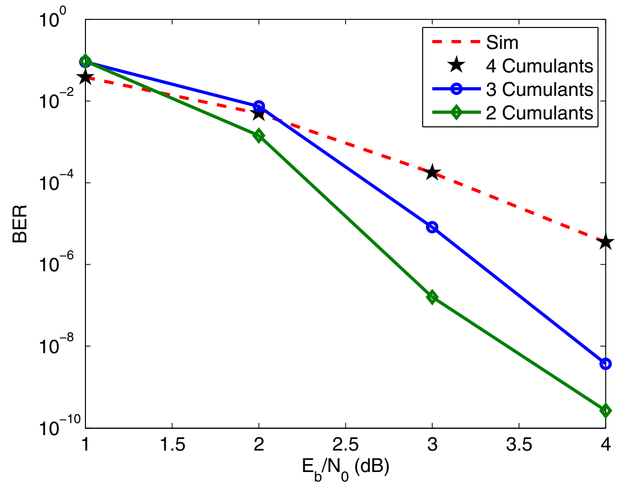

Fig. 1. BER curves for Turbo-Code of length 100 and rate $1/2$.

n	θ	p_c	p_m
10^4	0.0060	0.95	0.67
10^5	0.0060	0.96	0.70
10^6	0.0060	0.97	0.96
10^4	0.0020	0.94	0.66
10^5	0.0020	0.95	0.70
10^6	0.0020	0.96	0.96
10^4	0.0005	0.93	0.33
10^5	0.0005	0.94	0.68
10^6	0.0005	0.95	0.95

Table 1 : The Relationship between n (number of samples) and the PPEs ($p(|\Delta P_e| < \theta)$) at E_b/N_0=2dB for the Cumulant method p_c and the MC simulation p_m.

n	θ	p_c	p_m
10^6	10^{-6}	0.98	0.27
10^7	10^{-6}	0.99	0.96
10^8	10^{-6}	≈ 1	≈ 1
10^6	5×10^{-7}	0.94	0
10^7	5×10^{-7}	0.96	0.68
10^8	5×10^{-7}	≈ 1	0.99
10^6	10^{-7}	0.91	0
10^7	10^{-7}	0.92	0.08
10^8	10^{-7}	0.94	0.49

Table 2 : The Relationship between n (number of samples) and the PPEs ($p(|\Delta P_e| < \theta)$) at E_b/N_0=4dB for the Cumulant method p_c and the MC simulation p_m.

This method is similar to the one introduced in [25], where a suitable model for the *pdf* of bit LLR is suggested. The moment matching method with maximum entropy principle is then used to estimate the parameters of the suggested model for the *pdf*. In this case, a constrained maximization problem is solved using iterative Newton-Raphson method. At each iteration, solving a system of linear equations (with the same degree as the number of moments) as well as evaluating an integral of the exponential form is required. This renders the Cumulant method proposed here significantly less complex as compared to the moment method of [25].

Table-3 provides an example on how to decide on the required accuracy in estimating the Cumulants of different orders. Numerical values presented have been calculated using (30).

l	0	1	2	ΔP_e
$2l+1$	1	3	5	-
Δk_{2l+1}	10^{-4}	10^{-3}	10^{-2}	5.17×10^{-4}

Table 3 : The Relationship between error in Cumulant estimation and the error in BER estimation.

VI. Concluding Remarks

The problem of performance evaluation of a coded communication system with bit decoding algorithms in low BER regions is considered. The main ingredient of a bit decoding algorithm is the reliability information, i.e. the LLR. The pdf of the bit LLR is estimated using Cumulant matching technique. This method is based on estimating the characteristic function of the bit LLR using its Cumulants. In order to have an unbiased estimation of the Cumulants with minimum variance, the best choices are the K-statistics. Once the characteristic function of a random variable is known, the rest of the pdf computation is straightforward using the IFT. Numerical results demonstrate a close agreement between the theory and simulations. It is also shown that the error in BER estimation is bounded and may be reduced by increasing the accuracy of Cumulant estimation or equivalently increasing the number of samples. The complexity of this method is similar to the Monte-Carlo simulation with the advantage of providing similar accuracy using significantly fewer samples. The time consuming part of performance analysis is sample generation which is significantly decreased using the proposed method. Processing the samples and estimating the BER can be done very fast. Specifically in the provided example, 1 million samples are processed in less than a minute using an ordinary PC with AMD-Athlon 64-bit 3 GHz Processor running Windows XP.

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